1. Motion on a sphere $F_\theta = mR(\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta)$

$F_\phi = mR(2\dot{\theta}\dot{\phi}\cos\theta + \ddot{\phi}\sin\theta)$

Motion south at 100 km/h

$\dot{\phi}$ = Earth's rotation $= \frac{2\pi}{\text{day}} \cdot \frac{\text{day}}{24\text{h}} \cdot \frac{\text{h}}{3600\text{ s}} = 7.27 \times 10^{-5}\ \text{s}^{-1}$

$\ddot{\phi} = 0$

$\theta = 90° - 35°$ ← abq latitude $= 55°$ $\sin\theta = 0.82$, $\cos\theta = 0.57$

$\dot{\theta} = \frac{V}{R} = \frac{100\ \frac{\text{km}}{\text{h}}}{6366\ \text{km}} \times \frac{\text{h}}{3600\text{ s}} = 4.36 \times 10^{-6}\ \text{s}^{-1}$

$\ddot{\theta} = 0.$

$F_\theta = 1000\ \text{kg} \cdot 6.366\times10^{6}\ \text{m}\ [-(7.27\times10^{-5})^2 \cdot 0.82 \cdot 0.57] = -15.6\ \text{N}.$

NS Force on car is 15.6 N north. (This counters centrifugal force!)

$F_\phi = 1000\ \text{kg} \cdot 6.366\times10^{6}\ \text{m} \cdot [2 \cdot 7.27\times10^{-5} \cdot 4.36\times10^{-6} \cdot 0.57] = 2.3\ \text{N}$

EW force on car is 2.3 N $\hat{\phi}$, which is eastward.

Traveling East at 100 km

$\dot{\theta} = \ddot{\theta} = 0.$ $\dot{\phi} = \dot{\phi}_{earth} + \dot{\phi}_{CAR} = 7.71\times10^{-5}\ \text{s}^{-1}$ $\ddot{\phi} = 0.$

$F_\theta = 1000\ \text{kg} \cdot 6.366\times10^{6}\ \text{m} \cdot [-(7.71\times10^{-5})^2 \cdot 0.82 \cdot 0.57] = -17.7\ \text{N}.$

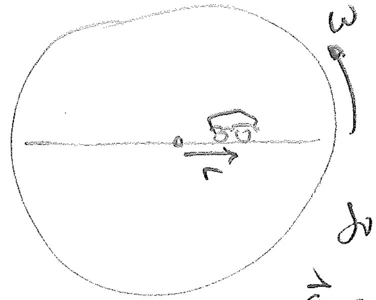
$F_\phi = 0$ because $\dot{\theta} = 0$ and $\ddot{\phi} = 0.$

At equator, $\theta = 90°$ $\cos\theta = 0.$

$\ddot{\phi} = 0$ so $F_\phi = 0.$ $\ddot{\theta} = 0$ so $F_\theta = 0.$

It's <u>almost</u> an inertial frame! (Motion along $\hat{r}$ is non-inertial.)

2.a)



Note that: $v_r = \frac{dr}{dt} = at$ $\quad \theta = \omega t$

$r = \frac{1}{2}at^2$ $\quad \dot{\theta} = \omega$

$\ddot{\theta} = 0$

so

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$= (a - \tfrac{1}{2}at^2\cdot\omega^2)\hat{r} + (\tfrac{1}{2}at^2\cdot 0 + 2at\cdot\omega)\hat{\theta}$$

The acceleration is caused by the frictional force

$\vec{F}_f = m\vec{a}$ $\quad$ so $\quad |\vec{a}| \leq \mu g$ $\quad$ or $\quad \vec{a}\cdot\vec{a} \leq \mu^2 g^2$

$|\vec{F}_f| \leq \mu m g$

$$a^2 - a^2\omega^2 t^2 + \tfrac{1}{4}a^2\omega^4 t^4 + 4a^2\omega^2 t^2 \leq \mu^2 g^2$$

Let $u = (\omega t)^2$, $\gamma = \mu g / a$.

$$1 + 3u + \tfrac{1}{4}u^2 \leq \gamma^2$$

(solve for =)

$$u^2 + 12u + 4(1-\gamma^2) = 0$$

$$u = \frac{-12 \pm \sqrt{144 - 16(1-\gamma^2)}}{2}$$ Need + root for $u > 0$.

$$u = -6 + \sqrt{32 + 4\gamma^2}$$

so $$\boxed{\omega t = \sqrt{\sqrt{32+4\gamma^2} - 6}}$$

Numbers: $\mu = 0.4$, $g = 10\ m/s^2$, $a\ 1\ m/s^2$

$\gamma = 4$. so

$\omega t = 1.949$. with $\omega = 1\,s^{-1}$, $t = 1.949\,s$.

