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Spontaneous symmetry breaking and finite-time singularities in d-dimensional incompressible flows with fractional dissipation

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Abstract – We investigate the formation of singularities in incompressible flows governed by Navier-Stokes equations in $d \geqslant 2$ dimensions with a fractional Laplacian $|\nabla|^{\alpha}$. We derive analytically a sufficient but not necessary condition for the solutions to remain always smooth and show that finite-time singularities cannot form for $\alpha \geqslant \alpha_c = 1 + d/2$. Moreover, initial singularities become unstable for $\alpha > \alpha_c$. The scale invariance symmetry intrinsic to the Navier-Stokes system becomes spontaneously broken, except at the critical point $\alpha = \alpha_c$.

Scale invariance symmetry [1–4] holds approximately for the nonlinear Navier-Stokes equations for the incompressible flow of a Newtonian fluid [5–7]. The nonlinearity and the scale invariance together can create conditions for the energy to cascade down to increasingly finer spatial and temporal scales, *e.g.*, turbulence [7]. In two dimensions, singularities cannot form [5]. However, more than a century since the discovery of these nonlinear parabolic partial differential equations, the question remains unanswered whether or not singularities can form in three dimensions, due to the crucial role played by scale invariance. This problem becomes even more difficult in higher dimensions $d > 3$. Incompressible flows in higher dimensions arise, for example, in the well-known Liouville description of Hamiltonian dynamics in phase space. Under what conditions does the flow remain free of singularities? How can we quantify the effects of dimensionality on the (possibly chaotic [2] and turbulent) cascades of energy to ever finer scales? Such fundamental and challenging problems remain the subject of ongoing investigations, due to their importance to a number of fields of physics and mathematics [5,7–20].

Here we adapt the concepts and methods of statistical physics to describe quantitatively the cascade process responsible for the formation of singularities. Our approach involves interpreting the cascade process as equivalent to the anomalous diffusion [21–30] of energy in Fourier space. This conceptual innovation enables us to determine in any dimension $d \geqslant 2$ a new sufficient condition for incompressible flows to remain smooth.

We briefly outline the main ideas underlying our analytical methodology before providing the complete details. We begin by investigating d-dimensional generalized Navier-Stokes equations with a fractional Laplacian operator [9]. The motivation for using fractional Laplacians stems from the ability that they confer to parameterize the strength of the dissipation necessary to disrupt the energy cascades, which become more violent as the number of dimensions increases. We then analytically derive a hard inequality based on the fact that no singularity can form if all partial space derivatives of all orders of the velocity field remain finite for all time. We obtain from this inequality three new results concerning the formation of singularities.

We first note the well known fact that the Fourier transform of a smooth function cannot decay algebraically but rather must decay rapidly. In terms of the d-dimensional Fourier transform $\tilde{\mathbf{v}}(\mathbf{k},t)$ of the velocity field $\mathbf{v}(\mathbf{x},t)$, we can write

$$(2\pi)^{d/2}\left|\frac{\partial^{n_1}}{\partial x_1^{n_1}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}\cdots\frac{\partial^{n_d}}{\partial x_d^{n_d}}\mathbf{v}(\mathbf{x})\right| = \left|\int_{-\infty}^{\infty}\mathbf{d}^d\mathbf{k}\exp[\mathrm{i}\mathbf{k}\cdot\mathbf{x}]\tilde{\mathbf{v}}(\mathbf{k})(\mathrm{i}k_1)^{n_1}(\mathrm{i}k_2)^{n_2}\ldots(\mathrm{i}k_d)^{n_d}\right| \leqslant \int_{-\infty}^{\infty}\mathbf{d}^d\mathbf{k}k^{n_1+n_2+\ldots n_d}|\tilde{\mathbf{v}}(\mathbf{k})|, \qquad k=|\mathbf{k}|. \tag{1}$$

