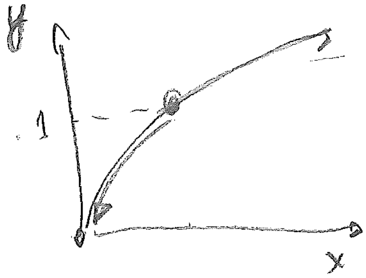
1. a)



b) $\frac{1}{2}\cancel{m}v^2 = \cancel{m}g(1-y)$ Cons of energy

$$V = [2g(1-y)]^{1/2} = \frac{ds}{dt}$$

$$ds = \sqrt{dx^2+dy^2} = \left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{1/2} dy$$

$kx = y^2$ so $x = y^2/k$

$$\frac{dx}{dy} = 2y/k$$

$$\frac{ds}{dt} = \left[\frac{4y^2}{k^2} + 1\right]^{1/2} \frac{dy}{dt} = [2g(1-y)]^{1/2}$$

4

$$\left(\frac{4y^2+1}{20(1-y)}\right)^{1/2} dy = dt$$

$$\boxed{\int_1^0 \left(\frac{4y^2+1}{20(1-y)}\right)^{1/2} dy = -t}$$

$T = 2\,s.$

2.a. $\omega = \frac{2\pi}{T} = \pi\ s^{-1}.$

$\delta = \frac{\pi}{8}$ $\quad \tan\delta = 0.4142 = \frac{2\beta\omega}{\omega_0^2 - \omega^2} = \frac{2 \cdot 1.95 \cdot \pi}{\omega_0^2 - \pi^2}$

$\omega_0^2 = 39.45$

$\omega_0 = 6.28 = 2\pi\ s^{-1}.$

b. $A = F/m = D\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$

$D = 0.02\,m$ from graph, while $F = 1\,N.$

$\frac{1}{m} = 0.02\sqrt{876.7 + 150.1}$

$m = 1.56\ kg$

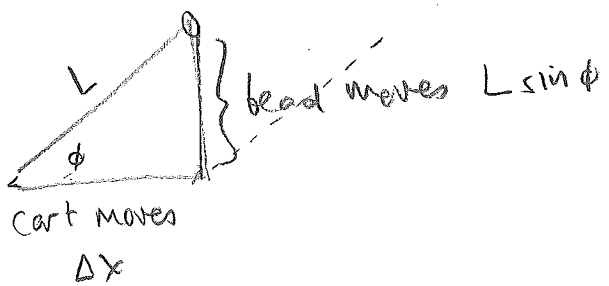
c. $\omega_0^2 = \frac{k}{m}$ $\quad k = m\omega_0^2 = 1.56 \cdot (2\pi)^2 = 61.6\ N/m.$

d. Underdamped, twice driven frequency, decay time 1/2 s.

$$\ddot{x} = \frac{m\sin\phi}{M + m\sin^2\phi} \cdot \cos\phi \cdot g$$

d) Max $\dot{x}$ when cart is massless, $M = 0$.

$$\ddot{x} = \frac{g}{\tan\phi}.$$ This means the bead goes straight down!



From geometry, $\frac{V_{bead}}{V_{cart}} = \tan\phi$.

$$\frac{1}{2} m V_{bead}^2 = mgh = mgL\sin\phi$$ Cons. of energy

$$V_{cart} = \frac{V_{bead}}{\tan\phi} = \frac{\sqrt{2gL\sin\phi}}{\tan\phi}.$$

Alternatively, $V_x^2 = 2a_x \Delta x$

$$= \frac{2g}{\tan\phi} \cdot L\cos\phi \left(\times \frac{\tan\phi}{\tan\phi} \right)$$

$$V_x^2 = \frac{2gL\sin\phi}{\tan^2\phi}.$$

3 a) $T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left[(\dot{x} + \dot{q}\cos\phi)^2 + (\dot{q}\sin\phi)^2\right]$

$= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m(\dot{x}^2 - 2\dot{x}\dot{q}\cos\phi + \dot{q}^2)$

b) $L = T - U$

$= T + mgq\sin\phi$

c) $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

$mg\sin\phi = \frac{d}{dt}\left(\frac{1}{2}m(-2\dot{x}\cos\phi + 2\dot{q})\right)$

$mg\sin\phi = -m\ddot{x}\cos\phi + m\ddot{q}$

$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

$0 = \frac{d}{dt}\left\{M\dot{x} + m\dot{x} - m\dot{q}\cos\phi\right\}$

$0 = (M+m)\ddot{x} - m\ddot{q}\cos\phi$

$m\ddot{q} = \frac{(M+m)\ddot{x}}{\cos\phi}$

$mg\sin\phi = -m\ddot{x}\cos\phi + \frac{(M+m)\ddot{x}}{\cos\phi}$

$= \left\{\frac{M+m}{\cos\phi} - \frac{m\cos^2\phi}{\cos\phi}\right\}\ddot{x} = \frac{M+m\sin^2\phi}{\cos\phi}\ddot{x}$

4a) θ is measured from perigee, $\theta = 120^\circ$ at $r = \infty$.

$$r = \frac{\alpha}{1+\varepsilon\cos\theta} \qquad \cos 120^\circ = -\frac{1}{2} \quad \text{so} \quad \varepsilon = 2.$$

b) Energy $= \frac{1}{2}mv^2$. (At ∞, $U = 0$)