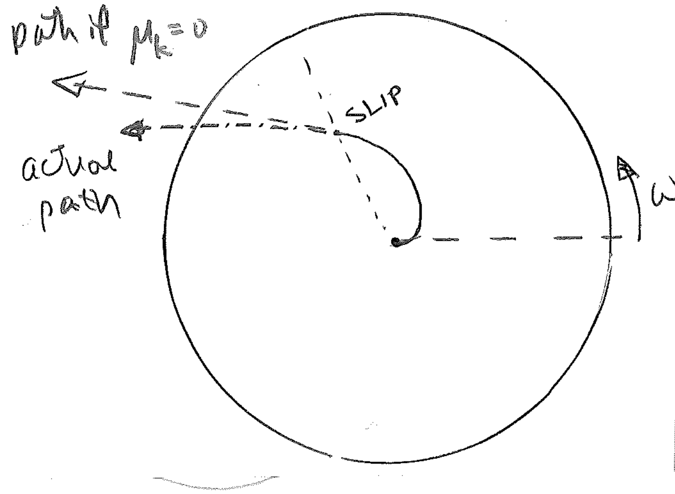
b) Assuming g stays the same, γ stays same.

ωt is still 1.068, but $\omega = 2\,s^{-1}$ so t = half as big = 0.974 s.

c) Place where it slips in part a

$r = \frac{1}{2}at^2 = 4.9\,m.$

$\theta = \omega t = 1.95°$ rad.



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3. $F = -cv^{3/2} = \frac{dv}{dt}$ $\quad \int \frac{dv}{v^{3/2}} = \int -c\,dt$ $\quad v^{-1/2} = -ct + k.$ $\quad K = v_0^{-1/2}.$

$v^{1/2} = \frac{1}{v_0^{-1/2} - ct}$ $\quad v = \frac{1}{(v_0^{-1/2} - ct)^2} = \frac{dx}{dt}$ $\quad$ (as $t \to \infty$, $v \to 0$)

$$\int \frac{dt}{(v_0^{-1/2} - ct)^2} = \int dx$$

$$= -\frac{1}{c}\frac{1}{v_0^{-1/2} - ct} = x + k_2.$$

When $t = 0$, $-\frac{1}{c}v_0^{1/2} = x_0 + k$ $\quad k = \frac{-v_0^{1/2}}{c} - x_0$

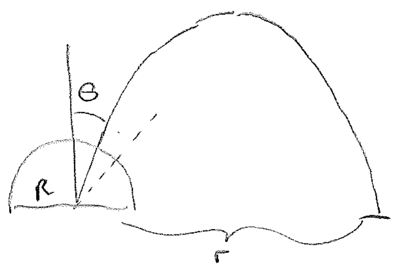
so $x = \frac{v_0^{1/2}}{c} + x_0 - \frac{1}{c}\frac{1}{v_0^{-1/2} - ct}$

As $t \to \infty$ $x \to x_0 + \frac{v_0^{1/2}}{c}.$ $\quad$ Stopping distance $= \frac{v_0^{1/2}}{c}.$

It never stops (in time), but goes a finite distance.

4. Sprinkler problem.

Let's use only θ from $0°$ to $45°$. ($45° \to 90°$ will water same area.)



Density of holes at angle θ = $\rho(\theta)$

Number of holes $dN = \rho(\theta)\, 2\pi R \sin\theta\, d\theta$.

This waters area $2\pi r\, dr$.

Holes per area watered = Constant = $\dfrac{\rho(\theta)\, \cancel{2\pi} R \sin\theta\, d\theta}{\cancel{2\pi}\, r\, dr}$

Range of a projectile $r = \dfrac{2v^2}{g} \cos\theta \sin\theta = \dfrac{v^2}{g} \sin 2\theta$

So

$$\rho(\theta) \propto \frac{r}{\sin\theta} \frac{dr}{d\theta} = \cos\theta \cdot \cos 2\theta.$$

