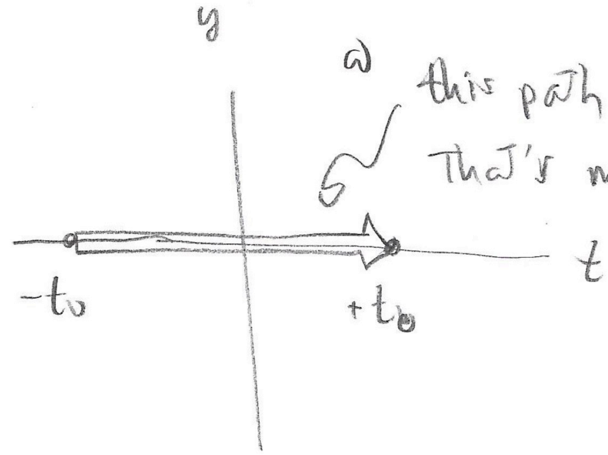
1.



a) this path has $T=0$ always, so $\int T\,dt = 0$

That's minimum, because $T \geq 0$.

b) If $U = mgy$, then $U=0$ and $\int U\,dt = 0$

c) $y = \alpha(t_0^2 - t^2)$

$\dot{y} = -2\alpha t$

$T = \frac{1}{2} m \dot{y}^2 = 2m\alpha^2 t^2$

$U = mgy = mg\alpha(t_0^2 - t^2)$

$$S = \int_{-t_0}^{+t_0} (T-U)\,dt = \int_{-t_0}^{+t_0} \left(2m\alpha^2 t^2 + mg\alpha t^2 - mg\alpha t_0^2\right) dt$$

$$S = (2m\alpha^2 + mg\alpha)\, 2 \cdot \frac{t_0^3}{3} - 2mg\alpha t_0^3$$

$$\frac{dS}{d\alpha} = 0 = (4m\alpha + mg)\left(\frac{2t_0^3}{3}\right) - 2mg\,t_0^3.$$

$$\frac{8}{3}\alpha + \frac{2}{3}g = 2g$$

$$8\alpha = 4g \quad \boxed{\alpha = g/2}$$

7.41.

$z = k\rho^2$. $\quad \dot z = 2k\rho\dot\rho$

$U = mgz = mgk\rho^2$.

$T = \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\dot z^2 + \frac{1}{2}m\rho^2\omega^2$

important! these are all $\perp$.

$= \frac{1}{2}m\left(\dot\rho^2 + 4k^2\rho^2\dot\rho^2 + \rho^2\omega^2\right)$

$\mathcal{L} = T - U$.

$$\frac{\partial\mathcal{L}}{\partial\rho} = \frac{1}{2}\left(8k^2\rho\dot\rho^2 + 2\omega^2\rho\right) - 2gk\rho = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\rho}\right)$$

$$= \frac{d}{dt}\left(\cancel{\tfrac{1}{2}}(1+4k^2\rho^2)\cancel{2}\dot\rho\right) = (1+4k^2\rho^2)\ddot\rho + 8k^2\rho\dot\rho^2$$

$$(4k^2\dot\rho^2 + \omega^2)\rho - 2gk\rho = (1+4k^2\rho^2)\ddot\rho + 8k^2\rho\dot\rho^2.$$

Equilibrium: $\dot\rho = 0$, $\ddot\rho = 0$. $\rightarrow (\omega^2 - 2gk)\rho = 0$

So $\rho = 0$ or $\omega^2 - 2gk = 0$, $\rho =$ anything.

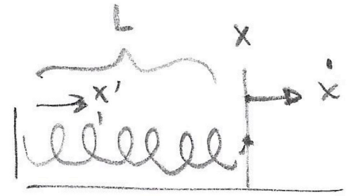
Stability? For ρ small and positive, $\dot\rho = 0$, check if $\ddot\rho \gtrless 0$?

$(\omega^2 - 2gk)\rho = (1+4k^2\rho^2)\ddot\rho$

If this is +, $\ddot\rho$ will be + and eqm is <u>unstable</u> $\omega > \sqrt{2gk}$

At $\omega = \sqrt{2gk}$, eqm is "neutral". If $\dot\rho = 0$, then $\ddot\rho = 0$.

