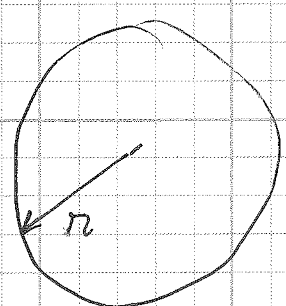
(1a) A homogeneous space does not imply isotropy. Consider the case where the expansion velocity $\vec{v}$ is given by:

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

with $v_x \neq v_y \neq v_z$.

Thus, expansion along the 3 axis occurs at different rates giving rise to anisotropy yet the universe maintains homogeneity.

(1b)



$$F = \frac{L}{4\pi r^2} \quad \text{for small distance}$$

# of objects inside sphere:

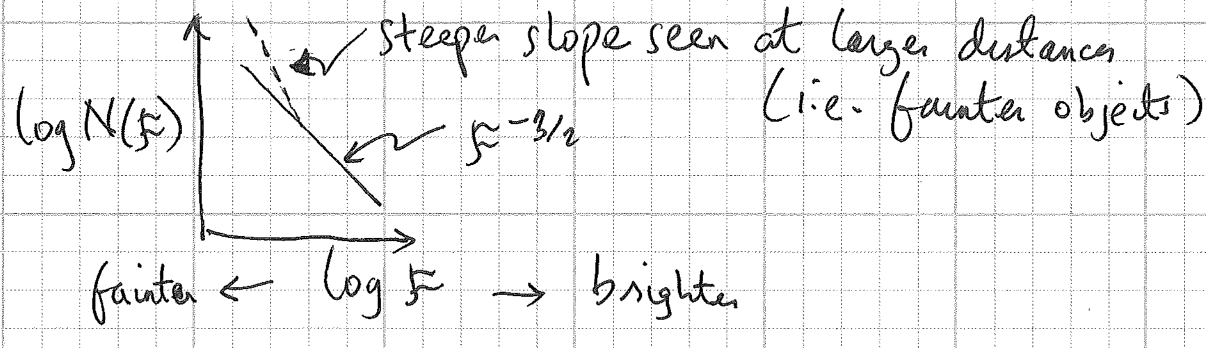
$$N = n\frac{4}{3}\pi r^3, \quad n \equiv \# \text{ density}$$

$$r = \left(\frac{3N}{4\pi}\right)^{1/3}$$; radius out to which one can observe objects of constant luminosity, L, with a limiting flux F.

$$F = \frac{L}{4\pi\left(\frac{3N}{4\pi}\right)^{2/3}}$$

$$N^{2/3} = \frac{L/F}{4\pi\left(3/4\pi\right)^{2/3}} \rightarrow N(F) = \frac{(L/F)^{3/2}}{(4\pi)^{3/2}(3/4\pi)}$$

$$\boxed{N(F) \propto F^{-3/2}}$$



The main reason for the deviation appears to be that the objects (galaxies) were more numerous in the past; so n, the # density, is not a constant but appears to evolve.

Any evidence for evolution, where properties of objects change with time, violates the "perfect cosmological principle" on which the original Steady State Theory was founded. This principle extends the cosmological principle we discussed in class to say that not only is the universe isotropic + homogeneous in space from every fundamental observer point of view, but also in time. Note that the Steady State theories are NOT static. There is expansion, but space is constantly being filled (matter is created spontaneously) to maintain constant density. Thus, at any place at any time in the Universe the properties of objects (large scale structures, densities of matter + radiation, etc) maintain the same values on average.

(2) Robertson-Walker metric:

$$ds^2 = c^2 dt^2 - dl^2, \quad \text{where}$$

$$dl^2 = R^2(t)\left[dr^2 + \mathcal{R}^2 \sin^2(r/\mathcal{R})\, d\Phi^2\right] \quad (a)$$

OR $$dl^2 = R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Phi^2\right] \quad (b)$$

where r & $R(t)$ are defined differently than in (a).

Here, $d\Phi^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

Surface area element:

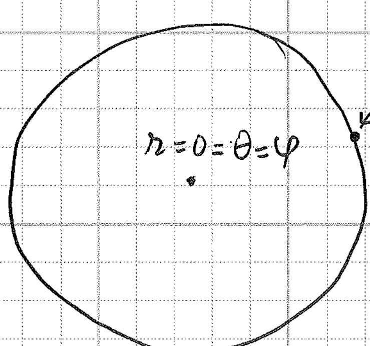
(a) $$dA = R^2 \mathcal{R}^2 \sin^2(r/\mathcal{R}) \sin\theta\, d\theta\, d\varphi$$

(b) $$dA = R^2 r^2 \sin\theta\, d\theta\, d\varphi$$

Volume element:

