3-39.

$$F(t)=\begin{cases}\sin\omega t & 0<t<\pi/\omega\\ 0 & \pi/\omega<t<2\pi/\omega\end{cases}$$

From Equations 3.89, 3.90, and 3.91, we have

$$F(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos n\omega t+b_n\sin n\omega t\right)$$

$$a_n=\frac{2}{\tau}\int_0^{\tau}F(t')\cos n\omega t'\,dt'$$

$$b_n=\frac{2}{\tau}\int_0^{\tau}F(t')\sin n\omega t'\,dt'$$

$$a_n=\frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t'\cos n\omega t'\,dt'$$

$$a_0=\frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t'\,dt'=\frac{2}{\pi}$$

$$a_1=\frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t'\cos\omega t'\,dt'=0$$

$$a_n(n\geq 2)=\frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t'\cos n\omega t'\,dt'=\frac{\omega}{\pi}\left[-\frac{\cos(1-n)\omega t'}{2(1-n)\omega}-\frac{\cos(1+n)\omega t}{2(1+n)\omega}\right]_0^{\pi/\omega}$$

Upon evaluating and simplifying, the result is

$$a_n=\begin{cases}\dfrac{2}{\pi\left(1-n^2\right)} & n\text{ even}\\ 0 & n\text{ odd}\end{cases}\qquad n=0,1,2,\ldots$$

$$b_0=0\text{ by inspection}$$

$$b_1=\frac{\omega}{\pi}\int_0^{\pi/\omega}\sin^2\omega t'\,dt'=\frac{1}{2}$$

$$b_n(n\geq 2)=\frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t'\sin n\omega t'\,dt'=\frac{\omega}{\pi}\left[\frac{\sin(1-n)\omega t'}{2(1-n)\omega}-\frac{\sin(1+n)\omega t'}{2(1+n)\omega}\right]_0^{\pi/\omega}=0$$

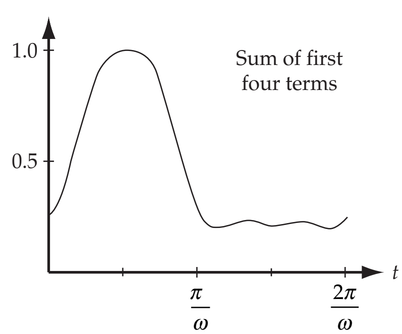
So

$$F(t)=\frac{1}{\pi}+\frac{1}{2}\sin^2\omega t+\sum_{n=2,4,6,\ldots}^{\infty}\frac{2}{\pi\left(1-n^2\right)}\cos n\omega t$$

or, letting $n\to 2n$

$$\boxed{F(t)=\frac{1}{\pi}+\frac{1}{2}\sin\omega t+\sum_{n=1,2,\ldots}^{\infty}\frac{2}{\pi\left(1-4n^2\right)}\cos 2n\omega t}$$

The following plot shows how well the first four terms in the series approximate the function.



for oscillator response, see notes below and .ppt slides on Green's functions and Fourier series.

3-38. The solution for $x(t)$ according to Green's method is

$$x(t)=\int_{-\infty}^{t}F(t')G(t,t')\,dt'$$

$$=\frac{F_0}{m\omega_1}\int_0^{t}e^{-\gamma t'}\sin\omega t'\,e^{-\beta(t-t')}\sin\omega_1(t-t')\,dt'$$

Using the trigonometric identity,

$$\sin\omega t'\sin\omega_1(t-t')=\frac{1}{2}\Big[\cos\left[(\omega_1+\omega)t'-\omega_1 t\right]-\cos\left[(\omega-\omega_1)t'+\omega_1 t\right]\Big]$$

we have

$$x(t)=\frac{F_0e^{-\beta t}}{2m\omega_1}\left[\int_0^t dt'e^{(\beta-\gamma)t'}\cos\left[(\omega+\omega_1)t'-\omega_1 t\right]-\int_0^t dt'e^{(\beta-\gamma)t'}\cos\left[(\omega-\omega_1)t'+\omega_1 t\right]\right]$$

