Name Solutions

1. Derive the Green's function (response to a unit impulse at time t') for a critically damped linear harmonic oscillator. What is the Green's function for $\omega_0 = 1\ s^{-1}$, m = 2 kg? (Don't worry about writing down the units – all numbers have appropriate MKS units.)

I'll solve for $t' = 0$, then offset my answer to any t'.

Initial conditions (just after impulse) $x(0) = 0$, $\dot{x}(0) = \frac{1}{m}$ ← unit impulse $= \Delta p = p_i$

Crit. damped $x(t) = (C_1 + C_2 t)e^{-\beta t}$.

Since $x(0) = 0$, $C_1 = 0$

$$\dot{x}(0) = C_2 e^{-\beta t} + -\beta C_2 t e^{-\beta t}\Big|_{t=0} = C_2 = \frac{1}{m}.$$

So $x(t) = \frac{t}{m} e^{-\beta t}$. ($\beta = \omega_0$, crit. damped)

And

$$G(t,t') = \begin{cases} \frac{t-t'}{m} e^{-\beta(t-t')} & t > t' \\ 0 & t < t' \end{cases}$$

2. Use your Green's function to find the response of the oscillator in 1 to the force

$F(t) = 4\ N \qquad 0 < t < 4$ s

$F(t) = 0 \qquad$ all other times.

Find the solution for t between 0 and 4 s. Your answer should contain only numbers and t. & $\beta = 1$

$$x(t) = \int_0^t F(t') G(t,t')\, dt'$$

$$= \int_0^t \frac{4}{\underset{=m}{2}} (t-t') \underbrace{e^{-\beta(t-t')}}_{= e^{-\beta t} e^{+\beta t'}} dt' = 2 \times \left[t e^{-\beta t} \int_0^t e^{\beta t'} dt' - e^{-\beta t} \int_0^t t' e^{\beta t'} dt' \right]$$

$$= 2\left[t e^{-\beta t}\left(\frac{e^{\beta t}}{\beta} - \frac{1}{\beta}\right) - e^{-\beta t}\left\{ e^{\beta t'}\left(\frac{t'}{\beta} - \frac{1}{\beta^2}\right)\right\}_0^t \right]$$

$$= 2\left[\frac{t}{\beta}\left(1 - e^{-\beta t}\right) - e^{-\beta t}\left\{ e^{\beta t}\left(\frac{t}{\beta} - \frac{1}{\beta^2}\right) + \frac{1}{\beta^2}\right\}\right]$$

$$= 2\left[\cancel{\frac{t}{\beta}} - \frac{t}{\beta} e^{-\beta t} - \cancel{\frac{t}{\beta}} + \frac{1}{\beta^2} - \frac{e^{-\beta t}}{\beta^2} \right]$$

$$= 2\left[\frac{1}{\beta^2} - \frac{e^{-\beta t}}{\beta^2} - \frac{\beta t}{\beta^2} e^{-\beta t} \right]$$

$$= \frac{2}{\beta^2}\left[1 - e^{-\beta t} - t e^{-\beta t}\right]$$

$$= 2\left[1 - e^{-t} - t e^{-t}\right] \quad \text{for } \beta = \omega_0 = 1\ s^{-1}.$$

3. *For this specific force* (i.e. a constant), there is another way to find the position for 0<t<4 s. Describe **in one or two sentences** the other approach.

A constant force simply shifts the equilibrium position. Find new eq'm position, take $x(0) = -X_{eqm}$, $\dot{x}(0) = 0$. } Answer

$\omega_0 = \sqrt{\frac{k}{m}}$ so $k = m\omega_0^2 = 2\ N/m$. if $F = 4N$, $\underbrace{kx_{eqm} = F_{ext}}_{\text{Spring force + Fext = 0}}$ $x_{eqm} = 2\ m$.

so $x(0) = -2\ m$, $\dot{x}(0) = 0$

$= c_1$ $\;\;\to c_2 = c_1$ $\quad x = -2e^{-t} - 2te^{-t}$,

measured from eqm position, +2!