1) Hubble's Law applies because $z \ll 1$:

$$v \approx zc = H_0 r$$

$$\rightarrow r = \frac{zc}{H_0} = \frac{0.12 \times 3\times10^5 \frac{km}{s}}{70 \frac{km}{s \cdot Mpc}}$$

$$\boxed{r = 514.3 \text{ Mpc}}$$

2) Feature in a spectrum of a galaxy at 588 nm has rest wavelength $\lambda_0 = 485.8$ nm.

a) How far is the galaxy?

$$z = \frac{588 - 485.8}{485.8} = 0.21$$

Hubble's Law:

$$\boxed{r = \frac{zc}{H_0} = 901.6 \text{ Mpc}}$$

b) If the galaxy has a random velocity of 400 km/s our estimate for r is off:
Our z = 0.21 corresponds to a recessional velocity of

$$v \approx cz \approx 6.3112 \times 10^4 \text{ km/s}$$

400 km/s is about 0.63 % of this, meaning our z is off by this & same with our r:

$$\boxed{\begin{aligned} \Delta r &\approx \frac{\Delta z c}{H_0} = 5.7 \text{ Mpc} \\ \text{So} \quad r &= 901.6 \pm 5.7 \text{ Mpc} \end{aligned}}$$

3 b) Cold universe means $T_0 = 2.725\,K$
Room $T = 293\,K$ (you can use $300\,K$ if you like). The number density of Blackbody ~~pa~~ photons goes as

$$n \propto T^3,$$

$$\therefore \quad \frac{n_{T=293}}{n_{T=2.7}} = \left(\frac{293}{2.725}\right)^3 = 1.243 \times 10^6$$ more photons ~~in~~ at room T, than CMBT!

3c) Adiabatic expansion of CMB means

$$T \propto \frac{1}{R} \leftarrow \text{scale factor}$$

$$\Rightarrow \frac{T}{T_0} = \frac{293}{2.725} = \frac{R_0}{R} = \frac{1}{R}$$

$$\Rightarrow \boxed{R = \frac{2.725}{293} = 0.009}$$

between fund. observers distance were ≈ 100 times shorter than today.

3d) Using answer from 3a we can write

$$n = \epsilon_\gamma / E_{mean} \; ; \; \epsilon_\gamma = \left(\frac{\pi^2 k^4}{15 \hbar^3 c^3}\right) T^4$$

and $E_{mean} \propto kT$

$$\longrightarrow E_{mean} = 2.7\,kT$$

Plugging in all the constants you get

$$\boxed{n = 4.1 \times 10^{8}\, m^{-3} = \frac{410}{cc}}$$

The other way to derive this was given in the hint:

$$du = S_\lambda \, d\lambda \equiv \frac{\text{energy}}{\text{volume}} \text{ in } \lambda \to \lambda + d\lambda$$

$$\Rightarrow \quad dn = \frac{du}{E_\lambda} = \frac{du}{hc/\lambda} = \frac{8\pi}{\lambda^4} \cdot \frac{1}{e^{hc/\lambda kT} - 1}$$

Integrating gives $n = 16\pi \underbrace{\zeta(3)}_{\substack{\text{Riemann} \\ \text{zeta funct, } \zeta(3) \approx 1.202}} \left(\frac{kT}{hc}\right)^3 \propto T^3$

4) $\boxed{\text{Redshift}}$ $\quad \frac{dE}{dr} = -K \cdot E$ $\Big\}$ energy loss per unit distance

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \uparrow$ constant.

