# Physics 570

**Homework #11**

Due Thursday, 19 April, 2007

**Solutions**

**1.** All stationary solutions with cylindrical symmetry can be characterized as having the two commuting Killing vectors, $\partial_\phi$ and $\partial_t$. Weyl and Papapetrou showed that the most general such solution of the vacuum field equations may be written in the following form utilizing the usual notions of cylindrical coordinates, $\{\rho, \phi, z, t\}$:

$$\mathbf{g} \;=\; e^{-2U}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\, d\phi^2\right] - e^{+2U}(dt - \omega\, d\phi)^2 \;,$$

where $U$, $\omega$, and $\gamma$ are (real-valued) functions of (only) $\rho$ and $z$, subject to being solutions of some partial differential equations.

The simplest case is obviously when the solution is actually static, so that $\omega = 0$. In that case the constraining equations are the following:

$$0 = \nabla^2 U \equiv \left\{\partial_\rho^2 + \tfrac{1}{\rho}\partial_\rho + \partial_z^2\right\} U \;,$$
$$(\partial_\rho + i\partial_z)\gamma = \rho[(\partial_\rho + i\partial_z)U]^2 \;.$$

The equation for $U$ is just the ordinary, azimuthally-symmetric, Laplace equation, and is **linear**. Once $U$ is chosen, then the equations for $\gamma$ is also linear, and the integrability condition for them to have a common solution is simply the requirement already given on $U$. Therefore, one may find many interesting solutions of the field equations in this way; the difficulty is mostly arranging that the solution should be finite and well-behaved on the $\hat{z}$-axis, i.e., when $\rho = 0$. A standard method to resolve these equations even more goes by the name of the Ernst equation, which can also be applied to various sorts of non-vacuum solutions; when $\omega$ vanishes, the (in general complex-valued) Ernst function—the solution of the Ernst equation—is given by $e^{+2U}$.

As the Schwarzschild solution has these symmetries, it must be capable of being written in terms of these coordinates. Please show that this can indeed be done, using the following coordinate transformations:

$$\rho = \sqrt{(r-m)^2 - m^2}\;\sin\theta \;, \quad z = (r-m)\cos\theta \;, \quad \phi = \varphi \;,$$

where the appropriate solution for the Laplace equation is given by

$$e^{+2U} = 1 - \frac{2m}{r} \;.$$

In the process, you will need to determine the function $\gamma$, which is most easily written out as a function of $r$ and $\theta$, rather than $\rho$ and $z$. [It is not required that you show that it satisfies the differential equations given above.]

Then show that the Schwarzschild horizon, at any fixed value of $t$, corresponds to that part of the $z$-axis between $z = \pm m$, i.e., when $\rho = 0$ and $z$ varies between $m$ and $-m$. Therefore, in these coordinates the horizon looks rather like a "short rod" along the axis, which is singular. This sort of a singularity is often referred to as a "strut," since in more complicated solutions it amounts to some sort of "extra force" holding two or more objects apart that would otherwise move toward (or away from) each other due to gravitational attractions.

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We may immediately note that with the given value of $e^{+2U}$, the function $g_{44}$ is just exactly what it should be in the Schwarzschild manifold; therefore, there is nothing more to do there. We can then move on to $g_{33}$, which is given, in these cylindrical coordinates, as

$$\frac{\rho^2}{1-2m/r} = \frac{r^2 - 2mr + m^2 - m^2}{1-2m/r}\sin^2\theta = r^2\sin^2\theta \ .$$