(a) $$dV = R^3 \mathcal{R}^2 \sin^2(r/\mathcal{R}) \sin\theta\, d\theta\, d\varphi\, dr$$

(b) $$dV = R^3 \frac{r^2\, dr}{(1-kr^2)} \sin\theta\, d\theta\, d\varphi$$



pt on sphere has coordinates r, θ, φ

Physical, or proper, radius of sphere:

(a) $$r' = \int_0^{r'} dr$$

(b) $$r' = \int_0^{r'} \frac{dr}{(1-kr^2)^{1/2}}$$

Calculate $\boxed{\text{Surface Area}}$:

(a) $A(r') = R^2 \mathcal{R}^2 \sin^2(r'/\mathcal{R}) \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi$

$= 4\pi R^2 \mathcal{R}^2 \sin^2(r'/\mathcal{R})$

Flat geometry, $\mathcal{R} \to \infty$, $\mathcal{R}^2 \sin^2(r'/\mathcal{R}) \to r'$

$\boxed{A(r') = 4\pi R^2 r'^2}$; as $r' \to \infty$ $A(r') \to \infty$

Unbounded (no edge), infinite

Closed ~~Open~~ geometry:

$$\boxed{A(r') = 4\pi R^2 \mathcal{R}^2 \sin^2(r'/\mathcal{R})}$$

the area increases as $(r'/\mathcal{R})$ increases, and reaches a maximum when $r'/\mathcal{R} = \pi/2$:

$$A_{max} = 4\pi R^2 \mathcal{R}^2 \text{ at } r'/\mathcal{R} = \pi/2$$

then $A \to 0$ as $r'/\mathcal{R} \to \pi$, and the cycle repeats

The space is closed, and $r'/\mathcal{R}$ going from $0 \to \pi/2$ ~~is~~ corresponds to traversing from the pole to the equator whence the area is a maximum (think of the circumference of a 2-sphere). Continuing traversal from $(r'/\mathcal{R}) = \frac{\pi}{2} \to \pi$ corresponds to traversing from equator to anti-pole, where the area (circumference in 2-sphere case) goes back to 0. Clearly the ~~are~~ space is finite yet one can continue traversing ($r'/\mathcal{R}$ can continue to increase, corresponding to traversing the great circle(s) from pole to pole ad infinitum) the space without "falling off": no edge → unbounded.

Open geometry : sin → sinh :

$$\boxed{A(r') = 4\pi a^2 R^2 \sinh(r'/R)}$$

as $(r'/R) \to \infty$, $A(r') \to \infty$: unbounded, infinite

(b) Using metric b we have to remember that the

~~Flat~~: ~~$A(r')=$~~ physical radius is:

$$r' = \int_0^{r'} \frac{dr}{(1-kr^2)^{1/2}}$$

For $k=0$, $r'=r$ (flat)

$k=+1$, $r' = \sin^{-1} r$ (closed ~~open~~)

$k=-1$, $r' = \sinh^{-1} r$ (open

Then the area $A(r')$:

flat: $\boxed{4\pi R^2 r'^2}$; as $r' \to \infty$, $A(r') \to \infty$; infinite, unbounded

closed: $\boxed{4\pi R^2 \sin^2(r')}$

As r' goes from $0 \to \pi/2$, $A(r')$ goes from 0 to $\boxed{A_{max} = 4\pi R^2}$

and as r' continues from $\pi/2 \to \pi$, $A(r')$ goes from $A_{max} \to 0$.

Same result as for metric a : finite, unbounded space

Open : $\boxed{4\pi R^2 \sinh^2(r')}$; as $r' \to \infty$, $A(r') \to \infty$; infinite, unbounded

~~Note that r' in metric b corresponds~~

Volumes :

(a) flat : $dV = R^3 r^2 \, dr \sin\theta \, d\theta \, d\varphi$

$$\boxed{V(r') = \frac{4\pi R^3 r'^3}{3}}$$

; $V(r') \to \infty$ as $r' \to \infty$ as expected from behavior of $A(r')$

Closed: $dV = R^3 \Re^2 \sin^2(r/\Re) \sin\theta \, d\theta \, d\varphi \, dr$

$$V(r') = 4\pi R^3 \Re^2 \int_0^{r'} \sin^2(r/\Re) \, dr$$

$$\boxed{V(r') = 4\pi R^3 \Re^2 \left[r'/2 - \frac{\Re}{4} \sin 2r'/\Re \right]}$$ Volume out to r'

The volume of the whole space corresponds to when $r'/\Re = \pi$ (~~A_{max}~~):

$$\boxed{V = 4\pi R^3 \Re^2 \left[\frac{\Re \pi}{2} - \frac{\Re}{4} \sin 2\pi \right]}$$

$$\boxed{V = 2\pi^2 R^3 \Re^3}$$ total volume of closed space with radius ~~$\Re$ today~~ equal to $R^3 \Re^3$.

