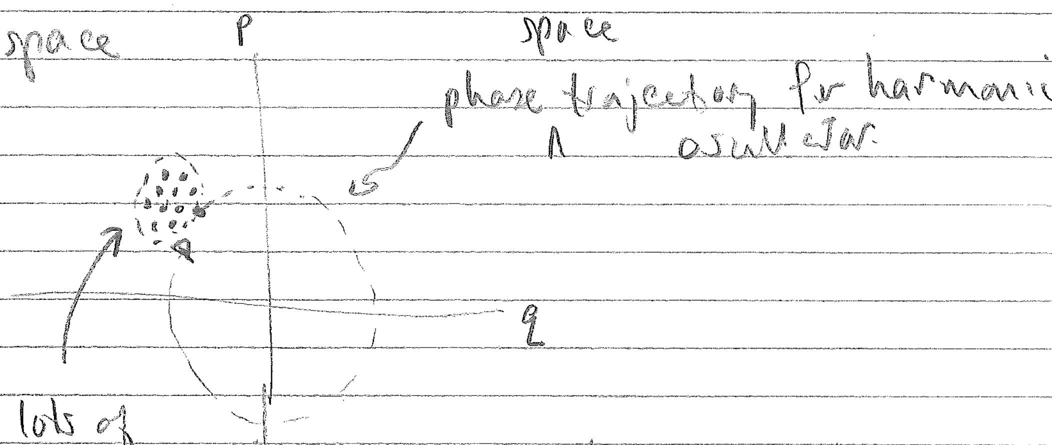
(Notes from Taylor)

Liouville's phase space theorem.

Recall: the canonical equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Where $H = \sum_i p_i \dot{q}_i - \mathcal{L}$, usually $H$ = energy.

Phase space




Consider lots of oscillators starting with nearby initial conditions, occupying area A in phase space.

How much area will they occupy later?

We can think of phase space filled with a moving fluid. Each "molecule", at $t=0$, is an initial condition, $p_0, q_0$.

The change in a small volume of fluid (flowing) is

$$dV = \left(\int_S \vec{v} \cdot \hat{n}\, dt \cdot dA\right)$$

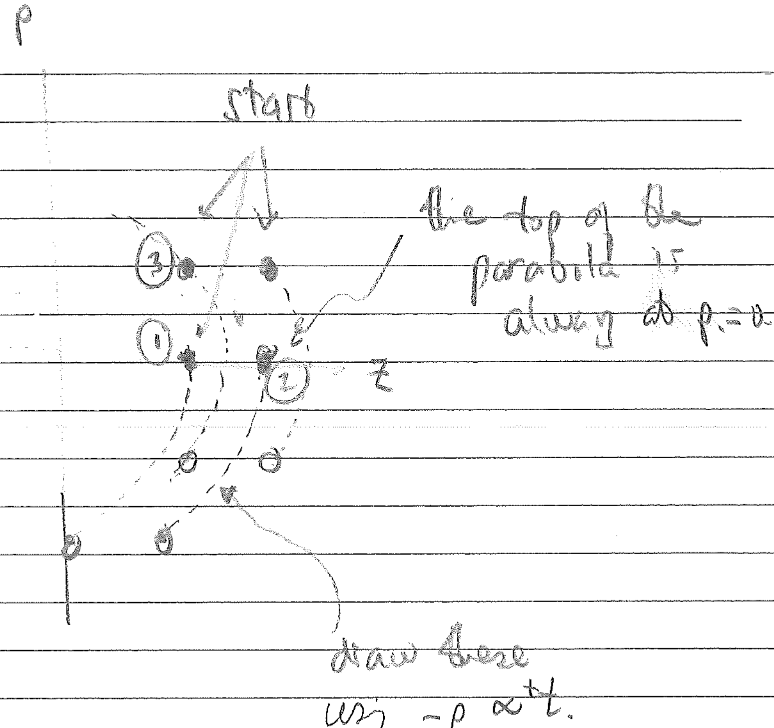
↳ velocity ⊥ to surface

$$\frac{dV}{dt} = \int_S \vec{v} \cdot \hat{n}\, dA = \int_V \vec{\nabla} \cdot \vec{v}\, dV \quad \text{Divergence thm.}$$

In phase space, "$\vec{v}$" $= (\dot{q}, \dot{p})$ so $\vec{\nabla} \cdot \vec{v} = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}$

$$= \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p}\left(\frac{\partial H}{\partial q_i}\right) = 0.$$

7-35 Gravity.