5. a) $F = ma = m\frac{dv}{dt} = -mg - bv \quad \int \frac{dv}{g+\beta v} = \int -dt \qquad \beta \equiv b/m.$

$$\frac{1}{\beta}\ln(g+\beta v) = -t + c.$$

$$g + \beta v = ce^{-\beta t} \qquad v = ce^{-\beta t} - g/\beta \qquad V_t = g/\beta$$

at $t=0, v=v_0$ $\quad v_0 = c - g/\beta. = c - V_t. \quad c = v_0 + V_t.$

$$v = (v_0+V_t)e^{-\beta t} - V_t. = \frac{dy}{dt}.$$

$$y + c = \int [(v_0+V_t)e^{-\beta t} - V_t]\,dt$$

$$y + c = -\frac{1}{\beta}(v_0+V_t)e^{-\beta t} - V_t t$$

At $t=0$, $y=0$, so $c = -\frac{1}{\beta}(v_0+V_t)$. $\quad y = \frac{1}{\beta}(v_0+V_t)\left[1-e^{-\beta t}\right] - V_t t.$

b) Highest point: $v=0$ $\quad (v_0+V_t)e^{-\beta t} = V_t$

$$e^{-\beta t} = \frac{V_t}{v_0+V_t} \qquad \underline{t = -\frac{1}{\beta}\ln\left(\frac{V_t}{v_0+V_t}\right).}$$

at top

$$y_{top} = \frac{1}{\beta}\cdot(v_0+V_t)\cdot\left[1 - \frac{V_t}{v_0+V_t}\right] + \frac{V_t}{\beta}\ln\left(\frac{V_t}{v_0+V_t}\right)$$

$$= \frac{v_0}{\beta} + \frac{V_t}{\beta}\ln\left(\frac{V_t}{v_0+V_t}\right) = \frac{v_0}{\beta} - \frac{V_t}{\beta}\cdot\ln\left(\frac{v_0+V_t}{V_t}\right)$$

c) As β gets small, $\frac{v_0}{V_t} \to 0$ so $y_{top} = \frac{v_0}{\beta} - \frac{V_t}{\beta}\cdot\left(\frac{v_0}{V_t} - \frac{1}{2}\frac{v_0^2}{V_t^2}\right)$

$$= +\frac{1}{2}\frac{v_0^2}{\beta V_t} = \frac{1}{2}\frac{v_0^2}{g}.$$

2-11. The equation of motion is

$$m\frac{d^2x}{dt^2} = -kmv^2 + mg$$

This equation can be solved exactly in the same way as in problem 2-12 and we find

$$x = \frac{1}{2k}\log\left[\frac{g - kv_0^2}{g - kv^2}\right]$$

where the origin is taken to be the point at which $v = v_0$ so that the initial condition is $x(v = v_0) = 0$. Thus, the distance from the point $v = v_0$ to the point $v = v_1$ is

$$\boxed{s(v_0 \to v_1) = \frac{1}{2k}\log\left[\frac{g - kv_0^2}{g - kv_1^2}\right]}$$

2-12. The equation of motion for the upward motion is

$$m\frac{d^2x}{dt^2} = -mkv^2 - mg \tag{1}$$

Using the relation

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{2}$$

we can rewrite (1) as

$$\frac{v\,dv}{kv^2 + g} = -dx \tag{3}$$

Integrating (3), we find

$$\frac{1}{2k}\log\left(kv^2 + g\right) = -x + C \tag{4}$$

where the constant C can be computed by using the initial condition that $v = v_0$ when $x = 0$:

$$C = \frac{1}{2k}\log\left(kv_0^2 + g\right) \tag{5}$$

Therefore,

$$x = \frac{1}{2k}\log\frac{kv_0^2 + g}{kv^2 + g} \tag{6}$$

Now, the equation of downward motion is

$$m\frac{d^2x}{dt^2} = -mkv^2 + mg \tag{7}$$

This can be rewritten as

$$\frac{v\,dv}{-kv^2 + g} = dx \tag{8}$$

Integrating (8) and using the initial condition that $x = 0$ at $v = 0$

$$x = \frac{1}{2k}\log\frac{g}{g - kv^2} \tag{9}$$

At the highest point the velocity of the particle must be zero.

substituting $v = 0$ in (6):

$$x_h = \frac{1}{2k}\log\frac{kv_0^2 + g}{g} \tag{10}$$

Then, substituting (10) into (9),

$$\frac{1}{2k}\log\frac{kv_0^2 + g}{g} = \frac{1}{2k}\log\frac{g}{g - kv^2} \tag{11}$$