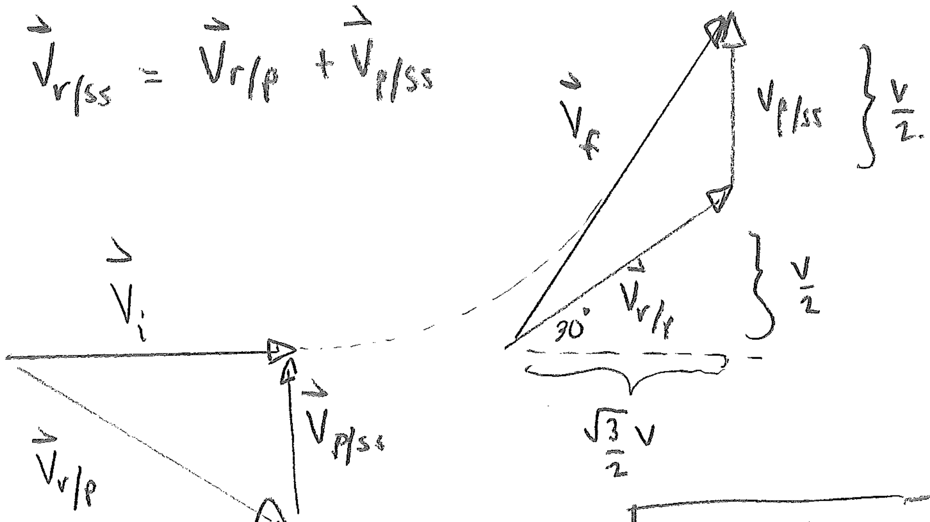
From eqns;
$$E = \frac{k^2\mu}{2\ell^2}(\varepsilon^2 - 1)$$
$$= \left(\frac{k}{2\alpha}\right)\cdot 3 = \frac{1}{2}mv^2$$

because $\alpha = \frac{\ell^2}{k\mu}$. Note $\mu = m$.

Solve for $\alpha = \frac{3k}{mv^2} = \frac{3GM}{v^2}$.

Then, perigee $= r_{min} = \frac{\alpha}{1+\varepsilon} = \frac{GM}{v^2}$.

c) $\vec{V}_{r/ss} = \vec{V}_{r/p} + \vec{V}_{p/ss}$



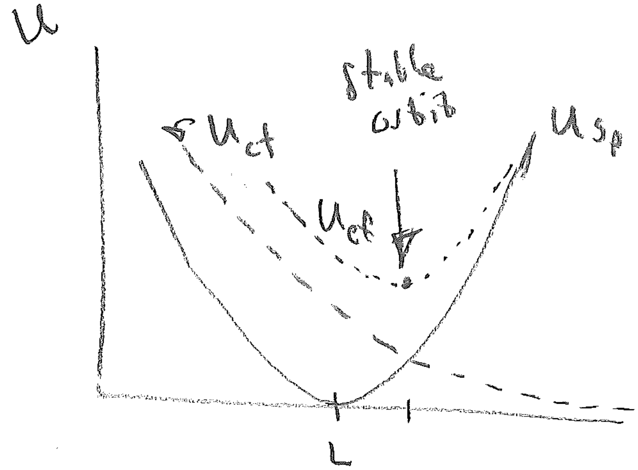
so $|\vec{V}_f| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}+\frac{1}{2}\right)^2}\; v$

$$= \sqrt{\frac{3}{4} + \frac{4}{4}}\; v = \sqrt{\frac{7}{4}}\; v.$$

5. a) $\vec{\ell} = \vec{r} \times \vec{p}$. $|\vec{\ell}| = \frac{L}{2} \cdot mv$, directed out of page.

b) $\mu = \frac{m^2}{m+M} = \frac{m}{2}$ so $|\vec{\ell}| = L\mu v$.

c) $U_{ef} = U_{sp} + U_{cf} = U_{sp} + \frac{\ell^2}{2\mu r^2} = \frac{1}{2}k(r-L)^2 + \frac{\ell^2}{2\mu r^2}$.



$F = 0$ at stable orbit $= -\frac{dU}{dr}$

$= -k(r-L) + \frac{\ell^2}{\mu r^3}$ w/ $r = (L+\delta)$

$0 = -k\delta + \frac{\ell^2}{\mu}\frac{1}{L^3}\left(1+\frac{\delta}{L}\right)^{-3} \approx -k\delta + \frac{\ell^2}{\mu L^3}\left(1 - \frac{3\delta}{L}\right)$

Solve for $\delta = \frac{\ell^2 L}{k\mu L^4 + 3\ell^2} \approx \frac{\ell^2}{k\mu L^3} = \frac{\cancel{L^2}\mu^{\cancel{2}} v^2}{k\cancel{\mu} L^{\cancel{3}}} = \frac{\mu v^2}{kL}$.

d) Amplitude $= \delta$.

e) $\frac{\partial^2 U}{\partial r^2} = -\frac{\partial F}{\partial r} = -\frac{\partial F}{\partial \delta} = k_{ef} = -\left(-k - \frac{3\ell^2}{\mu L^4}\right) = k + \frac{3\ell^2}{\mu L^4}$

$= k + \frac{3\cancel{L^2}\mu^{\cancel{2}} v^2}{\cancel{\mu} L^{\cancel{4}2}} = k + \frac{3\mu v^2}{L^2}$. $\omega = \sqrt{\frac{k_{ef}}{\mu}} = \omega_0\left(1 + \frac{3}{2}\frac{v^2}{L^2}\frac{\mu}{k}\right)$