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The velocity field will remain smooth so long as all such partial derivatives remain finite. To simplify the notation, let $\langle A, B\rangle$ represent the absolute integral $\int_{-\infty}^{\infty} |A(\mathbf{k})||B(\mathbf{k})|\mathbf{d}^d\mathbf{k}$. Then, the above inequality becomes

$$\left|\frac{\partial^{n_1}}{\partial x_1^{n_1}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}\cdots\frac{\partial^{n_d}}{\partial x_d^{n_d}}\mathbf{v}(\mathbf{x})\right| < \langle k^n, \tilde{\mathbf{v}}\rangle, \quad n=\sum_i^d n_i. \tag{2}$$

The integrated quantities represent modified Fourier analogs of the statistical moments typically encountered in the study of anomalous diffusion and Lévy flights [21–30]. Our inspiration comes from the theory of random walks. In what follows, we use this approach to study the formation of singularities, by deriving bounds on the rate of growth of these "absolute moments" $\langle k^n, \tilde{\mathbf{v}}\rangle$. From eq. (2), we can conclude that no singularity can form provided all such absolute moments in Fourier space remain finite.

The dissipation term in the Navier-Stokes equations contains a Laplacian for incompressible Newtonian fluids. Generalizing the Laplacian operator ∇^2 via Riesz fractional derivatives [23], we obtain the equations for the anomalous dissipation [9]:

$$\rho\left[\frac{\partial}{\partial t}v_i(\mathbf{x},t)+\mathbf{v}\cdot\nabla v_i(\mathbf{x},t)\right] = \mu_\alpha|\nabla|^\alpha v_i(\mathbf{x},t)+f_i(\mathbf{x}) - \frac{\partial}{\partial x_i}p(\mathbf{x},t), \tag{3}$$

$$\nabla\cdot\mathbf{v}(\mathbf{x},t)=0, \tag{4}$$

where p denotes the pressure, μ_α the viscosity, f the external forces, ρ the density, $\alpha>1$ the order of the fractional Laplacian and $i=1,\ldots,d$.

Fractional dissipation can arise due to the anomalous (*i.e.*, non-Fickian) transport of the momentum, *e.g.*, in non-Newtonian fluids. Generalizing the Laplacian operator in the (linear) diffusion equation with a fractional Laplacian with $\alpha<2$ leads to superdiffusive transport [23]. One obtains Lévy stable distributions for α in the semi-open interval $0<\alpha\leqslant 2$, with the marginal case $\alpha=2$ leading to normal diffusion with Gaussian propagators [23]. For $\alpha>2$, the propagator has a non-Gaussian functional form even though the variance remains finite, hence the central limit theorem renders values $\alpha>2$ unphysical for diffusing particles [23,31]. Nevertheless, in physical systems not subject to the central limit theorem, α can take on larger values. For instance, a stochastic partial differential equation having a fractional Laplacian with $\alpha=4$ correctly seems to describe slowly growing films in molecular beam epitaxy [31]. Similarly, for the viscous friction in fluids, values $\alpha>2$ give rise to hyper-dissipation of energy [9]. Specifically, α in eq. (3) parametrizes the relative strength with which very fine structures in the velocity field become "smeared" out in time, relative to larger structures (see eq. (6) below showing the Fourier transform of the dissipation term). In the context of Navier-Stokes systems, α can take on arbitrarily large values [9,14].

The substitution into eq. (3) of the scale transformation

$$\begin{aligned} \mathbf{v} &\to \lambda\mathbf{v},\\ \mathbf{x} &\to \lambda^{\frac{1}{1-\alpha}}\mathbf{x},\\ p &\to \lambda^2 p,\\ t &\to \lambda^{\frac{\alpha}{1-\alpha}}t, \end{aligned} \tag{5}$$

by a factor λ, leaves the Navier-Stokes equations unchanged (except for f_i). These scale-free properties leave open the possibility that solutions may conceivably possess structure at arbitrarily small scales, such that for $\alpha=2$, $d\geqslant 3$ we still do not know whether or not singularities can form in finite time given smooth initial conditions and external forces [7–9,14]. Our strategy consists in looking for a spontaneously broken scale invariance symmetry.