Again, this is the required result for the Schwarzschild solution, so we are done with that one. At this point then we need to calculate

$$d\rho = \sin\theta\,\frac{1-m/r}{\sqrt{1-2m/r}}\,dr + \sqrt{1-2m/r}\; r\cos\theta\, d\theta \ , \quad dz = \cos\theta\, dr - (r-m)\,\sin\theta\, d\theta \ ,$$

$$\Longrightarrow \quad d\rho^2 + dz^2 = \left[\sin^2\theta\,\frac{(1-m/r)^2}{1-2m/r} + \cos^2\theta\right] dr^2 + \left[(r-2m)r\,\cos^2\theta + (r-m)^2\,\sin^2\theta\right]\, d\theta^2$$

$$= [1 - 2m/r + (m/r)^2\sin^2\theta]\left\{\frac{dr^2}{1-2m/r} + r^2\, d\theta^2\right\} \ ,$$

which gives us the remainder of the metric in the form

$$e^{-2U+2\gamma}(d\rho^2 + d\theta^2) = e^{2\gamma}\left(\frac{1-2m/r + (m/r)^2\sin^2\theta}{1-2m/r}\right)\left\{\frac{dr^2}{1-2m/r} + r^2\, d\theta^2\right\} \ .$$

In order for this to be the Schwarzschild metric, then we must choose $\gamma = \gamma(r,\theta)$ as

$$e^{2\gamma} = \frac{1-2m/r}{1-2m/r+(m/r)^2\sin^2\theta} \ .$$

This allows us to infer that in these cylindrical coordinates we have

$$g_{\rho\rho} = g_{zz} = e^{-2U+2\gamma} = \frac{1}{1-2m/r+(m/r)^2\sin^2\theta} \ ,$$

which is singular on an oblate spheroid which has radius $2m$ at the poles and radius $m$ on the equator.

To proceed further, we now notice that on the horizon, $r = 2m$, we have $\rho = 0$. Because of the nature of cylindrical coordinates, this means that the azimuthal coordinate, $\phi$, no longer is relevant; this is of course the nature of the singularity of these coordinates even in flat space. Here, however, we have an additional problem because of the singularity already described above for the metric coordinates. To see better this singularity, we note that when $r = 2m$ we have $z = m\,\cos\theta$. Therefore, $\theta$ then is indeed allowed to vary, and as it does—between $+1$ and $-1$—the values of $z$ vary between $\pm m$, verifying that this piece of the $z$-axis, of length $2m$, corresponds to the spherically-symmetric horizon in the Schwarzschild coordinates.

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**2.** We are interested in the Weyl tensor, with 10 independent components, which we divide further into its self-dual and anti-self-dual parts via the following (obvious) identities:

$$C^{\mu\nu}{}_{\lambda\eta} = \frac{1}{2}\left\{C^{\mu\nu}{}_{\lambda\eta} + {}^*C^{\mu\nu}{}_{\lambda\eta}\right\} + \frac{1}{2}\left\{C^{\mu\nu}{}_{\lambda\eta} - {}^*C^{\mu\nu}{}_{\lambda\eta}\right\} \equiv \underset{SD}{C}{}^{\mu\nu}{}_{\lambda\eta} + \underset{ASD}{C}{}^{\mu\nu}{}_{\lambda\eta}\ ,$$
$${}^*C^{\mu\nu}{}_{\lambda\eta} \equiv \tfrac{i}{2}\eta^{\mu\nu\sigma\rho}g_{\sigma\alpha}g_{\rho\beta}C^{\alpha\beta}{}_{\lambda\eta}\ ,$$

where the things at the end of the first line simply give us names for the self-dual and anti-self-dual parts, while the second line reminds us how to effect the dual transform on the components of a 2-form. We are looking for "eigenvectors" of this tensor, which then in fact appear to be 2-forms—or second-rank, skew-symmetric tensors, $\Lambda^{\mu\nu}$:

$$\underset{SD}{C}{}^{\mu\nu}{}_{\lambda\eta}\,\Lambda^{\lambda\eta} = \alpha\,\Lambda^{\mu\nu}\ .$$