Making the change of variable, $z=(\omega+\omega_1)t'-\omega_1 t$, for the first integral and $y=(\omega-\omega_1)t'+\omega_1 t$ for the second integral, we find

$$x(t)=\frac{F_0e^{-\beta t}}{2m\omega_1}\left[\frac{e^{\frac{(\beta-\gamma)\omega_1 t}{\omega+\omega_1}}}{\omega+\omega_1}\int_{-\omega_1 t}^{\omega t}dz\,e^{\frac{(\beta-\gamma)z}{\omega+\omega_1}}\cos z-\frac{e^{\frac{-(\beta-\gamma)\omega_1 t}{\omega-\omega_1}}}{\omega-\omega_1}\int_{\omega_1 t}^{\omega t}dy\,e^{\frac{(\beta-\gamma)y}{\omega-\omega_1}}\cos y\right]$$

After evaluating the integrals and rearranging terms, we obtain

$$\boxed{\begin{aligned}x(t)=\frac{F_0}{m}&\frac{\omega}{\left[(\beta-\gamma)^2+(\omega+\omega_1)^2\right]\left[(\beta-\gamma)^2+(\omega-\omega_1)^2\right]}\\&\times\left[e^{-\gamma t}\left[2(\gamma-\beta)\cos\omega t+\left([\beta-\gamma]^2+\omega_1^2-\omega^2\right)\frac{\sin\omega t}{\omega}\right]\right.\\&\left.+e^{-\beta t}\left[2(\beta-\gamma)\cos\omega_1 t+\left([\beta-\gamma]^2+\omega^2-\omega_1^2\right)\frac{\sin\omega_1 t}{\omega}\right]\right]\end{aligned}}$$

3-42.

a)

$$m\left(x''+\omega_0{}^2x\right)-F_0\sin\omega t=0\qquad(1)$$

The most general solution is

$$x(t)=a\sin\omega_0 t+b\cos\omega_0 t+A\sin\omega t$$

where the last term is a particular solution.
To find A we put this particular solution (the last term) into (1) and find

$$A=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}$$

At $t=0$, $x=0$, so we find $b=0$, and then we have

$$x(t)=a\sin\omega_0 t+A\sin\omega t\quad\Rightarrow\quad v(t)=a\omega_0\cos\omega_0 t+A\omega\cos\omega t$$

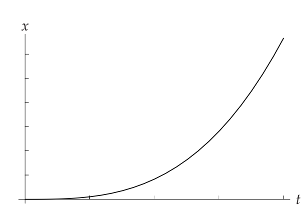
At $t=0$, $v=0\ \Rightarrow\ A=\dfrac{-a\omega_0}{\omega}\ \Rightarrow$

$$x(t)=\frac{F_0}{m\omega_0}\frac{1}{(\omega_0+\omega)(\omega_0-\omega)}\left(\omega_0\sin\omega t-\omega\sin\omega_0 t\right)$$

b) In the limit $\omega\to\omega_0$ one can see that

$$x(t)\to\frac{F_0t^3\omega_0}{6m}$$

The sketch of this function is shown below.



3-44. This oscillation must be underdamped oscillation (otherwise no period is present).

From Equation (3.40) we have

$$x(t) = A\exp(-\beta t)\cos(\omega_1 t - \delta)$$

so the initial amplitude (at $t = 0$) is A.

Now at $t = 4T = \dfrac{8\pi}{\omega_1}$

$$x(4T) = A\exp\left(-\beta\frac{8\pi}{\omega_1}\right)\cos(8\pi - \delta)$$

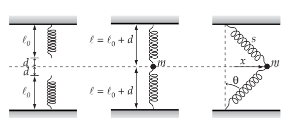
The amplitude now is $A\exp\left(-\beta\dfrac{8\pi}{\omega_1}\right)$, so we have

$$\frac{A\exp\left(-\beta\dfrac{8\pi}{\omega_1}\right)}{A} = \frac{1}{e}$$

and because $\beta = \sqrt{\omega_0^2 - \omega_1^2}$, we finally find

$$\frac{\omega_1}{\omega_0} = \frac{8\pi}{\sqrt{64\pi^2 + 1}}$$

4-1.