7.34. a)



with uniform stretch, the speed of any point is proportional to its distance x' away from the wall.

$$V(x') = \frac{x'}{L} \cdot \dot{x}$$

So the kinetic energy of a bit of spring is

$$dT = \frac{1}{2}\rho \cdot dx' \cdot (V(x'))^2 = \frac{1}{2}\rho \cdot dx' \cdot \frac{x'^2}{L^2}\dot{x}^2$$

Total

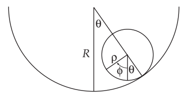
$$T = \frac{1}{2}\frac{\rho}{L^2}\dot{x}^2 \int_0^L x'^2 \, dx' = \frac{1}{2}\frac{\rho}{L^2}\dot{x}^2 \frac{L^3}{3} = \frac{1}{6}\rho L \dot{x}^2 = \frac{1}{6}M\dot{x}^2.$$

b) $$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{6}M\dot{x}^2 - \frac{1}{2}kx^2$$

$$= \frac{1}{2}\underbrace{\left(m + \frac{1}{3}M\right)}_{\text{effective mass}}\dot{x}^2 - \frac{1}{2}kx^2.$$

so $\omega = \sqrt{\dfrac{k}{m + \frac{M}{3}}}$

7-3.



If we take angles θ and ϕ as our generalized coordinates, the kinetic energy and the potential energy of the system are

$$T = \frac{1}{2} m \left[(R-\rho)\dot{\theta} \right]^2 + \frac{1}{2} I\, \dot{\phi}^2 \tag{1}$$

$$U = \left[R - (R-\rho)\cos\theta \right] mg \tag{2}$$

where m is the mass of the sphere and where $U = 0$ at the lowest position of the sphere. I is the moment of inertia of sphere with respect to any diameter. Since $I = (2/5)\, m\rho^2$, the Lagrangian becomes

$$L = T - U = \frac{1}{2} m (R-\rho)^2 \dot{\theta}^2 + \frac{1}{5} m\rho^2 \dot{\phi}^2 - \left[R - (R-\rho)\cos\theta \right] mg \tag{3}$$

When the sphere is at its lowest position, the points A and B coincide. The condition $A0 = B0$ gives the equation of constraint:

$$f(\theta, \phi) = (R-\rho)\theta - \rho\phi = 0 \tag{4}$$

Therefore, we have two Lagrange's equations with one undetermined multiplier:

$$\left. \begin{aligned} &\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}} \right] + \lambda \frac{\partial f}{\partial \theta} = 0 \\ &\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}} \right] + \lambda \frac{\partial f}{\partial \phi} = 0 \end{aligned} \right] \tag{5}$$

After substituting (3) and $\partial f / \partial \theta = R - \rho$ and $\partial f / \partial \phi = -\rho$ into (5), we find

$$-(R-\rho) mg \sin\theta - m(R-\rho)^2 \ddot{\theta} + \lambda (R-\rho) = 0 \tag{6}$$

$$-\frac{2}{5} m\rho^2 \ddot{\phi} - \lambda\rho = 0 \tag{7}$$

From (7) we find λ:

$$\lambda = -\frac{2}{5} m\rho\ddot{\phi} \tag{8}$$

or, if we use (4), we have

$$\lambda = -\frac{2}{5} m (R-\rho)\ddot{\theta} \tag{9}$$

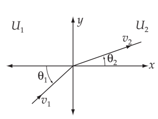
Substituting (9) into (6), we find the equation of motion with respect to θ:

$$\ddot{\theta} = -\omega^2 \sin\theta \tag{10}$$

where ω is the frequency of small oscillations, defined by

$$\omega = \sqrt{\frac{5g}{7(R-\rho)}} \tag{11}$$

7-8.



