Harmonic Oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Undamped:

$$x(t) = A\cos(\omega t - \delta)$$

$$x(t) = B_1\cos(\omega t) + B_2\sin(\omega t)$$

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

Overdamped: $x = C_\pm \mathrm{e}^{-(\beta \pm \sqrt{\beta^2 - \omega_0^2})t}$

Critical: $x = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$

Underdamped: $x = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

Driven:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$$

$$x(t) = D\cos(\omega t - \delta) + \text{homogeneous soln}$$

$$D^2 = \frac{A^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}$$

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

$$Q = \frac{\omega_0}{2\beta} = \pi\frac{\text{decay time}}{\text{period}}$$

Fourier Series

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t)$$

$$\left.\begin{aligned} a_n &= \frac{2}{\tau}\int_0^\tau F(t')\cos n\omega t'\,dt' \\ b_n &= \frac{2}{\tau}\int_0^\tau F(t')\sin n\omega t'\,dt' \end{aligned}\right\}$$

Green - underdamped

$$G(t, t') \equiv \begin{cases} \dfrac{1}{m\omega_1}e^{-\beta(t-t')}\sin\omega_1(t - t'), & t \geq t' \\ 0, & t < t' \end{cases}$$

$$\boxed{x(t) = \int_{-\infty}^{t} F(t')\,G(t, t')\,dt'}$$

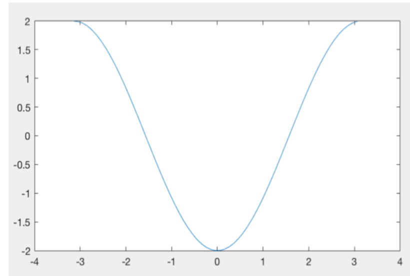
Calculus of Variations

$\int_a^b f\{y, y'; x\}\,dx$ is stationary when $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

Note that y is a function of x, and $y' \equiv \dfrac{dy}{dx}$.

Physics 303 Midterm Exam 2 Name______________________________

1a. The figure shows the potential energy U = -2cos(x) (in Joules), where x (in meters) is plotted horizontally and U is plotted vertically.



A mass m=0.125 kg moves in this potential without damping. What is the angular frequency ω_0 of small oscillations about x=0?

1b. For large oscillations (but with amplitude < π), what will be true:
a) the motion will be purely sinusoidal, because the potential is
b) the motion will include "overtones", both odd and even multiples of ω_0
c) the motion will include only odd harmonics (motion at ω_0, $3\omega_0$, $5\omega_0$, etc.)
d) the motion will include only even harmonics (motion at $2\omega_0$, $4\omega_0$, $6\omega_0$, etc.)
e) the motion will be chaotic, with no well-defined frequencies

1c. For large oscillations (but with amplitude < π), what else?
a) the fundamental frequency ω_0 will get smaller (longer period)
b) the fundamental frequency ω_0 will get bigger (shorter period)
c) the fundamental frequency ω_0 will be unchanged
d) because of the chaotic motion, it is no longer meaningful to talk about a fundamental frequency

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2a. A mass hangs on a spring (on Earth). What happens to the period of oscillation if the mass doubles?
a) it gets faster (shorter) by a factor of 2
b) it gets slower (longer) by a factor of 2
c) it gets faster by a factor of $\sqrt{2}$
d) it gets slower by a factor of $\sqrt{2}$
e) it is unchanged

2b. What happens to the period of oscillation if, **instead**, gravity (G) suddenly doubles?
a) it gets faster by a factor of 2
b) it gets slower by a factor of 2
c) it gets faster by a factor of $\sqrt{2}$
d) it gets slower by a factor of $\sqrt{2}$
e) it is unchanged

3a. If the distance to the moon were halved, how would it affect the gravitational force the moon exerts on you?

a) it would be 1/8th as much
b) it would be 1/4 as much
c) it would be half as much
d) it would be unchanged
e) it would be twice as big
f) it would be four times as big
g) it would be eight times as big

3b. If the distance to the moon were halved, how would it affect the tidal force from the moon?

a) it would be 1/8th as much
b) it would be 1/4 as much
c) it would be half as much
d) it would be unchanged
e) it would be twice as big
f) it would be four times as big
g) it would be eight times as big