Solving for v,

$$v = \sqrt{\frac{\frac{g}{k}v_0^2}{v_0^2 + \frac{g}{k}}} \tag{12}$$

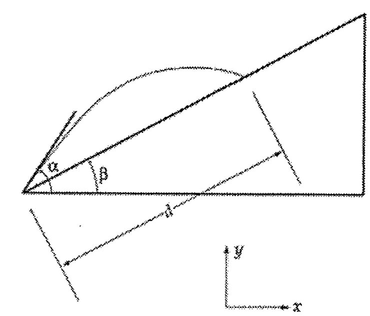
We can find the terminal velocity by putting $x \to \infty$ in(9). This gives

$$v_t = \sqrt{\frac{g}{k}} \tag{13}$$

Therefore,

$$\boxed{v = \frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}} \tag{14}$$

2-14.



a) The equations for the projectile are

$$x = v_0 \cos \alpha t$$

$$y = v_0 \sin \alpha t - \frac{1}{2}gt^2$$

Solving the first for t and substituting into the second gives

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v_0^2\cos^2\alpha}$$

Using $x = d\cos\beta$ and $y = d\sin\beta$ gives

$$d\sin\beta = d\cos\beta\tan\alpha - \frac{gd^2\cos^2\beta}{2v_0^2\cos^2\alpha}$$

$$0 = d\left[\frac{gd\cos^2\beta}{2v_0^2\cos^2\alpha} - \cos\beta\tan\alpha + \sin\beta\right]$$

Since the root $d = 0$ is not of interest, we have

$$d = \frac{2(\cos\beta\tan\alpha - \sin\beta)v_0^2\cos^2\alpha}{g\cos^2\beta}$$

$$= \frac{2v_0^2\cos\alpha(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{g\cos^2\beta}$$

$$\boxed{d = \frac{2v_0^2\cos\alpha\sin(\alpha - \beta)}{g\cos^2\beta}}$$

b) Maximize d with respect to α

$$\frac{d}{d\alpha}(d) = 0 = \frac{2v_0^2}{g\cos^2\beta}\left[-\sin\alpha\sin(\alpha-\beta) + \cos\alpha\cos(\alpha-\beta)\right]\cos(2\alpha-\beta)$$

$$\cos(2\alpha-\beta) = 0$$

$$2\alpha - \beta = \frac{\pi}{2}$$

$$\boxed{\alpha = \frac{\pi}{4} + \frac{\beta}{2}}$$

c) Substitute (2) into (1)

$$d_{max} = \frac{2v_0^2}{g\cos^2\beta}\left[\cos\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right]$$

Using the identity

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

we have

$$d_{max} = \frac{2v_0^2}{g\cos^2\beta}\cdot\frac{\sin\frac{\pi}{2} - \sin\beta}{2} = \frac{v_0^2}{g}\left[\frac{1-\sin\beta}{1-\sin^2\beta}\right]$$

$$\boxed{d_{max} = \frac{v_0^2}{g(1+\sin\beta)}}$$

2-19. The projectile's motion is described by

$$\left.\begin{aligned} x &= (v_0\cos\alpha)t \\ y &= (v_0\sin\alpha)t - \frac{1}{2}gt^2 \end{aligned}\right] \tag{1}$$

where v_0 is the initial velocity. The distance from the point of projection is

$$r = \sqrt{x^2+y^2} \tag{2}$$

Since r must always increase with time, we must have $\dot{r} > 0$:

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} > 0 \tag{3}$$

Using (1), we have

$$x\dot{x} + y\dot{y} = \frac{1}{2}g^2t^3 - \frac{3}{2}g(v_0\sin\alpha)t^2 + v_0^2t \tag{4}$$

Let us now find the value of t which yields $x\dot{x} + y\dot{y} = 0$ (i.e., $\dot{r} = 0$):

$$t = \frac{3}{2}\frac{v_0\sin\alpha}{g} \pm \frac{v_0}{2g}\sqrt{9\sin^2\alpha - 8} \tag{5}$$

For small values of α, the second term in (5) is imaginary. That is, $r = 0$ is never attained and the value of t resulting from the condition $\dot{r} = 0$ is unphysical.

Only for values of α greater than the value for which the radicand is zero does t become a physical time at which $\dot{r}$ does in fact vanish. Therefore, the maximum value of α that insures $\dot{r} > 0$ for all values of t is obtained from

$$9\sin^2\alpha_{max} - 8 = 0 \tag{6}$$

or,

$$\sin\alpha_{max} = \frac{2\sqrt{2}}{3} \tag{7}$$

so that

$$\boxed{\alpha_{max} \cong 70.5°} \tag{8}$$