However, as the eigen-2-form is complex-valued, it turns out that it may indeed be shown to specify a single null tangent vector in a way that is much clearer if we first re-describe everything in terms of spinors, which will also allow us to describe the 5 independent (complex) components of the self-dual part of the conformal tensor in terms of the 5 independent components of a **fourth-rank, totally symmetric** spinor, $C_{ABCD}$, which we create with 4 different simple spinors multiplied them together in all (24) possible orders and summed:

$$C_{ABCD} = b_{(A}\ell_B h_C f_{D)}\ ,$$

where (this time) the symmetrizing parentheses suggest the symmetrization over all 4 indices contained between them, and dividing the total by 4! (the number of distinct terms involved in the symmetric sum). Depending on the particular manifold it may be that not all 4 of these different spinors are linearly independent. Therefore, please consider a sum of our conformal spinor with (4 copies of) an arbitrary 1-spinor,

$$\zeta^A = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\ ,\quad \text{and consider the equation } C_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D = 0\ ,$$

which we treat as a polynomial equation in a single variable, say $\beta/\alpha$. Show that the number of distinct roots of this polynomial equation is the same as the number of distinct, i.e., non-proportional spinors needed to create the given conformal spinor via the sum described above, according to the scheme shown just below.

Next, let's apply this to the Schwarzschild metric, with which you are now intimately familiar: First create a null-basis for 1-forms for that metric via

$$\nu^1 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat\theta} + i\underset{\approx}{\omega}^{\hat\varphi}\right) \ , \quad \nu^2 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat\theta} - i\underset{\approx}{\omega}^{\hat\varphi}\right) \ ,$$
$$\nu^3 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat r} + \underset{\approx}{\omega}^{\hat t}\right) \ , \quad \nu^4 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat r} - \underset{\approx}{\omega}^{\hat t}\right) \ ,$$

and then determine the components of our conformal spinor via

$$\begin{aligned}
C_{0000} &= R_{3131} \ , \\
C_{0001} &= \tfrac{1}{2}\left(R_{1231} + R_{3431}\right) \ , \\
C_{0011} &= R_{4231} + \mathcal{R}/12 \ , \\
C_{0111} &= \tfrac{1}{2}\left(R_{1242} + R_{3442}\right) \ , \\
C_{1111} &= R_{4242} \ ,
\end{aligned}$$

This allows you to determine the two independent spinors $k^A$ and $\ell^B$ that factorize this system, and determine the associated null vectors and self-dual 2-forms for each one. Note that the determination of these two spinors is obviously indeterminate modulo multiplying one by some factor and dividing the other by the same factor, so try to make a reasonable, symmetric choice of their factors.

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The equation may be re-written easily in the form

$$\begin{aligned}
0 = C_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D &= C_{0000}\alpha^4 + 4\,C_{0001}\alpha^3\beta + 6\,C_{0011}\alpha^2\beta^2 + 4\,C_{0111}\alpha\beta^3 + C_{1111}\beta^4 \\
&= \alpha^4\left\{C_{0000} + 4\,C_{0001}z + 6\,C_{0011}z^2 + 4\,C_{0111}z^3 + C_{1111}z^4\right\} \ , \\
&= \alpha^4 C_{1111}\,(z_1 - z)(z_2 - z)(z_3 - z)(z_4 - z) \ , \qquad z \equiv \beta/\alpha \ ,
\end{aligned}$$

where the four $z_i$'s above are of course the 4 roots of this fourth-order polynomial equation, which are functions of the ratios of the various components of the conformal spinor. [Of course, if it turns out that $C_{1111}$ vanishes, then we must look for one of the other components to factor out. Only in the case of conformally flat space will all of them be zero.] For each of these roots we obtain a value of $z = \beta/\alpha$. We can normalize more or less as we like; for instance, we may choose $\alpha = 1$, provided it doesn't have to be zero. At any event, each root contributes a value for a spinor, and in the generic case we then have 4 spinors, none of them proportional. [As an aside, I note that we do NOT say that all 4 are linearly independent since, after all, in a 2-dimensional space one can only have two linearly-independent ones. Therefore, the correct phraseology in the generic case is that the meaning of having "4 different spinors" is that in each of the 6 different ways that one may choose two of these spinors that pair is linearly independent.]