Let us choose the x,y coordinates so that the two regions are divided by the y axis:

$$U(x) = \begin{cases} U_1 & x < 0 \\ U_2 & x > 0 \end{cases}$$

If we consider the potential energy as a function of x as above, the Lagrangian of the particle is

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - U(x) \tag{1}$$

Therefore, Lagrange's equations for the coordinates x and y are

$$m\ddot{x} + \frac{dU(x)}{dx} = 0 \tag{2}$$

$$m\ddot{y} = 0 \tag{3}$$

Using the relation

$$m\ddot{x} = \frac{d}{dt} m\dot{x} = \frac{dP_x}{dt} = \frac{dP_x}{dx}\frac{dx}{dt} = \frac{P_x}{m}\frac{dp_x}{dx} \tag{4}$$

(2) becomes

$$\frac{P_x}{m}\frac{dP_x}{dx} + \frac{dU(x)}{dx} = 0 \tag{5}$$

Integrating (5) from any point in the region 1 to any point in the region 2, we find

$$\int_1^2 \frac{P_x}{m}\frac{dP_x}{dx}\, dx + \int_1^2 \frac{dU(x)}{dx}\, dx = 0 \tag{6}$$

$$\frac{P_{x_2}^2}{2m} - \frac{P_{x_1}^2}{2m} + U_2 - U_1 = 0 \tag{7}$$

or, equivalently,

$$\frac{1}{2} m\dot{x}_1^2 + U_1 = \frac{1}{2} m\dot{x}_2^2 + U_2 \tag{8}$$

Now, from (3) we have

$$\frac{d}{dt} m\dot{y} = 0$$

and $m\dot{y}$ is constant. Therefore,

$$m\dot{y}_1 = m\dot{y}_2 \tag{9}$$

From (9) we have

$$\frac{1}{2} m\dot{y}_1^2 = \frac{1}{2} m\dot{y}_2^2 \tag{10}$$

Adding (8) and (10), we have

$$\frac{1}{2} mv_1^2 + U_1 = \frac{1}{2} mv_2^2 + U_2 \tag{11}$$

From (9) we also have

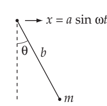
$$mv_1 \sin\theta_1 = mv_2 \sin\theta_2 \tag{12}$$

Substituting (11) into (12), we find

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \left[1 + \frac{U_1 - U_2}{T_1} \right]^{1/2}} \tag{13}$$

This problem is the mechanical analog of the refraction of light upon passing from a medium of a certain optical density into a medium with a different optical density.

7-16.



For mass m:

$$x = a\sin\omega t + b\sin\theta$$

$$y = -b\cos\theta$$

$$\dot{x} = a\omega\cos\omega t + b\dot{\theta}\cos\theta$$

$$\dot{y} = b\dot{\theta}\sin\theta$$

Substitute into

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 \right)$$

$$U = mgy$$

and the result is

$$L = T - U = \frac{1}{2} m\left(a^2\omega^2 \cos^2\omega t + 2ab\omega\dot{\theta}\cos\omega t\cos\theta + b^2\dot{\theta}^2 \right) + mgb\cos\theta$$

Lagrange's equation for θ gives

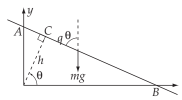
$$\frac{d}{dt}\left(mab\omega\cos\omega t\cos\theta + mb^2\dot{\theta} \right) = -mabw\dot{\theta}\cos\omega t\sin\theta - mgb\sin\theta$$

$$-ab\omega^2 \sin\omega t\cos\theta - ab\omega\dot{\theta}\cos\omega t\sin\theta + b^2\ddot{\theta} = -ab\omega\dot{\theta}\cos\omega t\sin\theta - gb\sin\theta$$

or

$$\boxed{\ddot{\theta} + \frac{g}{b}\sin\theta - \frac{a}{b}\omega^2\sin\omega t\cos\theta = 0}$$

7-17.



Using q and $\theta\,(=\omega t$ since $\theta(0)=0)$, the x,y coordinates of the particle are expressed as

$$\left.\begin{aligned} x &= h\cos\theta + q\sin\theta = h\cos\omega t + q(t)\sin\omega t \\ y &= h\sin\theta - q\cos\theta = h\sin\omega t - q(t)\cos\omega t \end{aligned}\right\} \tag{1}$$

from which

$$\left.\begin{aligned} \dot{x} &= -h\omega\sin\omega t + q\omega\cos\omega t + \dot{q}\sin\omega t \\ \dot{y} &= h\omega\cos\omega t + q\omega\sin\omega t - \dot{q}\cos\omega t \end{aligned}\right\} \tag{2}$$

Therefore, the kinetic energy of the particle is

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \\ &= \frac{1}{2}m\left(h^2\omega^2+q^2\omega^2+\dot{q}^2\right) - mh\omega\dot{q} \end{aligned} \tag{3}$$

