7.1 $\mathcal{L} = T - U$

$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) \Rightarrow$ $m\dot{x} = \text{const.}$ $m\dot{y} = \text{const.}$ $\frac{d}{dt}(m\dot{z}) = -mg$

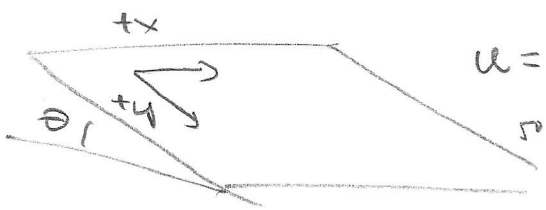
so $\ddot{z} = -g$

7.2 $F = -kx$ so $U = \frac{1}{2}kx^2$ $T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 = \mathcal{L}$

$\frac{\partial \mathcal{L}}{\partial x} = -kx$ $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$

$m\ddot{x} + kx = 0 \rightarrow x = A\cos(\omega t + \delta)$ $\omega = \sqrt{\frac{k}{m}}$.

7.4



$U = -mgy\sin\theta$

so $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy\sin\theta$

$\frac{\partial \mathcal{L}}{\partial x} = 0$ so $m\dot{x} = \text{constant}$

$\frac{\partial \mathcal{L}}{\partial y} = \underbrace{mg\sin\theta}_{F_y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{d}{dt}(m\dot{y}) = m\ddot{y}$

$F_y = m\ddot{y}$ as expected.

7.8 a)

$U = \frac{1}{2}k(x_1 - x_2 - l)^2$

$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(\underbrace{x_1 - x_2 - l}_{x})^2$.

$X = \frac{1}{2}(x_1 + x_2)$

so $2\dot{X} = \dot{x}_1 + \dot{x}_2$ while $\dot{x} = \dot{x}_1 - \dot{x}_2$ $(\dot{l} = 0)$

$2\dot{X} + \dot{x} = 2\dot{x}_1$ $\dot{x}_1 = \dot{X} + \frac{\dot{x}}{2}$ $\dot{x}_2 = \dot{X} - \frac{\dot{x}}{2}$

7.8 continued

so $\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}kx^2$

$$= \frac{1}{2}m\left(\dot{X}^2 + \cancel{\dot{X}\dot{x}} + \frac{\dot{x}^2}{4} + \dot{X}^2 - \cancel{\dot{X}\dot{x}} + \frac{\dot{x}^2}{4}\right) - \frac{1}{2}kx^2$$

$$= m\dot{X}^2 + \frac{1}{4}m\dot{x}^2 - \frac{1}{2}kx^2$$

for $\dot{X}$, $\frac{\partial \mathcal{L}}{\partial X} = 0$ so $m\dot{X}^2$ = constant. Center of mass!

for $\dot{x}$: $\frac{\partial \mathcal{L}}{\partial x} = -kx = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \underbrace{\frac{1}{2}m}_{\text{"reduced" mass}}\ddot{x}$

c. Center of mass moves at constant speed, masses oscillate with frequency $\sqrt{\frac{k}{\mu}}$ $\mu = \frac{m}{2}$

6-9. $\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}$ 3 indep variables; x, y, z

$$(\vec{\nabla}\phi)^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2$$

To minimize $\iiint \underbrace{(\vec{\nabla}\phi)^2}\, dx\,dy\,dz$, the Euler eqn is ✡

$$f\left(\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}; x, y, z\right) = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2$$

these need to be partials because multiple indep vars.

✡ $$\frac{\partial f}{\partial\phi} - \frac{\partial}{\partial x}\frac{\partial f}{\partial\left(\frac{\partial\phi}{\partial x}\right)} - \frac{\partial}{\partial y}\frac{\partial f}{\partial\left(\frac{\partial\phi}{\partial y}\right)} - \frac{\partial}{\partial z}\frac{\partial f}{\partial\left(\frac{\partial\phi}{\partial z}\right)} = 0$$ But $\frac{\partial f}{\partial\phi} = 0$.

Also $\frac{\partial f}{\partial\left(\frac{\partial\phi}{\partial x}\right)} = 2\left(\frac{\partial\phi}{\partial x}\right)$, etc so $-2\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_i}\right) = 0$

Thus $\nabla^2\phi = 0$. Laplace Eqn!