We label those 4 spinors so created by

$$\underset{i}{\lambda} \equiv \begin{pmatrix}\alpha_i \\ \beta_i\end{pmatrix} \ , \quad \beta_i = z_i\,\alpha_i \ , \quad i = 1,2,3,4 \qquad \Longrightarrow\ C_{ABCD} = \underset{1}{\lambda}_{(A}\underset{2}{\lambda}_B\underset{3}{\lambda}_C\underset{4}{\lambda}_{D)} \ .$$

As the number of such spinors is the same as the number of (distinct) roots, we are in principle done. It is however reasonable to note that if two of the roots were the same, then the ratio of $\beta/\alpha$ would be the same for the two spinors generated by them, so that those two spinors would be proportional, and therefore not acceptable as a linearly-independent basis pair.

Although it was probably stated that you didn't have to then study the individual cases in any detail, nonetheless, I will go ahead and describe the various subcases. The most general one, where all the roots are in fact distinct, corresponds to the form given just above, with all the 4 spinors being distinct: that case is labelled **Petrov Type I**.
All the other possibilities are referred to as *algebraically degenerate*, since they correspond to some sort of degeneration of the distinctness of the roots.

2. In the case where a single pair of the roots are the same, then we will determine two spinors that are proportional, and then two more, distinct ones. We go ahead and normalize those two proportional spinors so that they are actually the same, and we have

$$\textbf{Petrov Type II:} \quad C_{ABCD} = \underset{1}{\lambda}_{(A}\underset{1}{\lambda}_{B}\underset{2}{\lambda}_{C}\underset{3}{\lambda}_{D)} \, .$$

3. In the case where there are two pair of equal roots, then there are two pair of proportional spinors, that we normalize, in each case, to be equal:

$$\textbf{Petrov Type D:} \quad C_{ABCD} = \underset{1}{\lambda}_{(A}\underset{1}{\lambda}_{B}\underset{2}{\lambda}_{C}\underset{2}{\lambda}_{D)} \, .$$

4. In the case where three of the roots are the same and one is different, then again, the spinors have the same property:

$$\textbf{Petrov Type III:} \quad C_{ABCD} = \underset{1}{\lambda}_{(A}\underset{1}{\lambda}_{B}\underset{1}{\lambda}_{C}\underset{2}{\lambda}_{D)} \, .$$

5. Finally, in the case where all four of the roots are the same then the spinors have the same property, and we have the following form, where I have dropped the symmetrizing parentheses in this case, since they are no longer necessary:

$$\textbf{Petrov Type N:} \quad C_{ABCD} = \underset{1}{\lambda}_{A}\underset{1}{\lambda}_{B}\underset{1}{\lambda}_{C}\underset{1}{\lambda}_{D} \, .$$

Retreating back to the Schwarzschild metric, to apply all this, we must first determine the components of the conformal spinor, using the identities given for that. Prior yet to that we must determine the components of the conformal tensor, which is of course identical to the Riemann tensor in this case, since this is a solution of the vacuum field equations. Using the basis given in the problem, I simply inserted this basis into GRTensor II and asked it for the conformal tensor, which give me the following results, all other **non-equivalent** ones being zero:

$$C_{1212} = -2\eta = C_{3434} \; , \quad C_{1324} = C_{1423} = -\eta \; , \qquad \eta = \frac{m}{r^3} \; .$$

Inserting this into the components of the spinor form we find that all but the central one vanish, giving us

$$C_{0011} = C_{4231} = C_{1324} = -\eta = -\frac{m}{r^3} \; , \qquad \begin{cases} C_{0000} &= 0 \; , \\ C_{0001} &= 0 \; , \\ C_{0111} &= 0 \; , \\ C_{1111} &= 0 \; , \end{cases}$$

Having then only one non-zero value for the components of the conformal spinor, we may immediately write down our polynomial. However, for clarity of exposition we decide to use the symbols $k_A$ and $\ell_B$ to denote the two independent eigenspinors, earlier denoted as $\lambda_{1A}$ and $\lambda_{2B}$. This then gives us the equation:

$$-\eta = C_{0011} = k_{(0}k_0\ell_1\ell_{1)} = \frac{1}{6}\left[(k_0)^2(\ell_1)^2 + 4k_0k_1\ell_0\ell_1 + (k_1)^2(\ell_0)^2\right] \; .$$

