Physics 570

Homework #3

Due Thursday, 8 February, 2007

1. We have discussed the energy-momentum tensor for a perfect fluid,

$$\mathbf{T} \equiv (\rho + P)\widetilde{u} \otimes \widetilde{u} + P\mathbf{g} \; .$$

Please consider the general, special-relativistic case where the 4-velocity of the fluid is given by $\widetilde{u} = \gamma_v(\vec{v}, 1)^T$, and write out in detail the symmetric 4×4 matrix which constitutes the components of this tensor in a frame that observes the fluid moving with 3-velocity $\vec{v}$.
Do notice that under such conditions this tensor has off-diagonal terms which constitute tangential stresses on the system. Also note the contributions to the momentum density that come from the pressure. In fact, write down explicitly, from this calculation, the 4-momentum density of the moving fluid.

2. Consider the simple Lorentz boost from rest to 3-velocity $\vec{v} = v\hat{z}$.

a. Show that this 4×4 matrix may be obtained by evaluating the matrix $e^{\lambda Q_z}$, where

$$Q_z \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; , \qquad v \equiv \tanh\lambda \; .$$

b. Determine the eigenvalues and eigenvectors of this matrix, and comment on those eigenvalues, and their associated eigenvectors, which do not have the value 1.

3. Suppose that a particular curve on a (special-relativistic) manifold is given by

$$x = x(\lambda) = A\cosh(\lambda/A) \; , \quad t = t(\lambda) = A\sinh(\lambda/A) \; ,$$

where λ is the parameter along the curve and A is some constant.
Show the curve on a Minkowski diagram. Calculate the 4-velocity and the 4-acceleration, and the squares of each of these 4-vectors and their (mutual) scalar product. Show that λ is the proper time of a particle which has this curve as its worldline—which, among other things, requires that you show that the curve has an everywhere timelike tangent vector. Also please determine the physical meaning of the constant A.

4. Using the definitions of the $\eta^{\alpha\beta\gamma\delta}$ tensor constructed from the volume 4-form in 4-dimensional spacetime and/or the Levi-Civita symbol in 4-dimensional spacetime, use spherical, polar coordinates $\{r, \theta\,\varphi, t\}$ and determine explicitly the Hodge duals of the 1-forms

$$dr \; , \quad d\theta \, , \quad d\varphi \, , \quad dt \; .$$

Then determine the necessary values of the scalars α, β, and γ such that the following 2-forms are equal to their own duals:

$$dr \wedge d\theta + i\alpha d\varphi \wedge dt \; , \quad dr \wedge d\varphi + i\beta d\theta \wedge dt \; , \quad d\theta \wedge d\varphi + i\gamma dr \wedge dt \; .$$