Summary of Lorentz transformations of some 3-vectors

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From measurements made in frame S' to measurements made in frame S.

S measures S' to be moving with constant velocity $\vec{v}$, with $\gamma = \gamma_v \equiv 1/\sqrt{1-v^2}$.

1. For components of ordinary 4-vectors

a. for the location 4-vector, $\widetilde{x} = (\vec{r}, t)$:

$$\vec{r}_{||} = \gamma\left(\vec{r}\,'_{||} + \; t'\,\vec{v}\right)\ , \quad \vec{r}_\perp = \vec{r}\,'_\perp\ , \quad t = \gamma(t' + \vec{v}\cdot\vec{r}\,')\ ,$$

$$\text{or} \qquad \vec{r} = \gamma\left[(\vec{r}\,'\cdot\hat{v})\hat{v} + t'\vec{v}\right] + \hat{v}\times(\vec{r}\,'\times\hat{v}) \ = \ \vec{r}\,' + (\gamma-1)(\vec{r}\,'\cdot\hat{v})\hat{v} + \gamma t'\,\vec{v}\ .$$

b. for the energy-momentum 4-vector, $\widetilde{p} = (\vec{p}, E)$:

$$\vec{p}_{||} = \gamma\left(\vec{p}\,'_{||} + \; E'\,\vec{v}\right)\ , \quad \vec{p}_\perp = \vec{p}\,'_\perp\ , \quad E = \gamma(E' + \vec{v}\cdot\vec{p}\,')\ ,$$

$$\text{or} \qquad \vec{p} = \gamma\left[(\vec{p}\,'\cdot\hat{v})\hat{v} + E'\vec{v}\right] + \hat{v}\times(\vec{p}\,'\times\hat{v}) \ = \ \vec{p}\,' + (\gamma-1)(\vec{p}\,'\cdot\hat{v})\hat{v} + \gamma E'\,\vec{v}\ .$$

2. For the 3-velocity, $\vec{u}$ and its associated $\gamma_u \equiv 1/\sqrt{1-u^2}$, where the 4-velocity $\widetilde{u} = \gamma_u\,(\vec{u}\ ,\ 1)$:

$$\vec{u}_{||} = \frac{\vec{u}\,'_{||} + \vec{v}}{1+\vec{v}\cdot\vec{u}\,'}\ , \quad \vec{u}_\perp = \frac{\gamma^{-1}\,\vec{u}\,'_\perp}{1+\vec{v}\cdot\vec{u}\,'}\ , \qquad \gamma_u = \gamma_{u'}(1+\vec{v}\cdot\vec{u}\,')\ ,$$

$$\text{or} \qquad \vec{u} = \frac{\gamma^{-1}\vec{u}\,' + \vec{v} + (1-\gamma^{-1})(\hat{v}\cdot\vec{u}\,')\hat{v}}{1+\vec{v}\cdot\vec{u}\,'} = \frac{[\hat{v}\cdot\vec{u}\,' + v]\hat{v} + \gamma^{-1}\hat{v}\times(\vec{u}\,'\times\hat{v})}{1+\vec{v}\cdot\vec{u}\,'}$$

3. For the 3-acceleration, $\vec{a} \equiv d\vec{u}/dt$, where the 4-acceleration, $\widetilde{a} = \gamma_u^2\left(\vec{a} + \gamma_u^2(\vec{u}\cdot\vec{a})\ ,\ \gamma_u^2(\vec{u}\cdot\vec{a})\right)$:

$$\vec{a}_{||} = \frac{\gamma^{-3}}{(1+\vec{v}\cdot\vec{u}\,')^3}\vec{a}\,'_{||}\ , \quad \vec{a}_\perp = \frac{\gamma^{-2}}{(1+\vec{v}\cdot\vec{u}\,')^3}\left\{\vec{a}\,'_\perp + \vec{v}\times(\vec{a}\,'\times\vec{u}\,')\right\}$$

$$= \frac{1-v^2}{(1+\vec{v}\cdot\vec{u}\,')^2}\left\{\vec{a}\,'_\perp - \frac{\vec{v}\cdot\vec{a}\,'}{1+\vec{v}\cdot\vec{u}\,'}\vec{u}\,'_\perp\right\}$$

$$\text{or} \quad \vec{a} = \frac{\gamma^{-3}}{(1+\vec{v}\cdot\vec{u}\,')^3}\left\{\vec{a}\,'_{||} + \gamma\,\vec{a}\,'_\perp + \gamma\,\vec{v}\times(\vec{a}\,'\times\vec{u}\,')\right\}\ .$$

4. For the force-vector, $\vec{F}$ and its associated quantity, $\dot{E} = \vec{F}\cdot\vec{u}$, acting on an object with velocity $\vec{u}$, where the 4-vector equation is $\widetilde{K} = \gamma_u(\vec{F}\ ,\ \dot{E})$:

$$\vec{F}_{||} = \frac{\vec{F}\,'_{||} + (\vec{u}\,'\cdot\vec{F}\,')\vec{v}}{1+\vec{v}\cdot\vec{u}\,'}\ , \quad \vec{F}_\perp = \frac{\gamma^{-1}\vec{F}\,'_\perp}{1+\vec{v}\cdot\vec{u}\,'}\ ,$$

$$\text{or} \quad \vec{F} = \frac{[(\hat{v}\cdot\vec{F}\,') + v(\vec{u}\,'\cdot\vec{F}\,')]\hat{v} + \gamma^{-1}\hat{v}\times(\vec{F}\,'\times\hat{v})}{1+\vec{v}\cdot\vec{u}\,'} = \frac{\gamma^{-1}\vec{F}\,' + (1-\gamma^{-1})(\hat{v}\cdot\vec{F}\,')\hat{v} + (\vec{u}\,'\cdot\vec{F}\,')\vec{v}}{1+\vec{v}\cdot\vec{u}\,'}$$

$$\text{and} \quad \vec{F} \ = \gamma_u\,m\left[\vec{a} + \gamma_u^2(\vec{u}\cdot\vec{a})\vec{u}\right] \ \Rightarrow\ \vec{F}_{||\vec{u}} = \gamma_u^3\,m\,\vec{a}_{||\vec{u}}\ ,\ \vec{F}_{\perp\vec{u}} = \gamma_u\,m\,\vec{a}_{\perp\vec{u}}\ .$$