1. $A = F/M = 8/2 = 4$. $\omega_0 = 5\,s^{-1}$, $\omega = 6\,s^{-1}$.

a) $$D = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}} = \frac{4}{\sqrt{11^2 + 4\cdot 3^2 \cdot 6^2}} = 0.106\text{ m.}$$

b) $$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right) = \arctan\left(\frac{2\cdot 3\cdot 6}{-11}\right) = -73^\circ + 180^\circ = 107^\circ.$$

c) See next page.

d) We need the underdamped free response, matching initial conditions

$t<0$. $x = D\cos(\omega t - \delta)$ at $t=0$, $x = D\cos(\delta) = -0.031$ m.

$\dot{x} = -\omega D\sin(\omega t - \delta)$ at $t=0$, $\dot{x} = \omega D\sin(\delta) = 0.608$ m/s

$t>0$ $x = Ae^{-\beta t}\cos(\omega_1 t - \delta^*)$. At $t=0$, $x = A\cos(-\delta^*) = A\cos(\delta^*)$

* this δ is not the same as δ in part a–c.!

$$\dot{x} = Ae^{-\beta t}(-\omega_1)\sin(\omega_1 t - \delta) + -\beta Ae^{-\beta t}(\cos\omega_1 t - \delta)$$

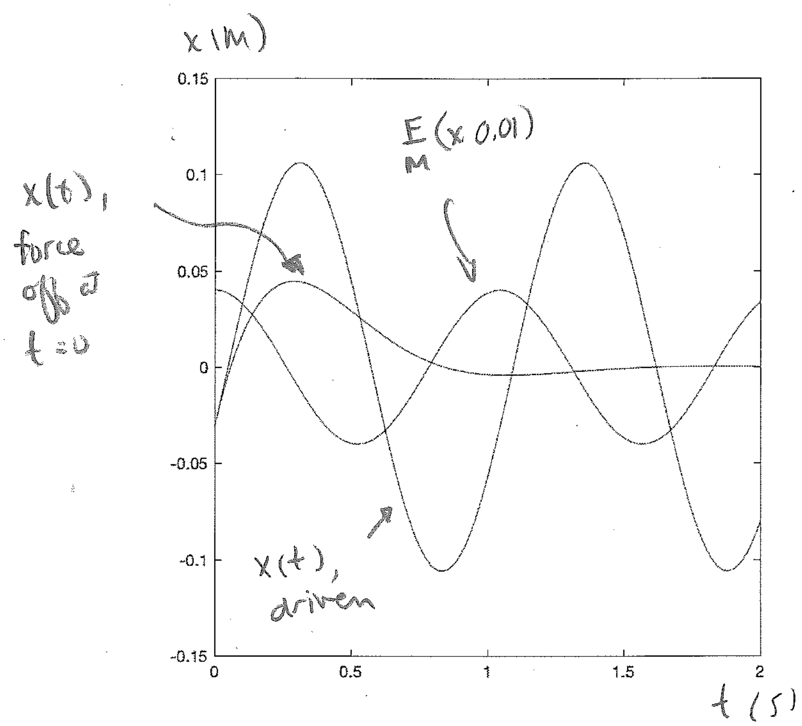
at $t=0$ $\dot{x} = +\omega_1 A\sin\delta - \beta A\cos(-\delta)$

So $$\frac{\dot{x}}{x} = \omega_1\tan\delta - \beta$$

$$\tan\delta = \frac{1}{\omega_1}\left(\frac{\dot{x}}{x} + \beta\right) = \frac{1}{4}\left[\left(\frac{0.608}{-0.031}\right) + 3\right] = -4.15 \quad \delta = -76^\circ \text{ or } 103.5^\circ$$

$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = 4\,s^{-1}$

2


x (m)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
0.5
1
1.5
2
t (s)
E/M (x 0.01)
x(t), force off at t = 0
x(t), driven

Choose $\delta = 103.5°$ so $\cos\delta < 0$ so $A > 0$.

$\cos\delta = -0.233$. $A\cos\delta = x_0 = -0.031\,m$ $A = 0.133\,m$.

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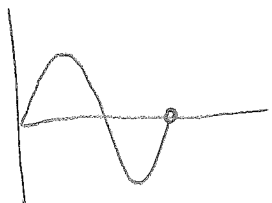
2. Since $\omega_1 = 4$, $\frac{\pi}{2}\,s = \tau = \frac{2\pi}{\omega_1}$.