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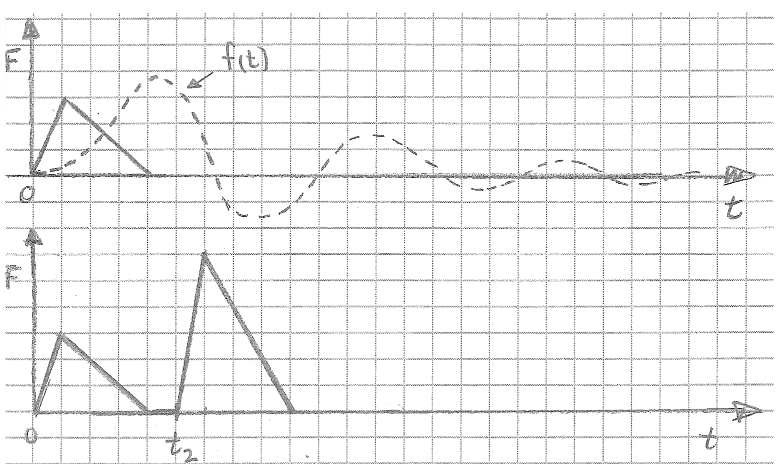
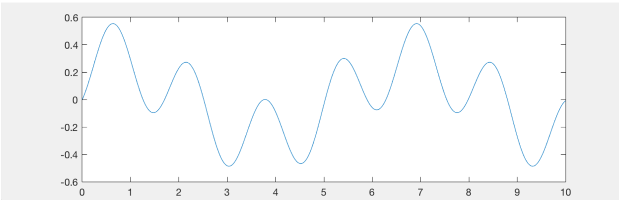
4. A dynamical system is described by a linear differential equation. When subject to the triangular pulse driving force shown (solid line), which begins at t=0, the response is the dashed line, f(t).

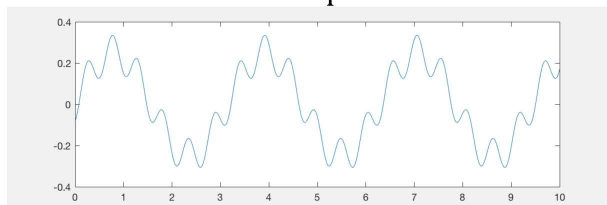
In addition to the original driving force, a second "pulse" is applied at t_2, with an amplitude twice as large as the original.

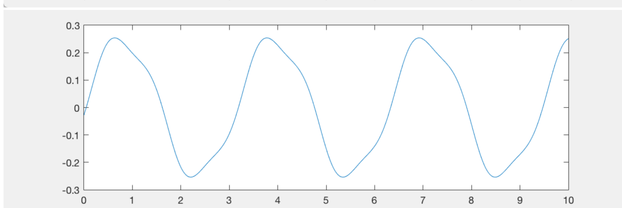
Is it possible to write down the response of the system to the two pulses, in terms of f and t_2? (Your answer would be a function of t, or course.) If so, write it down. If not, explain why it is not possible.

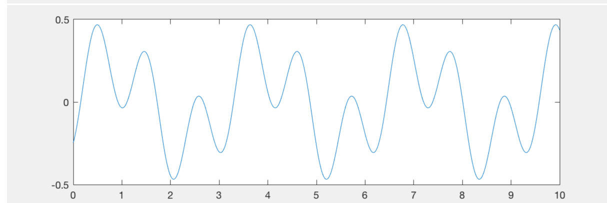


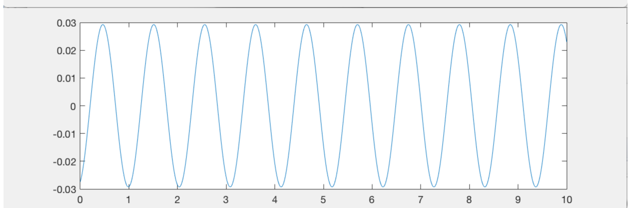
5a. A linear harmonic oscillator has ω_0 = 2 s^{-1} and β = 1 s^{-1}. It is driven with $F/m = \cos 2t + \cos 6t$ (in N/kg; for all t). Which plot shows the motion? The horizontal axis is time in seconds, vertical x in m. Circle the correct plot.











5b. The same harmonic oscillator is driven with F/m =cos 4.377t (for all time.) The resulting motion is given by $x(t) = D\cos(\omega^* t - \delta)$.
What are ω^* and δ? (I'll give you D = 0.057 m; you don't need it for this part.)

5c. Instead, this force is turned on at t=0. Find the position x(t) for t>0.

6a. Find the Green's function (response to a unit impulse at time t') for the critically damped harmonic oscillator. Your Green's function may contain β, t, t', m.

6b. The critically damped oscillator is subject to a force $=\alpha mt$ for $0 < t < 6$ s. m = mass, α is a constant. Write down (but do not solve) an integral that gives the motion for $0 < t < 6$ s.

7a. Find the differential equation of the path that minimizes $\int_{t1}^{t2} f(x,x';t)\,dt$ where $f(x,x';t) = \frac{1}{2}x'^2 + x$.

7b. Solve the differential equation. You should have two undetermined constants.

7c. Solve for your undetermined constants, given that x(0) = 0 and x(4) = 0.