Clearly this equation has many solutions, and only ratios are really important. Nonetheless a very common approach to such things is to choose (irrelevant) components to vanish, although one must do this in such a way as to ensure that the resultant spinors are not proportional. Therefore, we (arbitrarily) choose $k_1 = 0 = \ell_0$, which reduces our equation to simply $(k_0)^2(\ell_1)^2 = -6\eta = -6m/r^3$. A symmetric solution for this is then

$$k_A = \begin{pmatrix} \nu \\ 0 \end{pmatrix} \; , \quad \ell_B = \begin{pmatrix} 0 \\ \nu \end{pmatrix} \; , \qquad \nu = \sqrt[4]{-6\eta} = e^{i\pi/4}\,\sigma \; , \; \sigma \equiv \sqrt[4]{6\eta} \; .$$

To determine the associated null 4-vector for each of these, we must first state the meaning of $\sqrt[4]{-1}$, which we take to mean $e^{+i\pi/4}$. Then we may determine the two associated Hermitean 2-spinors that map to a (null) 4-vector:

$$k^A = k_B\epsilon^{BA} = \begin{pmatrix} 0 \\ \nu \end{pmatrix} \; , \qquad \ell^A = \ell_B\epsilon^{BA} = \begin{pmatrix} -\nu \\ 0 \end{pmatrix} \; ,$$

$$k^A k^{\dot{B}} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \;\Longrightarrow\; -\sqrt{6m/r^3}\,[\widetilde{e}_{\hat{r}} + \widetilde{e}_{\hat{t}}] \; , \qquad \ell^A \ell^{\dot{B}} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} \;\Longrightarrow\; +\sqrt{6m/r^3}\,[\widetilde{e}_{\hat{r}} - \widetilde{e}_{\hat{t}}] \; ,$$

where the identifications of the $2 \times 2$ matrices to the 4-vectors was made by using the standard forms for Hermitean spinor matrices given in the problem statement. These two null vectors, one outgoing and one incoming, both radial, constitute the two *principal null vectors* for this geometry.

We may also go ahead and find the "eigen-2-forms" (of the conformal tensor) corresponding to these spinors:

$$k^A k^B = i\begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \;\Longrightarrow\; i\sqrt{6m/r^3}\,\nu^3 \wedge \nu^1 \; , \quad \ell^A \ell^B = i\begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} \;\Longrightarrow\; i\sqrt{6m/r^3}\,\nu^4 \wedge \nu^2 \; .$$

Again as an aside, one might wonder—as I originally did—why we have used the $C_{1423}$ component of the conformal tensor in creating its spinorial equivalent, but not the other (equal) ones, $C_{1212}$ and $C_{3434}$. The answer turns out to be that those components are NOT independent of the one that we did use, and ONLY the 10 independent ones are used in the spinorial version. Of course, in the Schwarzschild case it's rather obvious that they are not really independent, but that's not a good "proof" of this statement. Therefore, instead, consider the fact that, in complete generality, the conformal tensor is required to be traceless. Therefore we consider one set of traces, remembering to calculate this in the null basis currently being used:

$$0 = g^{\mu\nu} C_{\mu\lambda\nu\eta} = C_{1\lambda 2\eta} + C_{2\lambda 1\eta} + C_{3\lambda 4\eta} + C_{4\lambda 3\eta} \ ,$$

which must be true for all values of $\lambda$ and $\nu$. Therefore, we next consider two separate, interesting cases:

$$\begin{aligned} \lambda = 1 \ \eta = 2 \ &: \quad 0 = -C_{1212} + C_{3142} + C_{4132} \implies \quad C_{1212} = C_{3142} + C_{4132} \ , \\ \lambda = 3, \ \eta = 4 \ &: \quad 0 = C_{1324} + C_{2314} - C_{3434} \implies \quad C_{3434} = C_{1324} + C_{1423} \ , \end{aligned}$$

so that we see that not only are those two (additional) components not truly independent, in all generality, but that they are even equal to each other as well. Therefore, while GRTensor II goes ahead and tells us their values, they do not truly count as part of the independent set of components of the tensor, and are therefore not directly relevant to the values of the conformal spinor, nor its complex conjugate, the anti-self-dual part.

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