So oscillator will cross zero at $\frac{\pi}{2}\,s$.



Since $x(0) = 0$, $x = Ae^{-\beta t}\sin\omega_1 t$. (equivalently, $\delta = \pi/2$)

$$\dot{x} = Ae^{-\beta t}\omega_1\cos\omega_1 t - \beta A e^{-\beta t}\sin\omega_1 t$$

$\dot{x}$ when $x = 0$ (when $\sin\omega_1 t = 0$) $= Ae^{-\beta t}\omega_1\cos\omega_1 t = \pm\omega_1 A e^{-\beta t}$

So over time τ, $\dot{x}$ decreases by a factor of $e^{-\beta\tau}$.

$\beta = 3\,s^{-1}$. $\tau = \frac{\pi}{2}\,s$. $e^{-\beta\tau} = 0.009$

So impulse must be $-0.009 \times 6\,\frac{kg\,m}{s} = -0.054\,\frac{kg\,m}{s}$.

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3. a) Critical damping $x = (C_1 + tC_2)e^{-\beta t}$.

$x(0) = 0 = C_1$.

$\dot{x} = (C_1 + tC_2)(-\beta)e^{-\beta t} + C_2 e^{-\beta t}$

At $t = 0$, with $C_1 = 0$, $\dot{x} = C_2$.

So an initial unit impulse gives a response $G(t) = \frac{t}{m}e^{-\beta t}$.

$1 = m\dot{x} = mC_2$ $C_2 = \frac{1}{m}$. $G(t,t') \to \frac{(t-t')}{m}e^{-\beta(t-t')}$

b) $x = \int_0^t F(t')\,G(t,t')\,dt' = \int_0^t 8t'\cdot\frac{(t-t')}{m}e^{-\beta(t-t')}\,dt'$

4

$$x = \frac{8e^{-\beta t}}{M}\left\{ t\int_0^t t' e^{+\beta t'} dt' - \int_0^t t'^2 e^{+\beta t'} dt' \right]$$

$$= \frac{8e^{-\beta t}}{M}\left\{ t\cdot \frac{(\beta t'-1)e^{\beta t'}}{\beta^2}\Bigg|_0^t - \frac{(\beta^2 t'^2 - 2\beta t' + 2)\, e^{\beta t'}}{\beta^3}\Bigg|_0^t \right\}$$

$$= \frac{8\cancel{e^{-\beta t}}}{M}\left\{ t\cdot\left[\frac{(\beta t-1)\cancel{e^{\beta t}}}{\beta^2} + \frac{e^{-\beta t}}{\beta^2}\right] - \left[\frac{\beta^2 t^2 - 2\beta t + 2}{\beta^3}\cancel{e^{\beta t}} - \frac{e^{-\beta t}\, 2}{\beta^3}\right]\right\}$$

$$= \frac{8}{M}\left\{ \frac{\cancel{\beta t^2} - t + t e^{-\beta t}}{\beta^2} - \cancel{\frac{\beta t^2}{\beta^2}} + \frac{2t}{\beta^2} - \frac{2}{\beta^3} + \frac{2e^{-\beta t}}{\beta^3} \right\}$$

$$= \frac{8}{M}\left\{ \frac{t}{\beta^2} - \frac{2}{\beta^3} + \frac{(\beta t+2)e^{-\beta t}}{\beta^3} \right\}$$

$$x(t) = \frac{8}{M\beta^3}\left\{(\beta t - 2) + (\beta t + 2)e^{-\beta t}\right\} \qquad \dot{x}(t) = \frac{8}{M\beta^3}\left[\beta + \beta e^{-\beta t} - \beta^2 t e^{-\beta t}\right]$$

Put in values for $t=1$, $M=4$, $\beta = \omega_0 = 4$

$$x(1) = \frac{8}{4\cdot 64}\left\{2 + 6e^{-4}\right\} \qquad \dot{x}(1) = \frac{8}{4\cdot 64}\left[4 + 4e^{-4} - 16\cdot 1\cdot e^{-4}\right]$$

$$= 0.26\ \text{m}. \qquad\qquad = 0.12\ \text{m/s}.$$

5-2. $g = \frac{GM}{r^2}$ M = mass enclosed = $\int_0^r \rho(r')\, 4\pi r'^2\, dr'$

if g is indep of r, this integral must be $\propto r^2$.