Integrate to find the energy vs. r:

$$\underbrace{\int_{E_0}^{E} \frac{dE'}{E'}}_{\ln\left(\frac{E}{E_0}\right)} = -K \int_0^r dr = -Kr$$

$$\ln\left(\frac{E}{E_0}\right) = -Kr \Rightarrow \boxed{E(r) = E_0 e^{-Kr}}$$

Wavelength - redshift - energy are related by:

$$E = \frac{hc}{\lambda} \; ; \quad z \equiv \frac{\lambda_{observed} - \lambda_{emitted}}{\lambda_{emitted}}$$

We're interested in very low redshift. This means that our boxed equation on prev. page:

$$E(r) \cong E_0 (1 - Kr) \; ; \quad Kr \ll 1$$

$$\Rightarrow \quad \frac{1}{\lambda(r)} = \frac{1}{\lambda(0)}(1 - Kr)$$

or $$\lambda(0) = \lambda(r)\left[1 - Kr\right]$$

So, $$z = \frac{\lambda(r) - \lambda(0)}{\lambda(0)} = \frac{\lambda(r)}{\lambda(0)}\left[1 - 1 + Kr\right] = \frac{\lambda(r)}{\lambda(0)} Kr$$

$$\frac{\lambda(r)}{\lambda(0)} = \frac{1}{1 - Kr} \approx 1 + Kr$$

$$\Rightarrow \quad \boxed{z \approx (1 + Kr) Kr \approx Kr \; !}$$

Hubble: $v = zc = H_0 r \rightarrow z = \frac{H_0}{c} r$

$\therefore$ if $K \equiv H_0 / c$ we have the linear redshift-dist. Hubble law. Note that $\boxed{K = \frac{H_0}{c} = \frac{70}{3 \times 10^5} \cdot \frac{1}{Mpc} \approx \frac{1}{4285 \, Mpc}}$

You'll start to see deviations from linear relation at $r \approx$ few 1000 Mpc.

(b) Compton formula: $\Delta\lambda = \frac{h}{mc}(1-\cos\theta) = .0024(1-\cos\theta)\,\text{nm}$

(i) Assuming a typical scattering $\theta = 90°$, we have

$\Delta\lambda \approx .0024\ \text{nm}$; we want a

$\Delta\lambda_{tot} = 100\ \text{nm}$ $(500 \to 600\text{nm})$

$$N_{scatterings} = \frac{100\ \text{nm}}{.0024\ \text{nm}} \approx \boxed{41{,}666\ \text{scatterings}}$$

(ii) The probability of having 1 scatter in a density n_e, using the Thompson x-section, σ_e:

$$n_e \sigma_e \lambda = 1$$

λ ← mean-free-path

We want $\approx 41{,}700$ scatterings, so the ~~path~~ distance the photon travelled is d and must agree with Hubble's Law!

$$cz = H_0 d \to d = cz/H_0$$

$$z = \frac{600-500}{500} = 1/5$$

$$d = cz/H_0 = 3\times10^5 \cdot \frac{1}{5} \cdot \frac{1}{70} = 857\ \text{Mpc}$$

$1\ \text{Mpc} = 3.086\times10^{22}\ \text{m}$, so $\boxed{d = 2.6\times10^{25}\ \text{m}}$

~~Therefore~~ Now we can find n_e, the e^- density required to give $N_{scatt} = 41{,}700$ over a distance d:

$$n_e \sigma_e d = N_{scatt} \quad ; \quad \sigma_e = 6.65 \times 10^{-29}\ m^2$$

$$n_e = N_{scatt} / \sigma_e d$$

$$n_e = \frac{41{,}700}{(6.65 \times 10^{-29}\ m^2)(2.6 \times 10^{25}\ m)} = 2.4 \times 10^{7}\ m^{-3} \ !!$$

This is a ridiculously large e^- density!

(iii) Assuming a proton for every e^- (i.e. a fully ionized hydrogen-only medium), we get

$$n_p = n_e \ ; \ \text{so} \ \rho = m_p n_p$$

$$\rho = 1.67 \times 10^{-27}\ kg \times 2.4 \times 10^{7}\ m^{-3}$$

$$\boxed{\rho = 4.03 \times 10^{-20}\ kg/m^3}$$

(iv) Recall that critical density needed for flat universe is $\rho_c = \frac{3H^2}{8\pi G} \approx 10^{-26}\ kg/m^3$

So, our $\rho >> \rho_c$, & is unphysically high.
This is much larger than what is observed; Closed universe.

Such a large ρ would not have allowed the universe to reach an age of ~ 15 Gyrs; expansion would have halted long ago, leading to big crunch.