Fourier transforming eqs. (3) and (4) to eliminate the space derivatives, we obtain

$$\rho\left[\frac{\partial}{\partial t}\tilde{v}_i(\mathbf{k},t)+\sum_j^d\tilde{v}_j(\mathbf{k},t)*(\mathtt{i}k_j\tilde{v}_i(\mathbf{k},t))\right] = -\mu_\alpha k^\alpha\tilde{v}_i(\mathbf{k},t)-\mathtt{i}k_i\tilde{p}(\mathbf{k},t)+\tilde{f}_i(\mathbf{k}), \tag{6}$$

$$\mathbf{k}\cdot\tilde{\mathbf{v}}(\mathbf{k},t)=0, \tag{7}$$

with the nonlinear term becoming a nonlocal convolution term in Fourier space. Here onwards, we always define $k=|\mathbf{k}|$ as in eq. (1) and $\mathtt{i}=\sqrt{-1}$, whereas i denotes a component index. We now note that for any function $g(t)$, if $\mathrm{d}g/\mathrm{d}t=A(t)-B(t)g(t)$ and $B\geqslant 0$ then $\frac{\mathrm{d}}{\mathrm{d}t}|g|\leqslant |A|-B|g|$. Identifying g with $\rho\tilde{v}_i$ and B with $\mu_\alpha k^\alpha$, we get from eq. (6),

$$\begin{aligned}\rho\frac{\partial}{\partial t}|\tilde{v}_i(\mathbf{k},t)| \leqslant{}& \rho\left|\sum_j\tilde{v}_j(\mathbf{k},t)*(\mathtt{i}k_j\tilde{v}_i(\mathbf{k},t))\right|\\ &-\mu_\alpha k^\alpha|\tilde{v}_i(\mathbf{k},t)|\\ &+|k_i\tilde{p}(\mathbf{k},t)|+|\tilde{f}_i(\mathbf{k},t)|.\end{aligned} \tag{8}$$

Multiplying by k^n $(n=0,1,2,\ldots)$ and integrating out $\mathbf{k}$ over the entire Fourier domain, we get,

$$\begin{aligned}&\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\mathbf{d}^d\mathbf{k}\,k^n|\tilde{v}_i(\mathbf{k},t)|\leqslant\\ &\rho\int_{-\infty}^{\infty}\mathbf{d}^d\mathbf{k}\,k^n\sum_j|\tilde{v}_j(\mathbf{k},t)*(k_j\tilde{v}_i(\mathbf{k},t))|\\ &\quad-\mu_\alpha\int_{-\infty}^{\infty}\mathbf{d}^d\mathbf{k}\,k^{\alpha+n}|\tilde{v}_i(\mathbf{k},t)|\\ &\quad+\int_{-\infty}^{\infty}\mathbf{d}^d\mathbf{k}\,k^n\left(|k\tilde{p}_i(\mathbf{k},t)|+|\tilde{f}_i(\mathbf{k},t)|\right).\end{aligned} \tag{9}$$

Using the shorthand notation $\langle\cdot,\cdot\rangle$ introduced earlier, eq. (9) leads to

$$\begin{aligned}\rho\frac{\mathrm{d}}{\mathrm{d}t}\langle k^n,\tilde{v}_i\rangle &\leqslant \rho\sum_j\int \mathbf{d}^d\mathbf{k}'\langle|\mathbf{k}+\mathbf{k}'|^n,\tilde{v}_j\rangle\left|k_j'\tilde{v}_i(\mathbf{k}',t)\right| \\ &\quad -\mu_\alpha\langle k^{\alpha+n},\tilde{v}_i\rangle + C_{i,n}(t) \\ &\leqslant \rho\sum_j\int \mathbf{d}^d\mathbf{k}'\sum_\ell^n\binom{n}{\ell} \\ &\quad \times\left[\langle k^\ell,\tilde{v}_j\rangle|k'|^{n-\ell}\left|k_j'\tilde{v}_i(\mathbf{k}',t)\right|\right] \\ &\quad -\mu_\alpha\langle k^{\alpha+n},\tilde{v}_i\rangle + C_{i,n}(t),\end{aligned}$$

