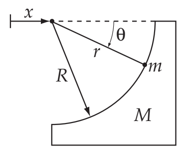
Homework 14 Solutions.
There is a typo in eqn (2) in the solution to 7-34.

7-34.



The coordinates of the wedge and the particle are

$$\begin{aligned} x_M &= x \qquad & x_m &= r\cos\theta + x \\ y_M &= 0 \qquad & y_m &= -r\sin\theta \end{aligned} \tag{1}$$

The Lagrangian is then

$$L = \frac{M+m}{2}\dot{x}^2 + \frac{m}{r}\left(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{x}\dot{r}\cos\theta - 2\dot{x}r\dot{\theta}\sin\theta\right) + mgr\sin\theta \tag{2}$$

Note that we do not take r to be constant since we want the reaction of the wedge on the particle. The constraint equation is $f(x,\theta,r) = r - R = 0$.

a) Right now, however, we may take $r = R$ and $\dot{r} = \ddot{r} = 0$ to get the equations of motion for x and θ. Using Lagrange's equations,

$$\ddot{x} = aR\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \tag{3}$$

$$\ddot{\theta} = \frac{\ddot{x}\sin\theta + g\cos\theta}{R} \tag{4}$$

where $a \equiv m/(M+m)$.

b) We can get the reaction of the wedge from the Lagrange equation for r

$$\lambda = m\ddot{x}\cos\theta - mR\dot{\theta}^2 - mg\sin\theta \tag{5}$$

We can use equations (3) and (4) to express $\ddot{x}$ in terms of θ and $\dot{\theta}$, and substitute the resulting expression into (5) to obtain

$$\lambda = \left[\frac{a-1}{1-a\sin^2\theta}\right]\left(R\dot{\theta}^2 + g\sin\theta\right) \tag{6}$$

To get an expression for $\dot{\theta}$, let us use the conservation of energy

$$H = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 - 2\dot{x}R\dot{\theta}\sin\theta\right) - mgR\sin\theta = -mgR\sin\theta_0 \tag{7}$$

where θ_0 is defined by the initial position of the particle, and $-mgR\sin\theta_0$ is the total energy of the system (assuming we start at rest). We may integrate the expression (3) to obtain $\dot{x} = aR\dot{\theta}\sin\theta$, and substitute this into the energy equation to obtain an expression for $\dot{\theta}$

$$\dot{\theta}^2 = \frac{2g\left(\sin\theta - \sin\theta_0\right)}{R\left(1-a\sin^2\theta\right)} \tag{8}$$

Finally, we can solve for the reaction in terms of only θ and θ_0

$$\lambda = -\frac{mMg\left(3\sin\theta - a\sin^3\theta - 2\sin\theta_0\right)}{(M+m)\left(1-a\sin^2\theta\right)^2} \tag{9}$$

7-35. We use z_i and p_i as our generalized coordinates, the subscript i corresponding to the ith particle. For a uniform field in the z direction the trajectories $z = z(t)$ and momenta $p = p(t)$ are given by

$$\left.\begin{aligned} z_i &= z_{i0} + v_{i0}t - \frac{1}{2}gt^2 \\ p_i &= p_{i0} - mgt \end{aligned}\right] \tag{1}$$

where z_{i0}, p_{i0}, and $v_{i0} = p_{i0}/m$ are the initial displacement, momentum, and velocity of the ith particle.

Using the initial conditions given, we have

$$z_1 = z_0 + \frac{p_0 t}{m} - \frac{1}{2}gt^2 \tag{2a}$$

$$p_1 = p_0 - mgt \tag{2b}$$

$$z_2 = z_0 + \Delta z_0 + \frac{p_0 t}{m} - \frac{1}{2}gt^2 \tag{2c}$$

$$p_2 = p_0 - mgt \tag{2d}$$

$$z_3 = z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2}gt^2 \tag{2e}$$

$$p_3 = p_0 + \Delta p_0 - mgt \tag{2f}$$

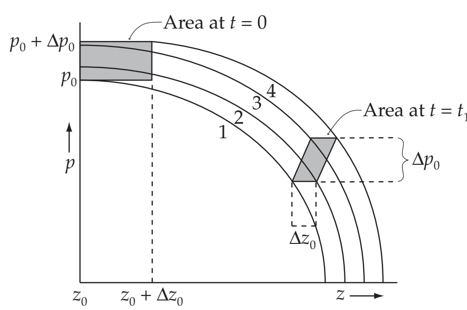
$$z_4 = z_0 + \Delta Z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2}gt^2 \tag{2g}$$

$$p_4 = p_0 + \Delta p_0 - mgt \tag{2h}$$

The Hamiltonian function corresponding to the ith particle is

$$H_i = T_i + V_i = \frac{m\dot{z}_i^2}{2} + mgz_i = \frac{p_i^2}{2m} + mgz_i = \text{const.} \tag{3}$$

From (3) the phase space diagram for any of the four particles is a parabola as shown below.