The potential energy is

$$U = mgy = mg\left(h\sin\omega t - q\cos\omega t\right) \tag{4}$$

Then, the Lagrangian for the particle is

$$L = \frac{1}{2}mh^2\omega^2 + \frac{1}{2}mq^2\omega^2 + \frac{1}{2}m\dot{q}^2 - mgh\sin\omega t + mgq\cos\omega t - mh\omega\dot{q} \tag{5}$$

Lagrange's equation for the coordinate is

$$\ddot{q} - \omega^2 q = g\cos\omega t \tag{6}$$

The complementary solution and the particular solution for (6) are written as

$$\left.\begin{aligned} q_c(t) &= A\cos\left(i\omega t+\delta\right) \\ q_P(t) &= -\frac{g}{2\omega^2}\cos\omega t \end{aligned}\right\} \tag{7}$$

so that the general solution is

$$q(t) = A\cos\left(i\omega t+\delta\right) - \frac{g}{2\omega^2}\cos\omega t \tag{8}$$

Using the initial conditions, we have

$$\left.\begin{aligned} q(0) &= A\cos\delta - \frac{g}{2\omega^2} = 0 \\ \dot{q}(0) &= -i\omega A\sin\delta = 0 \end{aligned}\right\} \tag{9}$$

Therefore,

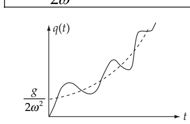
$$\delta = 0,\; A = \frac{g}{2\omega^2} \tag{10}$$

and

$$q(t) = \frac{g}{2\omega^2}\left(\cos i\omega t - \cos\omega t\right) \tag{11}$$

or,

$$\boxed{q(t) = \frac{g}{2\omega^2}\left(\cosh\omega t - \cos\omega t\right)} \tag{12}$$



In order to compute the Hamiltonian, we first find the canonical momentum of q. This is obtained by

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} - m\omega h \tag{13}$$

Therefore, the Hamiltonian becomes

$$\begin{aligned} H &= p\dot{q} - L \\ &= m\dot{q}^2 - m\omega h\dot{q} - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 - \frac{1}{2}m\dot{q}^2 + mgh\sin\omega t - mgq\cos\omega t + m\omega\dot{q}h \end{aligned}$$

so that

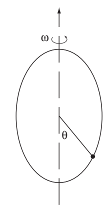
$$H = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t \tag{14}$$

Solving (13) for $\dot{q}$ and substituting gives

$$\boxed{H = \frac{p^2}{2m} + \omega hp - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t} \tag{15}$$

The Hamiltonian is therefore different from the total energy, $T + U$. The energy is not conserved in this problem since the Hamiltonian contains time explicitly. (The particle gains energy from the gravitational field.)

7-21.



From the figure, we can easily write down the Lagrangian for this system.

$$T = \frac{mR^2}{2}\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) \tag{1}$$

$$U = -mgR\cos\theta \tag{2}$$

The resulting equation of motion for θ is

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0 \tag{3}$$

The equilibrium positions are found by finding the values of θ for which

$$0 = \ddot{\theta}\Big|_{\theta=\theta_0} = \left(\omega^2\cos\theta_0 - \frac{g}{R}\right)\sin\theta_0 \tag{4}$$

Note first that 0 and π are equilibrium, and a third is defined by the condition

$$\cos\theta_0 = \frac{g}{\omega^2 R} \tag{5}$$

To investigate the stability of each of these, expand using $\varepsilon = \theta - \theta_0$

$$\ddot{\varepsilon} = \omega^2\left(\cos\theta_0 - \frac{g}{\omega^2 R} - \varepsilon\sin\theta_0\right)\left(\sin\theta_0 + \varepsilon\cos\theta_0\right) \tag{6}$$