where $C_{i,n}(t)\equiv\langle k^{n+1},\tilde{p}_i\rangle+\langle k^n,\tilde{f}_i\rangle$. We thus arrive at the key inequality from which we will derive three interesting results:

$$\begin{aligned}\rho\frac{\mathrm{d}}{\mathrm{d}t}\langle k^n,\tilde{v}_i\rangle &\leqslant \rho\sum_j\sum_\ell^n\binom{n}{\ell}\langle k^\ell,\tilde{v}_j\rangle\langle k^{n-\ell+1},\tilde{v}_i\rangle \\ &\quad -\mu_\alpha\langle k^{\alpha+n},\tilde{v}_i\rangle + C_{i,n}(t). \qquad (10)\end{aligned}$$

Notice that no approximation has gone into the exact derivation of this hard inequality. We briefly comment on its physical significance. The growth of the moments $\langle k^n,\tilde{v}_i(t)\rangle$ indicates a broadening or widening of $\tilde{v}_i(\mathbf{k},t)$ in Fourier space. The broadening of $|\tilde{\mathbf{v}}(\mathbf{k},t)|^2$ has a physical interpretation as (anomalous) diffusion of the energy density in Fourier space. We have chosen to work with the moments of the amplitudes $|\tilde{v}_i|$ rather than of the energy density $|\tilde{\mathbf{v}}|^2$ due to convenience, since we can show that both kinds of moments quantify equivalent properties. Given smooth initial conditions, the following picture thus emerges of the cascade process: excitations of the Fourier modes can pseudorandomly "diffuse" anomalously from near the origin $k=0$ outwards, towards large k. We now ask whether for bounded energy E the moments $\langle k^n,\tilde{v}_i\rangle$ can diverge in finite time. They cannot diverge at infinite time because nonzero dissipation implies that energy decays asymptotically to zero. Only the "center-of-mass" velocity (associated with the $\mathbf{k}=0$ mode) does not decay.

The terms $C(t)$ do not matter in the context of singularities. We briefly review the known reasons for their irrelevance. Due to the lack of a separate pressure evolution equation and the special role of pressure in incompressible flows, we know that the absolute moments of pressure cannot diverge unless the moments of velocity diverge. Indeed, Leray projection methods can eliminate pressure [7,8], because in incompressible flows pressure merely serves to ensure eq. (4). It does not contribute to the cascade process, *e.g.*, in $d=3$ pressure does not appear in the vorticity equation (see below). Similarly, we can ignore smooth external force terms because they do not possess scale invariance. By their very definition, smooth external forces have a minimum scale, *i.e.*, they act only within a well-defined band of spatial frequencies, so $\langle k^n,f_i\rangle<\infty$ for all $n\geqslant 0$. From now on, we altogether ignore the terms $C(t)$.

The interesting physics only concerns which process wins the competition between the dissipation effect that damps high-frequency (large k) modes and the nonlocal interaction that transfers (or cascades) the energy to the higher-frequency modes. In inequality (10), the n-th moment $\langle k^n,\tilde{v}_i\rangle$ can only grow provided the dissipation term containing α remains smaller than the inertial terms containing products of moments. We thus arrive at a sufficient but not necessary condition for all moments to remain finite, our first result:

$$\lim_{\langle k^n,\tilde{\mathbf{v}}\rangle\to\infty}\frac{\langle k^\ell,\tilde{v}_j\rangle\langle k^{n-\ell+1},\tilde{v}_i\rangle}{\langle k^{n+\alpha},\tilde{v}_i\rangle}=0, \qquad (11)$$

for all n,i,j and $0\leqslant\ell\leqslant n$. This condition guarantees that structures of arbitrarily small spatial scale will dissipate. However, if the condition does not hold, then inequality (10) may in fact allow a finite-time singularity of the form $\langle k^n,\tilde{\mathbf{v}}\rangle\sim|t_c-t|^{-\delta_n}$. The original question of whether or not singularities can form now reduces to the more manageable problem of whether or not eq. (11) holds.

We next look for evidence of spontaneous symmetry breaking. As the excitations in the Fourier modes cascade to larger k, the moments grow due to two related but distinct reasons: i) an "entropic" effect, due to the pseudo-random or chaotic spreading or nonlocal diffusion of the excitations in Fourier space and ii) an energy renormalization effect, due to conservation of energy, which leads to approximate L^2 normalization of $\tilde{\mathbf{v}}(\mathbf{k})$: $\langle\tilde{\mathbf{v}},\tilde{\mathbf{v}}\rangle\leqslant E/\rho$. The equality holds only at $t=0$ due to energy dissipation. As an illustrative example of effect ii), if the energy in a single mode with initial amplitude $\tilde{v}_0$ becomes evenly split into N^d modes, then the sum of the amplitudes of the N^d modes grows to $N^{d/2}\tilde{v}_0$. Moreover, this energetic growth affects not only $\langle k^0,\tilde{\mathbf{v}}\rangle$ but all moments $\langle k^n,\tilde{\mathbf{v}}\rangle$. The interplay between these entropic and energetic effects lies at the root of the fundamental physical mechanism of singularity formation. As the moments grow sufficiently large, we know that $\langle k^m,\tilde{\mathbf{v}}\rangle\gg\langle k^n,\tilde{\mathbf{v}}\rangle$ if $m>n$. The highest moment appears in the dissipation term, since $\alpha>1$. Nevertheless, the overgrowth of the numerator of eq. (11) relative to the higher moment of order $n+\alpha$ in the denominator may cause the limit not to vanish. We may then lose all control and a finite-time singularity becomes a possibility.

The worst case scenario of the fastest possible overgrowth for the lower moments in the numerator of eq. (11) corresponds to isotropic cascade with a moment $\langle k^m,\tilde{\mathbf{v}}\rangle$ that diverges in finite time. Moments can diverge only due to fat tails (*i.e.*, asymptotic power law decays) in $|\tilde{\mathbf{v}}(\mathbf{k})|$. Hence, we arrive at a multifractal [3,4] scaling relation for moments in the limit of large $\langle k^m,\tilde{v}_i\rangle$ for the worst case scenario:

$$\langle k^n,\tilde{v}_i\rangle\sim\langle k^m,\tilde{v}_j\rangle^{\frac{n+d/2}{m+d/2}}, \quad n\geqslant m, \qquad (12)$$

for all i, j and $m \geqslant 0$. This scaling relation follows directly from the conservation of energy and the terms $d/2$ in the exponent come from the energy renormalization effect, as discussed earlier. Substituting eq. (12) into (11), we obtain the value of the upper critical α_c that guarantees finite moments:

$$\alpha_c = 1 + d/2. \tag{13}$$

This critical value constitutes our second important result: singularities cannot form if $\alpha \geqslant \alpha_c$. For the marginal case $\alpha = \alpha_c$ we see from inequality (10) that dissipation will hold off the singularity by "buying time" for the energy to decay. In fact, the total energy decays always with time, as mentioned earlier. However, we ignore this effect except for the marginal case because we can easily show its relative impotence to prevent singularity formation in finite time for the nonmarginal cases. We thus obtain $\alpha_c = 5/2$ for the important special case $d = 3$, in agreement with the result obtained in refs. [9,14] using different approaches. Equation (13) also correctly gives the known value $\alpha_c = 2$ in $d = 2$. Moreover, eqs. (11) and (12) together guarantee that pre-existing singularities become unstable under small perturbations for $\alpha > \alpha_c$.