The laplace eqn gives the electric potential in a region of space where there are no charges. Since $\vec{E} = -\vec{\nabla}\phi$ and electric field energy varies as the square of the field, we have shown that, for any given boundary conditions, the electric field arranges to minimize field energy!

6-10. min. surface area of a cylinder of fixed volume

$$A = f = 2\pi r^2 + 2\pi r h = f(r, \dot{r}, h, \dot{h}; t), \quad \text{but } \frac{\partial f}{\partial \dot{r}} = 0 \ \& \ \frac{\partial f}{\partial \dot{h}} = 0,$$

w/o constraint, area is minimized when

$$\frac{\partial A}{\partial r} - \frac{d}{dt}\left(\frac{\partial A}{\partial \dot{r}}\right) = 0 \quad \rightarrow \quad 4\pi r = 0 \qquad r = 0. \ \text{Also, } h = 0. \ \text{Duh.}$$

Constraint $\quad V = \pi r^2 h$

So, with undetermined multiplier

$$\frac{\partial A}{\partial r} + \lambda \frac{\partial V}{\partial r} = 0 \quad \rightarrow \quad \cancel{4\pi} r^{2} + 2\pi h + \lambda \cdot \cancel{2\pi} r h = 0 \rightarrow 2 + \frac{h}{r} + \lambda h = 0$$

$$\frac{\partial A}{\partial h} + \lambda \frac{\partial V}{\partial h} = 0 \quad \rightarrow \quad 2\cancel{\pi} r + \lambda \cancel{\pi} r^2 = 0 \rightarrow 2 + \lambda r = 0$$

$$\text{So } \lambda = -\frac{2}{r} \quad \& \quad 2 + \frac{h}{r} + -\frac{2h}{r} = 0 \qquad \boxed{\frac{h}{r} = 2 \quad \text{or} \quad r = \frac{h}{2}}$$

6.14. Shortest path from $(0, -1)$ to $(0, 1)$ on volcano $z = 1 - r$. (x, y)

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \qquad \text{Start at } r = 1, \ \theta = -\frac{\pi}{2}, \text{ end at } +\frac{\pi}{2}.$$

Since $z = 1 - r \quad dz = -dr \quad dz^2 = dr^2$

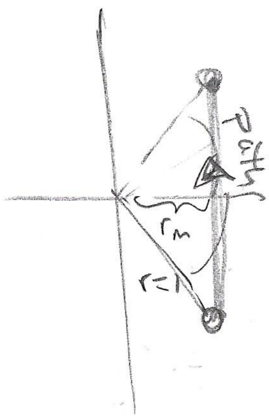
$$ds^2 = 2\,dr^2 + r^2 d\theta^2$$

Define a "pseudoangle" $\beta = \frac{\theta}{\sqrt{2}}$. $\theta = \sqrt{2}\,\beta$ $d\theta = \sqrt{2}\,d\beta$

then $d\theta^2 = 2\,d\beta^2$. Now

$$\triangle\triangle \quad ds^2 = 2\left(dr^2 + r^2 d\beta^2\right)$$

this is the distance in a plane, described in polar coordinates r, β. So answer is a straight line in r, β



$\beta_f = +\frac{\pi}{2\sqrt{2}}$

$\beta_i = -\frac{\pi}{2\sqrt{2}}$

On this path $\frac{r_m}{r} = \cos\beta$

So $r = \frac{r_m}{\cos\beta}$

Thus $r_m = 1 \cdot \cos\left(\frac{\pi}{2\sqrt{2}}\right)$

the length of this path can be found from trig

$$l = 2 \cdot \sqrt{1^2 - r_m^2}$$

$$= 2\sqrt{1^2 - \cos^2\left(\frac{\pi}{2\sqrt{2}}\right)} = 2\sin\frac{\pi}{2\sqrt{2}}.$$

this is the length of a path described by the term in parenthesis ($\triangle\triangle$). Our actual path is longer by factor of $\sqrt{2}$.

$$\text{Path length} = 2\sqrt{2}\,\sin\frac{\pi}{2\sqrt{2}}.$$