Open: $dV = R^3 \Re^2 \sinh(r/\Re) \sin\theta \, d\theta \, d\varphi \, dr$

$$V(r') = 4\pi R^3 \Re^2 \int_0^{r'} \sinh^2(r/\Re) \, dr$$

$$\boxed{V(r') = 4\pi R^3 \Re^2 \left[\frac{\Re}{4} \sinh \frac{2r'}{\Re} - \frac{r'}{2} \right]}$$ $\to \infty$ as $r' \to \infty$

$\boxed{\text{Expansion of Results}}$: This only applies to the Closed & Open cases.

$\boxed{\text{Closed}}$: Expanding $\sin 2r'/\Re \approx \frac{2r'}{\Re} - \left(\frac{2r'}{\Re}\right)^3 / 3! + \left(\frac{2r'}{\Re}\right)^5 / 5! \cdots$

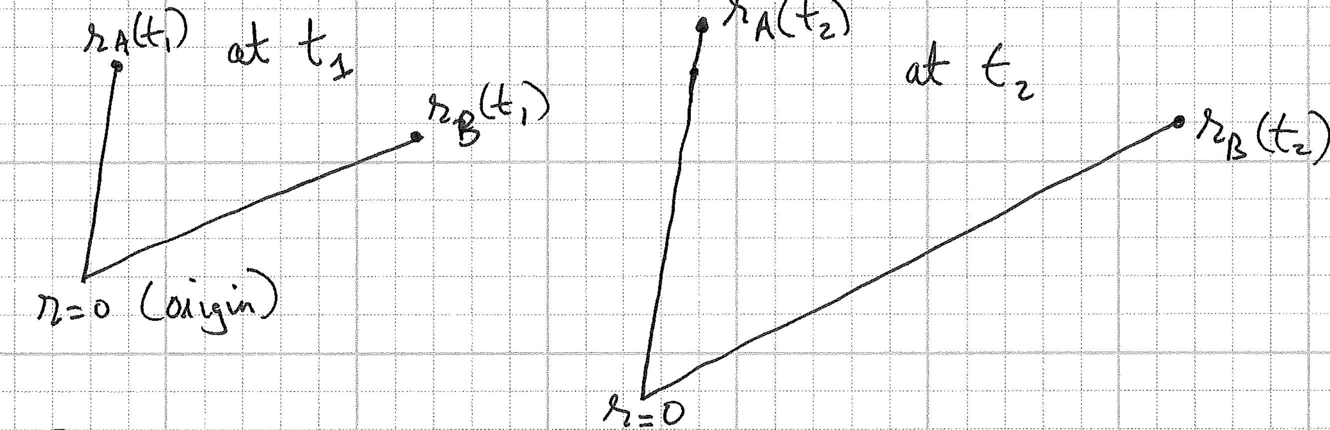
$$\left[r'/2 - (\Re/4) \sin 2r'/\Re \right] \approx + \frac{\Re}{4} \left[\left(\frac{8r'^3}{6\Re^3} \right) + \left(\frac{2^5 r'^5}{5! \Re^5} \right) + \cdots \right]$$

$$V(r') \approx 4\pi R^3 \Re^3 \left[+ \left(\frac{8r'^3}{6\Re^3} \right) + \left(\frac{2^5 r'^5}{5! \Re^5} \right) + \cdots \right]$$

$$V(r') \approx \frac{4\pi R^3 r'^3}{3} \left[1 - \frac{1}{5} \frac{r'^2}{\Re^2} + \mathcal{O}\left(\frac{r'}{\Re}\right)^4 \right]$$

$\boxed{\text{Open}}$: $V(r') \approx \frac{4\pi R^3 r'^3}{3} \left[1 + \frac{1}{5} \frac{r'^2}{\Re^2} + \mathcal{O}\left(\frac{r'}{\Re}\right)^4 \right]$

3 In a uniformly expanding, isotropic and homogeneous Universe the distance to any 2 points should increase by the same factor in a given time interval:



Thus:

$$\frac{r_A(t_1)}{r_A(t_2)} = \frac{r_B(t_1)}{r_B(t_2)} = \dots = \frac{r_n(t_1)}{r_n(t_2)} \equiv \alpha$$

, a constant independent of time, but depends on Δt.

