# Homework 6 Solutions

1. The energy eigenvalues are $\varepsilon_k = \frac{\hbar^2}{2m}k^2$. The mean value over the volume of a sphere in k space is

$$<\varepsilon> = \frac{\hbar^2}{2m}\frac{\int k^2 dk \cdot k^2}{\int k^2 dk} = \frac{3}{5}\cdot\frac{\hbar^2}{2m}k_F{}^2 = \frac{3}{5}\varepsilon_F.$$

The total energy of N electrons is :

$$U_0 = N\cdot\frac{3}{5}\varepsilon_F.$$

2a. In general p = $-\partial U/\partial V$ at constant entropy. At absolute zero all processes are at constant entropy (the Third Law), so that $p = -dU_0/dV$, where $U_0 = \frac{3}{5}N\varepsilon_F = \frac{3}{5}N\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$, whence $p = \frac{2}{3}\cdot\frac{U_0}{V}$. (b) Bulk modulus

$$B = -V\frac{dp}{dV} = V\left(-\frac{2}{3}\frac{U_0}{V^2} + \frac{2}{3V}\frac{dU_0}{dV}\right) = \frac{2}{3}\cdot\frac{U_0}{V} + \left(\frac{2}{3}\right)^2\frac{U_0}{V} = \frac{10}{9}\frac{U_0}{V}.$$

(c) For Li,

$$\frac{U_0}{V} = \frac{3}{5}(4.7\times 10^{22}\ \mathrm{cm}^{-3})(4.7\ \mathrm{eV})(1.6\times 10^{-12}\ \mathrm{erg/eV})$$
$$= 2.1\times 10^{11}\ \mathrm{erg\ cm}^{-3} = 2.1\times 10^{11}\ \mathrm{dyne\ cm}^{-2},$$

whence B = $2.3\times 10^{11}$ dyne cm$^{-2}$. By experiment (Table 3.3), B = $1.2\times 10^{11}$ dyne cm$^{-2}$.

3. The number of electrons is, per unit volume, $n = \int_0^\infty d\varepsilon\ D(\varepsilon)\cdot\frac{1}{e^{(\varepsilon-\mu)/\tau}+1}$, where D(ε) is the density of orbitals. In two dimensions

$$n = \frac{m}{\pi\hbar^2}\int_0^\infty d\varepsilon\frac{1}{e^{(\varepsilon-\mu)/\tau}+1}$$
$$= \frac{m}{\pi\hbar^2}(\mu + \tau\ \log(1 + e^{-\mu/\tau})),$$

Use wolfram or equiv to look up integral.

4a. In the sun there are $\frac{2\times 10^{33}}{1.7\times 10^{-24}} \simeq 10^{57}$ nucleons, and roughly an equal number of electrons. In a white dwarf star of volume

$$\frac{4\pi}{3}(2\times 10^9)^3 \approx 3\times 10^{28}\ \mathrm{cm}^3$$

the electron concentration is $\approx \frac{10^{57}}{3\times 10^{28}} \approx 3\times 10^{28}\ \mathrm{cm}^{-3}$. Thus

$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \approx \frac{1}{2}10^{-27}\cdot 10^{20} \approx \frac{1}{2}10^{-7}$ ergs, or $\approx 3.10^4$ eV. (b) The value of $k_F$ is not affected by relativity and is $\approx n^{1/3}$, where n is the electron concentration. Thus $\varepsilon_F \simeq \hbar c k_F \simeq \hbar c^3 \sqrt{}n$. (c) A change of radius to 10 km = $10^6$ cm makes the volume $\approx 4\times 10^{18}$ cm$^3$ and the concentration $\approx 3\times 10^{38}$ cm $^{-}$ 3. Thus $\varepsilon_F \approx 10^{-27}(3.10^{10})(10^{13}) \approx 2.10^{-4}\ \mathrm{erg} \approx 10^8\ \mathrm{eV}$. (The energy is relativistic.)