We next proceed to our third and final result. Critical points often (but not always) separate phases with different symmetries. Intuitively, we expect that a spontaneously broken scale invariance symmetry should lead to the emergence of a special characteristic scale that breaks the symmetry. We next show that this may happen. If $\alpha \neq \alpha_c$ then inequality (10) suggests the existence of a characteristic length scale $\eta_c = 2\pi/k_c$ for which the inertial and dissipative effects can cancel each other. Except for the special case $\alpha = \alpha_c$, either the nonlinear inertial effects or the dissipation effects can dominate at spatial scales smaller than η_c. This statement constitutes our third and final result. Since only an inequality relation holds, rather than an equality relation, we cannot estimate this characteristic scale. For $\alpha < \alpha_c$ $(\alpha > \alpha_c)$, we can make the adimensional equivalent of μ_α very large (small) to obtain the dependence of η_c on known quantities:

$$\eta_c \propto \left(\frac{E\rho}{\mu_\alpha^2}\right)^{\frac{1}{2(\alpha_c - \alpha)}}, \quad \eta_c > 0. \tag{14}$$

Note that the units of μ_α depend not only on d but also on α. For $\alpha > \alpha_c$, the length η_c represents the smallest scale to which energy can cascade. For $\alpha < \alpha_c$, the value η_c represents the scale below which inertial effects may dominate at arbitrarily small scales, conceivably allowing singularities. Note that η_c depends on energy, which decreases with time.

We briefly discuss each of these three results. Note that our first result, eq. (11), represents a sufficient but not necessary condition for solutions to remain free of singularities. We emphasize the impossibility of obtaining the analogous necessary condition from inequality (10). The second result, concerning the upper critical dimension, shows precisely the crucial role that dimensionality has on singularity formation. This advance has deep physical significance because of the importance of α_c in relation to the Hausdorff dimension of the singular set at the time of the first blow-up [9]. The third result has a bearing on spontaneous symmetry breaking, which we address next.

Although the Navier-Stokes equations possess scale invariance symmetry, yet our results, (13) and (14), indicate spontaneous symmetry breaking. A trivial characteristic length scale $\eta = 0$ always exists for any α, corresponding to singular solutions, but our results indicate the existence of a second, nontrivial, length scale η_c. Whereas inequality (10) does not rule out stable singularities for $\alpha < \alpha_c$, on the other hand, for $\alpha > \alpha_c$ dissipation always overcomes inertial effects at scales smaller than η_c, rendering singularities unstable. The characteristic scale η_c spontaneously breaks the scale invariance symmetry. In contrast, for $\alpha = \alpha_c$, inertia and dissipation have exactly the same scaling, so that the system becomes truly scale invariant, *i.e.*, the original scale invariance symmetry remains unbroken.

In this context, we note that Tao[1] has pointed out the crucial role of energy and how it behaves under scale transformations. If the velocity in a region could evolve to become larger by a factor λ, as discussed earlier, then a (possibly intermittent) cascade of ever smaller rescaled solutions could repeat the process iteratively until the velocity blows up in finite time. For $\alpha = 2$ and $d = 2$, the energy remains invariant under scale transformations, hence energy considerations by themselves prohibit the formation of singularities. On the other hand, in $d = 3$ the energy goes to zero as $\lambda \to \infty$, so energy arguments fail. In the general case of eqs. (3), (5), the energy transforms according to $E \to \lambda^{2 - \frac{d}{(\alpha-1)}} E$, which we can express in terms of α_c as

$$E \to \lambda^{\frac{2(\alpha - \alpha_c)}{\alpha - 1}} E. \tag{15}$$

Thus, the physical interpretation of α_c becomes clear: for $\alpha \geqslant \alpha_c$ singularities become energetically forbidden.