(a) **(b)** **(c)**

The unextended length of each spring is ℓ_0, as shown in (a). In order to attach the mass m, each spring must be stretched a distance d, as indicated in (b). When the mass is moved a distance x, as in (c), the force acting on the mass (neglecting gravity) is

$$F = -2k(s - \ell_0)\sin\theta \qquad (1)$$

where

$$s = \sqrt{\ell^2 + x^2} \qquad (2)$$

and

$$\sin\theta = \frac{x}{\sqrt{\ell^2 + x^2}} \qquad (3)$$

Then,

$$F(x) = -\frac{2kx}{\sqrt{\ell^2 + x^2}}\left[\sqrt{\ell^2 + x^2} - \ell_0\right] = -\frac{2kx}{\sqrt{\ell^2 + x^2}}\left[\sqrt{\ell^2 + x^2} - (\ell - d)\right]$$

$$= -2kx\left[1 - \frac{\ell - d}{\sqrt{\ell^2 + x^2}}\right] = -2kx\left[1 - \frac{\ell - d}{\ell}\left[1 + \frac{x^2}{\ell^2}\right]^{-1/2}\right] \qquad (4)$$

Expanding the radical in powers of x^2/ℓ^2 and retaining only the first two terms, we have

$$F(x) \cong -2kx\left[1 - \frac{\ell - d}{\ell}\left[1 - \frac{1}{2}\frac{x^2}{\ell^2}\right]\right]$$

$$= -2kx\left[1 - \left[1 - \frac{d}{\ell}\right] + \frac{1}{2}\frac{\ell - d}{\ell}\frac{x^2}{\ell^2}\right]$$

$$= -\frac{2kd}{\ell}x - \frac{k(\ell - d)}{\ell^3}x^3$$

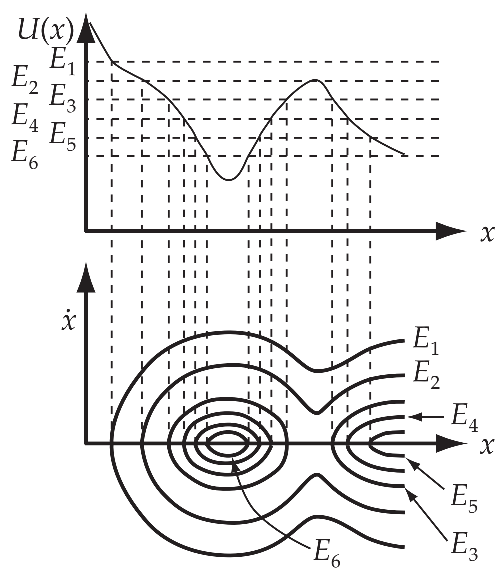
The potential is given by

$$U(x) = -\int F(x)\,dx$$

so that

$$\boxed{U(x) = \frac{kd}{\ell}x^2 + \frac{k(\ell - d)}{4\ell^3}x^4}$$

4.2 (See section 4.3 in Marion)



Oscillator Response to a "Fourier Series" driving force

Each term in F(t) fourier series contributes a sinusoidal response. Each term is of the form $A\cos(\omega_n t + \phi_n)$

1st term $\omega_n = 0$, $\phi_n = 0$ $A = 1/\pi$.

2nd term $\omega_n = \omega$ $\phi_n = -\pi/2$ $A = 1/2$.

3rd term $\omega_n = 2\omega$ $\phi_n = 0$ $A = 2/\pi(-3) = -\frac{2}{3}\pi$.

4th term. $\omega_n = 4\omega$ $\phi_n = 0$ $A = 2/\pi(-15) = -\frac{2}{15}\pi$.

δ_n is found from $\delta_n = \tan^{-1}\left(\frac{2\beta\omega_n}{\omega_0^2 - \omega_n^2}\right)$

$$D_n = \frac{A}{((\omega_0^2-\omega_n^2)^2 + 4\beta^2\omega_n^2)^{1/2}}$$

Each response term is $D_n \cos(\omega_n t + \phi_n - \delta_n)$

$$x = \frac{1}{\pi\omega_0^2} + \frac{1}{2}\cdot\frac{1}{((\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2)^{1/2}} \sin(\omega t - \delta_n) + \sum_1 \text{ etc}$$

For specific choice of ω_0, ω, β you can plot using Matlab, if you like. See ppt slides on Green's Functions + Fourier Series.