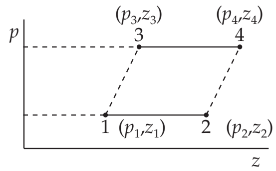


From this diagram (as well as from 2b, 2d, 2f, and 2h) it can be seen that *for any time t*,

$$p_1 = p_2 \tag{4}$$

$$p_3 = p_4 \tag{5}$$

Then for a certain time t the shape of the area described by the representative points will be of the general form



where the base $\overline{1\ 2}$ must parallel to the top $\overline{3\ 4}$. At time $t = 0$ the area is given by $\Delta z_0 \Delta p_0$, since it corresponds to a rectangle of base Δz_0 and height Δp_0. At any other time the area will be given by

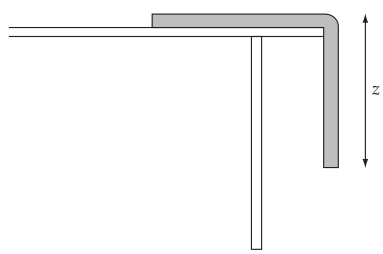
$$A = \left\{ \text{base of parallelogram}\big|_{t=t_1} = (z_2 - z_1)\big|_{t=t_1} = (z_4 - z_3)\big|_{t=t_1} = \Delta z_0 \right\}$$

$$x \ \left\{ \text{height of parallelogram}\big|_{t=t_1} = (p_3 - p_1)\big|_{t=t_1} = (p_4 - p_2)\big|_{t=t_1} = \Delta p_0 \right\}$$

$$= \Delta p_0\ \Delta z_0 \tag{6}$$

Thus, the area occupied in the phase plane is constant in time.

7-39.



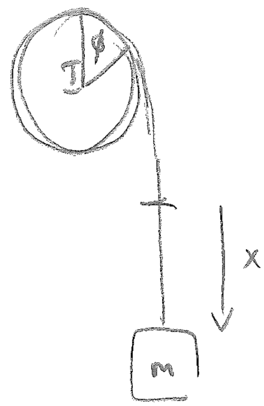
The Lagrangian of the rope is

$$L = T - U = \frac{1}{2} m \left(\frac{dz}{dt} \right)^2 - \left(-\frac{mz}{b} g \frac{z}{2} \right) = \frac{1}{2} m \left(\frac{dz}{dt} \right)^2 + \frac{mgz^2}{2b}$$

from which follows the equation of motion

$$\frac{\partial L}{\partial z} = \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} \Rightarrow \frac{mgz}{b} = m \frac{d^2 z}{dt^2}$$

7.52



$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = -mgx$$

$$\mathcal{L} = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2 + mgx$$

Constraint: $x = \phi R$, or $\phi R - x = 0 = f(x, R)$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + \lambda \frac{\partial f}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) + \lambda \frac{\partial f}{\partial \phi} = 0$$

$$mg - \frac{d}{dt}(m\dot{x}) - \lambda = 0 \qquad 0 - \frac{d}{dt}(I\dot{\phi}) + \lambda R = 0$$

① $mg - m\ddot{x} - \lambda = 0$ ② $I\ddot{\phi} = \lambda R$

Now, from constraint $R\ddot{\phi} = +\ddot{x}$ so $\lambda = +\frac{I\ddot{x}}{R^2}$.

Then,

$$mg = m\ddot{x} + \frac{I\ddot{x}}{R^2} = \ddot{x}\left(m + \frac{I}{R^2}\right)$$

$$\boxed{\ddot{x} = \frac{mR^2}{mR^2 + I} g}$$ Downward acceleration.

The constraint force $= \lambda \frac{\partial f}{\partial x} = -\lambda = \frac{-I\ddot{x}}{R^2} = \frac{-mI}{mR^2+I} g.$

↗ upward on mass m.

Newton:

$F = ma = mg - T = m\ddot{x}$ cf. eq. ①

↑ tension

Wheel.

$\tau = I\alpha = T \cdot R = I\ddot{\phi}$ cf. eq. ②

$\lambda \frac{\partial f}{\partial \phi} = T \cdot R$ = constraint <u>Torque</u>

↑ tension