For $\theta_0 = \pi$, we have

$$\ddot{\varepsilon} = \omega^2\left(1 + \frac{g}{\omega^2 R}\right)\varepsilon \tag{7}$$

indicating that it is unstable. For $\theta_0 = 0$, we have

$$\ddot{\varepsilon} = \omega^2\left(1 - \frac{g}{\omega^2 R}\right)\varepsilon \tag{8}$$

which is stable if $\omega^2 < g/R$ and unstable if $\omega^2 > g/R$. When stable, the frequency of small oscillations is $\sqrt{\omega^2 - g/R}$. For the final candidate,

$$\ddot{\varepsilon} = -\omega^2\sin^2\theta_0\varepsilon \tag{9}$$

with a frequency of oscillations of $\sqrt{\omega^2 - \left(g/\omega R\right)^2}$, when it exists. Defining a critical frequency $\omega_c^2 \equiv g/R$, we have a stable equilibrium at $\theta_0 = 0$ when $\omega < \omega_c$, and a stable equilibrium at $\theta_0 = \cos^{-1}\left(\omega_c^2/\omega^2\right)$ when $\omega \geq \omega_c$. The frequencies of small oscillations are then $\omega\sqrt{1-\left(\omega_c/\omega\right)^2}$ and $\omega\sqrt{1-\left(\omega_c/\omega\right)^4}$, respectively.

To construct the phase diagram, we need the Hamiltonian

$$H \equiv \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L \tag{10}$$

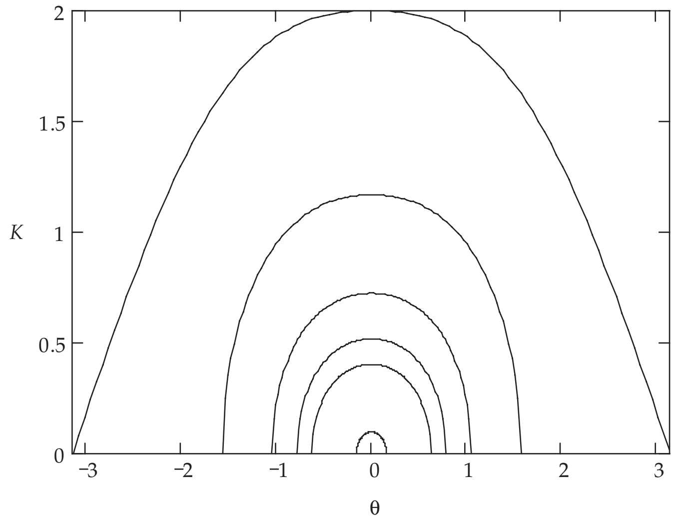
which is not the total energy in this case. A convenient parameter that describes the trajectory for a particular value of H is

$$K \equiv \frac{H}{m\omega_c^2R^2} = \frac{1}{2}\left[\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\right)^2\sin^2\theta\right] - \cos\theta \tag{11}$$

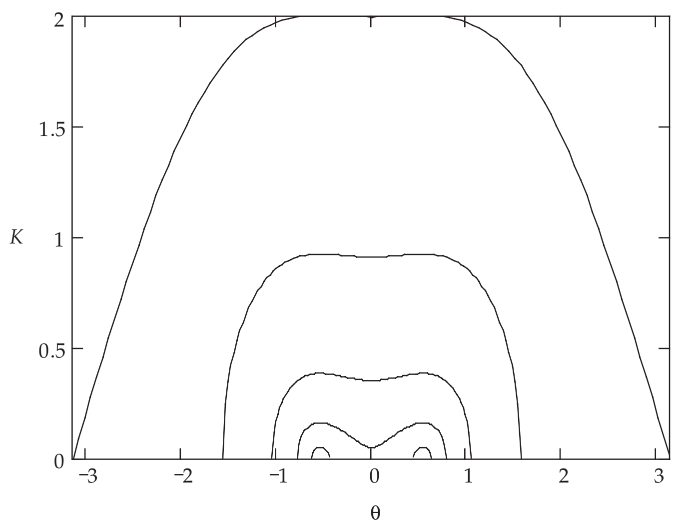
so that we'll end up plotting

$$\left(\frac{\dot{\theta}}{\omega_c}\right)^2 = 2\left(K + \cos\theta\right) + \left(\frac{\omega}{\omega_c}\right)^2\sin^2\theta \tag{12}$$

for a particular value of ω and for various values of K. The results for $\omega < \omega_c$ are shown in figure (b), and those for $\omega > \omega_c$ are shown in figure (c). Note how the origin turns from an attractor into a separatrix as ω increases through ω_c. As such, the system could exhibit chaotic behavior in the presence of damping.



(b)



(c)