$p = mv_z$

$\frac{dp}{dt} = ma_z = -mg$

$p = p_0 - mgt$

$z = z_0 + v_0 t - \frac{1}{2} g t^2$

Now $v_0 = \frac{p_0}{m}$

if $p_0 = 0$, $p \propto t$.

Area starts as rectangle, becomes parallelogram
It will be

$$\Delta z(t) \times \Delta p(t)$$

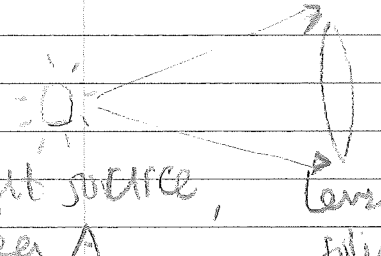
$\Delta z$:

$$\left.\begin{aligned} z_1 &= z_0 - \tfrac{1}{2} g t^2 \\ z_2 &= z_0 + \Delta z_0 - \tfrac{1}{2} g t^2 \end{aligned}\right\} \quad \Delta z = \Delta z_0 \text{ for all } t.$$

$\Delta p$:

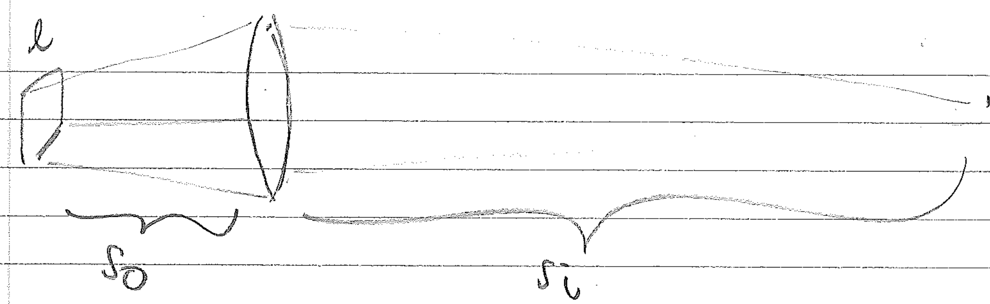
$$\left.\begin{aligned} p_1 &= 0 - mgt \quad (\text{I took } p_0 = 0) \\ p_2 &= \Delta p_0 - mgt \end{aligned}\right\} \quad \Delta p = \Delta p_0 \ \forall t.$$

Liouville's thm is important in stat mech & optics.




angle $\propto \vec{p}_{transverse}$

So $A\Omega$ = constant.




You know $M = \frac{S_i}{S_o}$ so $area_i \propto M^2 l^2$

Solid angle subtended by the lens $= \frac{A_{lens}}{4\pi S^2}$

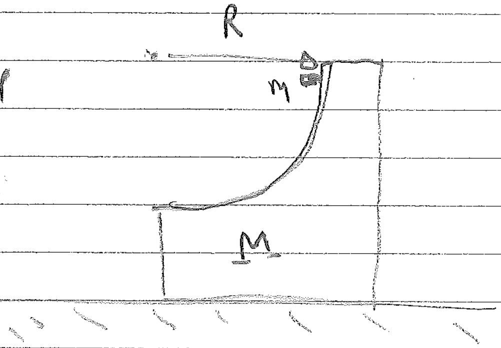
$V_{\phi_o}$ = Area object $\cdot \Omega_o = l^2 \cdot \frac{A_{lens}}{4\pi S_o^2}$

$V_{\phi_i}$ = Area image $\cdot \Omega_i = M^2 l^2 \frac{A_{lens}}{4\pi S_i^2}$ $\quad S_i^2 = M^2 S_o^2$ so $V_{\phi,i} = V_{\phi,o}$.

For your microscope, you can get a 50 W lamp with an arc size of 1 mm × 1 mm, or a 200 W lamp with an arc size of 2 mm × 4 mm.

Which is "brighter"?

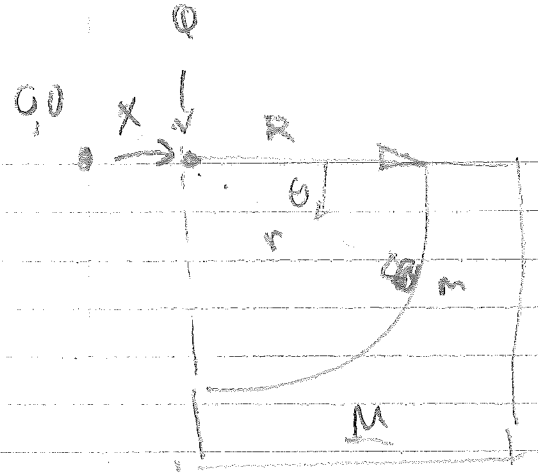
7.34




① You don't need Lagrangian mechanics for this. You know that the lost potential energy as m drops → kinetic energy of M and m.