From this, the recession velocity of A:

$$v_A = \frac{r_A(t_2) - r_A(t_1)}{t_2 - t_1} = \frac{r_A(t_1)}{t_2 - t_1}\left[\frac{r_A(t_2)}{r_A(t_1)} - 1\right]$$

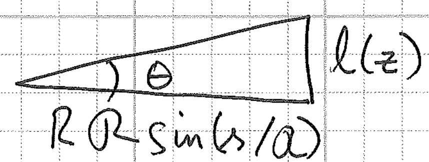
$$= \frac{r_A(t_1)}{t_2 - t_1}\left[\frac{1}{\alpha} - 1\right] = f(\Delta t)\, r_A(t_1)$$

So, $\boxed{v_A \propto r_A}$ and $f(\Delta t) = H_0$

4. An object with size today given by l_0 and which expands with the Universe has size $l(z)$ at redshift z given by

$$l(z) = l_0 / (1+z)$$

It's angular size at z is given by:



$$\theta = \frac{l(z)}{R\,\mathcal{R}\sin(r/\mathcal{R})}$$

or:

$$\boxed{\theta = \frac{(1+z)\,l(z)}{\mathcal{R}\sin(r/\mathcal{R})} = \frac{l_0}{\mathcal{R}\sin(r/\mathcal{R})}}$$

; its dependence on z is strictly through $r(z)$

Expanding $\sin(r/\mathcal{R})$ and subst. $r(z)$ approximation from class:

$\sin(r/\mathcal{R}) \approx r/\mathcal{R}$ at low redshift

so $\mathcal{R}\sin(r/\mathcal{R}) \approx r$

From class, we derived: $r(z) \approx \frac{c}{H_0 \underbrace{R_0}_{=1}}\left[z - \frac{1}{2}(1+q_0)z^2\right]$

So:

$$\boxed{\theta \approx \frac{H_0 l_0}{c\left[z - \frac{1}{2}(1+q_0)z^2\right]}}$$

for small z.

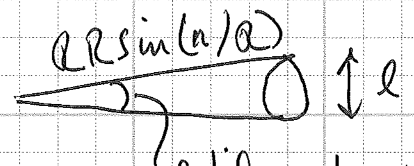
Continuing even further: $\theta \approx \dfrac{H_0 l_0}{cz\left[1 - \underbrace{\frac{1}{2}(1+q_0)z}_{\ll 1 \text{ for } z \ll 1}\right]}$

$$\boxed{\theta \approx \frac{H_0 l_0}{cz}\left[1 + \frac{1}{2}(1+q_0)z\right]}$$

5 Surface brightness, SB:

$$SB \equiv \frac{\text{Flux } (\text{ergs } s^{-1} m^{-2})}{\text{angular size } \Delta\Omega \ (\text{arcsec}^2)}$$

$$F = \frac{\mathcal{L}}{4\pi d_L^2} \ ; \quad d_L = \mathcal{R}\sin(r/\mathcal{R})\,(1+z)$$



solid angle $\Delta\Omega = \frac{\pi}{4}\Delta\theta^2$

for a spherical object of diameter l.

from diagram:

$$\Delta\theta = \frac{l\,(1+z)}{\mathcal{R}\sin(r/\mathcal{R})}$$

$$\Delta\Omega = \frac{\pi}{4}\Delta\theta^2 = \frac{\pi}{4}\frac{l^2(1+z)^2}{\mathcal{R}^2\sin^2(r/\mathcal{R})}$$

$$\therefore \quad S.B. = \left[\frac{\mathcal{L}}{4\pi\mathcal{R}^2\sin^2(r/\mathcal{R})\,(1+z)^2}\right] \Bigg/ \frac{\pi}{4}\frac{l^2(1+z)^2}{\mathcal{R}^2\sin^2(r/\mathcal{R})}$$

$$\boxed{S.B. = \frac{\overbrace{\mathcal{L}}^{\text{intrinsic luminosity}}}{\underbrace{\pi^2 l^2}_{\text{related to intrinsic area}}(1+z)^4}} \Longrightarrow \underline{\text{COSMOLOGY}} \ \underline{\text{INDEPENDENT}}!$$

Uses of this test: main thing I was looking for was that, unlike other tests which contain dependence on cosmol. models, this test is a pure test of <u>evolution</u>. Independently of Cosmology one can look for objects which obey $SB \propto \frac{1}{(1+z)^4}$

and know that they are standard candles and rods. For other "tests", which depend on cosmol. models, one has to rely on other data or theoretical arguments to make the case that some class of objects are standard anything. If the S.B. $\propto \frac{1}{(1+z)^4}$ for some class of objects, one can then use them for other tests of cosmology which rely on their being standard candles or rods.

This relationship $SB \propto \frac{1}{(1+z)^4}$ has not definitively been demonstrated for any class of objects!

Note that the $\frac{1}{(1+z)^4}$ behavior is specific to an expanding-universe explanation for redshift and ~~not~~ for other explanations for redshift (e.g. tired light) would give different behaviors for S.B. Thus seeing S.B. $\propto \frac{1}{(1+z)^4}$ would independently show that redshift $\leftrightarrow$ expansion.