We next discuss the critical scale η_c for the important case $d = 3$ and $\alpha = 2$. In the limit of low Reynolds number Re, η_c scales as $\sim (Re)^2$. No matter how viscous or how small we take Re, once a structure forms at length scales smaller than η_c, we cannot rule out further cascade to ever finer scales. This somewhat counter-intuitive result suggests that although the parabolic Navier-Stokes equations represent a singular perturbation of the hyperbolic Euler equations, yet for certain classes of solutions the behavior of ultra-fine structures possibly remains relatively unperturbed by dissipation at length scales $\eta \ll \eta_c$ for $\alpha < \alpha_c$. So inertial effects may remain relatively unperturbed by dissipation at arbitrarily small scales, with nothing to stop a scale-invariant cascade.

[1] `http://terrytao.wordpress.com`.

The well-known idea of smaller eddies nested hierarchically within larger eddies captures the main ingredient of such cascades, *viz*, the scale invariance. One may well ask whether this interpretation agrees with the standard picture of turbulence, where cascades also take place (at least for high Reynolds numbers), yet one typically assumes that very fine scale structures become smeared out. Mean-field theories of turbulence assume an average rate of energy dissipation. The neglect of spatial fluctuations in the dissipation rate leads to a uniform lower cutoff for spatial structures, *i.e.* cascades never become singular in mean-field theories. In contrast, our approach —based on worst case scenarios— does not introduce mean-field approximations or attempt to treat nonlinearities perturbatively. We also note that the situation where the critical scale slightly falls short of the effective system size does not imply large-scale laminar motion together with small-scale turbulent motion. Rather, the critical scale merely represents the lower threshold that a cascade must reach before it can attempt to form singularities. The cascade itself might have initiated at a much larger scale. In contrast, cascades must always terminate for $\alpha > \alpha_c$.

Finally, we comment on the case $\alpha = 2$, in terms of the relevant dynamical processes. In $d = 3$, the vorticity $\vec{\omega} \equiv \nabla \times \mathbf{v}$ satisfies the vorticity equation

$$\frac{\partial}{\partial t}\vec{\omega} + \mathbf{v} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \mathbf{v}, \tag{16}$$

for inviscid flow [7]. The last term generates vorticity stretching, which can increase vorticity and excite higher frequency modes. In $d = 2$ the vorticity stretching term vanishes, but in $d = 3$ it allows energy and vorticity to cascade and this behavior carries over to viscous flow. The well-known criterion of Beale-Kato-Majda [18] states that singularities in the Euler equation in $d = 3$ can only form if the time integral of the maximum vorticity diverges. Whether or not singularities can form in $d = 3$ for the Euler flow, however, still remains an open question. In $d > 3$ we expect even more violent energy cascades. So eqs. (13) and (14) suggest that in $d = 3$ perhaps the more important question concerns the singularity formation for the Euler equations, since $\alpha < \alpha_c$. Equation (14) conceivably makes room for singularities in Euler flow, if they in fact occur, to persist in the dissipative case. Numerical results have suggested (but not shown) that the Euler equations in $d = 3$ may in fact allow singularities [17]. Singularities possibly persist even in the Navier-Stokes case [12] under special circumstances.

In summary, we have investigated the behavior of solutions of Navier-Stokes equations with fractional Laplacians $|\nabla|^\alpha$ in d dimensions and shown analytically that singularities cannot form for $\alpha \geqslant \alpha_c$. Our approach relies on studying the growth of absolute moments $\langle k^n, \mathbf{v}\rangle$ in Fourier space. We have used neither mean-field approaches nor introduced stochasticity. Our results represent hard limits, since they correspond to the worst imaginable scenarios. We hope that the approach developed here may inspire future application towards a deeper understanding of other nonlinear partial differential equations.

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