Also, $\vec{P}_M + \vec{P}_m = 0$

Let's use Lagrangian formalism, though.




Wedge at X.

mass m, $x_m = X + r\cos\theta$

$y_m = -r\sin\theta$

$$T_m = \frac{1}{2} m\left(\dot{x}_m^2 + \dot{y}_m^2\right)$$

$$= \frac{1}{2} m\left[\left(\dot{X} + \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}\right)^2 + \left(-\dot{r}\sin\theta - r\cos\theta\,\dot{\theta}\right)^2\right]$$

$$= \frac{1}{2} m\Big[\dot{X}^2 + \dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2 + 2\dot{X}\dot{r}\cos\theta - 2\dot{X}\dot{\theta}\,r\sin\theta - 2r\dot{r}\cancel{\cos\theta\sin\theta\,\dot{\theta}}$$
$$+ \dot{r}^2\sin^2\theta + r^2\cos^2\theta\,\dot{\theta}^2 + \cancel{2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}}\Big]$$

$$T = T_m + T_M$$

$$\mathcal{L} = T - U = \frac{1}{2}(m+M)\dot{X}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{r}\cos\theta - 2\dot{X}\dot{\theta}\,r\sin\theta\right)$$
$$+ mgr\sin\theta$$

Constraint $r - R = 0$.

To find motion, use $r = R$, $\dot{r} = \ddot{r} = 0$, find Lagrange eqn's for $\ddot{X}$, $\ddot{\theta}$.

Results for $\ddot{X}$: $$\ddot{X} = \frac{mR}{M+m}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$

To find constraint force, use an undetermined multiplier.

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right) + \lambda \frac{\partial f}{\partial q} = 0$$

$$\underset{\substack{\uparrow \\ \text{Generalized} \\ \text{force}}}{\frac{\partial \mathcal{L}}{\partial q}} \qquad \underset{\frac{dp}{dt}}{\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right)}} \qquad \underset{\substack{\text{Constraint} \\ \text{force}}}{\underbrace{\lambda \frac{\partial f}{\partial q}}}$$

Here, we are interested in the force that keeps $r = R$.

So

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) + \lambda \frac{\partial f}{\partial r} = 0 \qquad f = r - R = 0$$

$$\frac{\partial f}{\partial r} = 1$$

$$\cancel{\tfrac{1}{2}} m \cdot \left[\cancel{2} r \dot\theta^2 - \cancel{2}\dot x \dot\theta \sin\theta\right] + mg\sin\theta$$

$$- \underbrace{\frac{d}{dt}\left[\cancel{\tfrac{1}{2}} m \left(\cancel{2}\dot r + \cancel{2}\dot x \cos\theta\right)\right]}_{m\left(\ddot r + \ddot x \cos\theta - \dot x \sin\theta\, \dot\theta\right)} + \lambda = 0$$

cancel

Now (after taking derivatives), we can impose $r = R = $ constant

so $\ddot r = 0$.

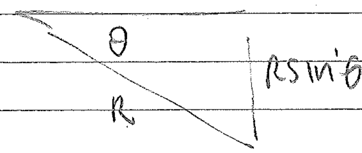
$$m r \dot\theta^2 + mg\sin\theta - m\ddot x \cos\theta + \lambda = 0$$

You can use Lagrange eqn for $\ddot x$ to get in terms of $\theta$.

Result: $\lambda = \dfrac{a-1}{1 - a\sin^2\theta}\left(R\dot\theta^2 + g\sin\theta\right) \qquad a = \dfrac{m}{M+m}$

It's actually easier to use conservation of energy to solve for the motion (than to solve coupled Lagrange eqns.)

At angle $\theta$,




$$\star \quad T = \underbrace{mgR\sin\theta}_{\text{If released from top.}} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 + 2\dot{x}R\dot{\theta}\sin\theta\right)$$

Now since $\ddot{x} = \frac{mR}{M+m}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$

$$\frac{d}{dt}(\dot{x}) = \frac{mR}{M+m}\underbrace{\left[\frac{d}{dt}(\dot{\theta})\cdot\sin\theta + \dot{\theta}^2\cos\theta\right]}_{= \frac{d}{dt}(\dot{\theta}\sin\theta)}$$

Thus

$$\dot{x} = \frac{mR}{M+m}\dot{\theta}\sin\theta$$

By the way, this is conservation of x momentum!

Now equation $\star$ can be solved for $\dot{\theta}^2$ in terms of $\theta$ and put into eqn. for $\lambda$.