so $\rho(r') \sim \frac{1}{r'}$ so that $\int_0^r \frac{1}{r'} \cdot 4\pi r'^2\, dr' = \int_0^r 4\pi r'\, dr' = 2\pi r^2$

$$\boxed{\rho(r) = \frac{c}{r}}$$

5.5 $E = \frac{1}{2} m v^2 - \frac{GMm}{r}$ so $v = \sqrt{\frac{2}{m}\left(E + \frac{GMm}{r}\right)} = \frac{dr}{dt}$

$$t = \int dt = \int_{\sim 0}^{R} \frac{dr}{\sqrt{\frac{2}{m}\left(E + \frac{GMm}{r}\right)}} = \int_0^R \frac{dr}{\sqrt{\frac{2}{\cancel{m}}\left(-\frac{GM\cancel{m}}{R} + \frac{GM\cancel{m}}{r}\right)}}$$

$$= \frac{1}{\sqrt{2GM}} \int_{\sim 0}^{R} \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}}$$

$$t = -\frac{1}{\sqrt{2GM}} \int_R^0 \sqrt{\frac{rR}{R-r}}\, dr$$

let $r = y^2$, $dr = 2y\,dy$, use E.7 appendix E

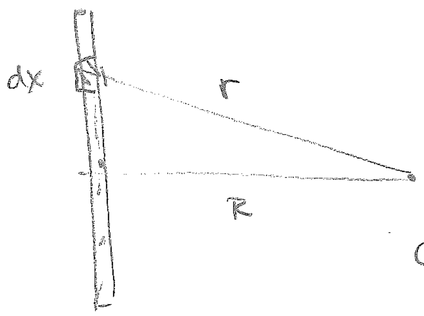
$$t = \sqrt{\frac{R}{2GM}} \left[\sqrt{r(R-r)} - R \sin^{-1}\left(\sqrt{\frac{r}{R}}\right) \right]_R^0$$

$$t = \frac{\sqrt{R}}{\sqrt{GM}} (+R\pi) = \pi \frac{R^{3/2}}{\sqrt{GM}}.$$

$t_{1/2}$, use $R/2$ as limit, then $t = \frac{1}{\sqrt{GM}} \left(\frac{R}{2}\right)^{3/2} \left(1 + \frac{\pi}{2}\right)$

Ratio is $\frac{t_{1/2}}{t} = \frac{1 + \pi/2}{\pi} \approx \frac{9}{11}.$

5-7.



$$d\phi = \frac{-G\lambda\,dx}{r}$$

$$\phi = 2\int_0^{\ell/2} \frac{-G\lambda\,dx}{(R^2+x^2)^{1/2}}$$

$$= -2G\lambda \left[\ln \left| \frac{\sqrt{x^2+R^2}+x}{R} \right| \right]_0^{\ell/2}$$

$$= -\frac{2GM}{\ell} \ln\left[\frac{\sqrt{(\ell/2)^2+R^2}+\ell/2}{R}\right]$$

5-15. $g = \frac{GM}{r^2}$. M = mass closer to center.

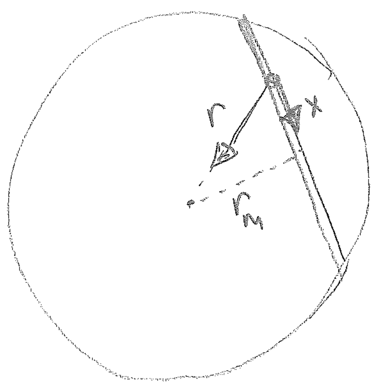
$M = \frac{4}{3}\pi r^3 \cdot \rho$ so $g = G \cdot \frac{4}{3}\pi r \rho \;\; \alpha\; r.$ → simple harmonic motion.

$F = mg = Gm \cdot \frac{4}{3}\pi r\rho$ so $k = \frac{4}{3}\pi\rho G m$ & $m = m$.

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{3}\pi\rho G}$ ρ = density of earth $= 5.51\ g/cm^3$
$= 5.51 \times 10^3\ kg/m^3$.

$$\omega_0 = \sqrt{\frac{4}{3} \cdot \pi \cdot 5.51\times10^3 \cdot 6.67\times10^{-11}} = 0.00124\ s^{-1}$$

$\frac{2\pi}{\omega_0} = \tau = 5064\ s = 84.4$ min.



For NY-SF tunnel, call x the distance from the center of the tunnel the force of gravity has an x-component

$$F_{g,x} = F_g \cdot \frac{x}{r} = Gm\frac{4}{3}\pi x\rho.$$

so motion is simple harmonic with same "spring constant"!