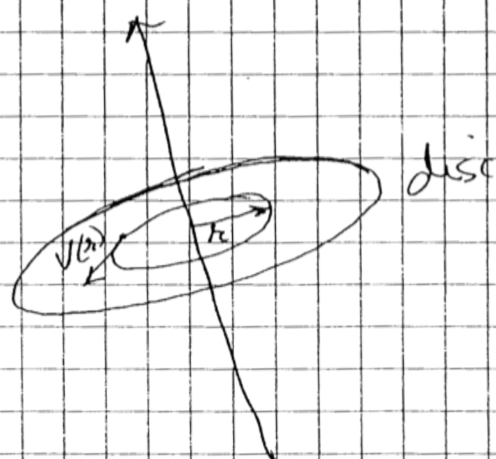
ASTRO 536, HW # 3 due 10/18/17 in class

Each problem is worth 15 points AND ½ of this HW is for Xtra-credit

(1) Consider a spiral galaxy such as M31. Think of it as a circular disc with a velocity distribution $V(r)$ and a mass density distribution $\sigma(r)$. Also define $M(r)$ as the total mass interior to radius r.



We can obtain $V(r)$ from Doppler effect in selected emission lines. $M(r)$ is then determined from the formula:

$$\boxed{M(r) = \frac{2}{G\pi}\int_0^r \frac{V^2(a)\, a\, da}{(r^2-a^2)^{1/2}}}$$

DERIVE THIS FORMULA!

The following will be helpful!

a) A circular disc is the limiting form of an oblate spheroid of semi-major axis a, and semi-minor axis c (eccentricity $k = \frac{(a^2-c^2)^{1/2}}{a}$ as $k \rightarrow 1$.)

b) The gravitational field vanishes inside a shell of uniform density bounded by 2 ellipsoids with the same axes and eccentricity. This is a generalization of the familiar elementary result for spherical shells.

c) The gravitational potential of an oblate spheroid, of uniform density, when evaluated on the median plane at a distance $r > a$ from the center is,

$$\phi(r) = -2\pi G a \frac{(1-k^2)^{1/2}}{k^2} \rho \left[\frac{1}{2}\sqrt{r^2-k^2a^2} + \left(ka - \frac{r^2}{2ka}\right) \arcsin \frac{ka}{r} \right]$$

ρ = density

d) If $\quad \psi(x) = \frac{1}{\pi^{1/2}} \int_0^x \frac{f(t)\,dt}{(x-t)^{1/2}}$

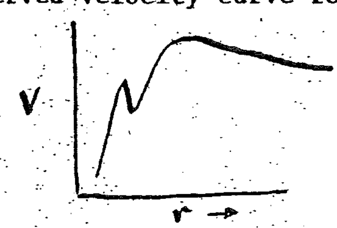
and $\quad \psi(0) = 0$

then:

$$f(x) = \frac{1}{\pi^{1/2}} \frac{d}{dx} \int_0^x \frac{\psi(t)\,dt}{(x-t)^{1/2}}$$

(This last is called Abel's integral equation. Try to derive it from properties of the gamma function.)

The observed velocity curve for M 31 looks like this:



In practice one approximates the curve with a polynomial and evaluates the integral numerically to find M(r).

2) Fun with Hydrogen 21 cm line from Hyperfine splitting.

a) Write down the Hamiltonian as a sum of terms, where: 0^{th} order is KE + PE; 1^{st} order is relativistic correction (1^{st} term in Taylor series expansion of the relativistic energy formula); 2^{nd} order is spin-orbit coupling; 3^{rd} order is spin-spin coupling. Estimate the contributions from each of these terms. Show from this that the energy for the spin-spin coupling is in the ball-park corresponding to a wavelength of 21 cm (very approximate!)

b) Rates of transitions. Take an oscillating dipole (**d** is the dipole moment, which is the charge q times the Bohr radius, a_0) at a frequency corresponding to the 21 cm wavelength (ν = 1420 MHz). Look up the classical (E&M) rate of energy loss (emission) of such a dipole, and set it equal to the rate of transitions, Q, times hbar •ω. Calculate Q (answer has units of 1/time). You'll find its VERY large. The reason is that a proper QM treatment will show that the electric dipole radiation is forbidden, and that the lowest non-forbidden radiative process is magnetic dipole radiation. Find (look up) the formula for magnetic dipole radiation and calculate the rate Q for this. You'll find a number close to that given in class.