Physics 570

Homework #3

Due Thursday, 8 February, 2007

Solutions

1. We have discussed the energy-momentum tensor for a perfect fluid,

$$\mathbf{T} \equiv (\rho + P)\widetilde{u} \otimes \widetilde{u} + P\mathbf{g} \,.$$

Please consider the general, special-relativistic case where the 4-velocity of the fluid is given by $\widetilde{u} = \gamma_v(\vec{v}, 1)^T$, and write out in detail the symmetric 4×4 matrix which constitutes the components of this tensor in a frame that observes the fluid moving with 3-velocity $\vec{v}$.
Do notice that under such conditions this tensor has off-diagonal terms which constitute tangential stresses on the system. Also note the contributions to the momentum density that come from the pressure. In fact, write down explicitly, from this calculation, the 4-momentum density of the moving fluid.

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In the general case, we have simply that our tensor has components that may be represented by the following matrix:

$$\mathbf{T} \Longrightarrow \quad (\rho+P)\gamma_v^2 \begin{pmatrix} v^x \\ v^y \\ v^z \\ 1 \end{pmatrix} \begin{pmatrix} v^x & v^y & v^z & 1 \end{pmatrix} + P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= (\rho+P)\gamma_v^2 \begin{pmatrix} (v^x)^2 & v^x v^y & v^x v^z & v^x \\ v^x v^y & (v^y)^2 & v^y v^z & v^y \\ v^z v^x & v^z v^y & (v^z)^2 & v^z \\ v^x & v^y & v^z & 1 \end{pmatrix} + P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \gamma_v^2 \begin{pmatrix} \rho(v^x)^2 + P[1-(v^y)^2-(v^z)^2] & (\rho+P)v^x v^y & (\rho+P)v^x v^z & (\rho+P)v^x \\ (\rho+P)v^y v^x & \rho(v^y)^2 + P[1-(v^x)^2-(v^z)^2] & (\rho+P)v^y v^z & (\rho+P)v^y \\ (\rho+P)v^z v^x & (\rho+P)v^z v^y & \rho(v^z)^2 + P[1-(v^x)^2-(v^y)^2] & (\rho+P)v^z \\ (\rho+P)v^x & (\rho+P)v^y & (\rho+P)v^z & \rho + Pv^2 \end{pmatrix} .$$

The 4-momentum density is the last column, or row:

$$\gamma_v^2 \begin{pmatrix} (\rho+P)v^x \\ (\rho+P)v^y \\ (\rho+P)v^z \\ \rho + Pv^2 \end{pmatrix} ,$$

We might, perhaps, have expected that the 3-vector part of the momentum density would be $\rho\vec{v}$, or, perhaps $\gamma_v \rho \vec{v}$. In fact we see two differences from this:

a.) there is actually an overall factor of γ_v^2 instead of just γ_v. The additional factor of γ_v comes from the Lorentz transformation behavior of the 3-volume.

b.) in addition, there is an additional term due to the pressure, which enters in the same way as the mass density, although it is important to remember that it is really P/c^2 which is involved. The pressure contributes to the overall energy density in the material.

c.) Coming to the fourth component, which one would perhaps surmise would just be the energy density we again find an additional contribution due to the pressure, multiplied of course by $(v/c)^2$.

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2. Consider the simple Lorentz boost from rest to 3-velocity $\vec{v} = v\hat{z}$.

a. Show that this 4×4 matrix may be obtained by evaluating the matrix $e^{\lambda Q_z}$, where

$$Q_z \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \qquad v \equiv \tanh\lambda \, .$$

b. Determine the eigenvalues and eigenvectors of this matrix, and comment on those eigenvalues, and their associated eigenvectors, which do not have the value 1.

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Beginning with λQ_z, one could ask some algebraic computer program to determine the exponential thereof, or, being more interested in details or being more truthful as to one's own efforts, one could actually evaluate the sum oneself. [NOTE to grader: either method is acceptable as a correct solution!] Performing the sum oneself, one first finds the following:

$$Q_z^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad Q_z^3 = Q_z \, .$$

Using these two relations it is actually not difficult to sum the exponential series, which gives

$$e^{\lambda Q_z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\lambda & \sinh\lambda \\ 0 & 0 & \sinh\lambda & \cosh\lambda \end{pmatrix} .$$

Using the given $\tanh\lambda = v$, we calculate the associated $\gamma_v = 1/\sqrt{1-v^2} = \cosh\lambda$, which then give us the matrix in the expected form:

$$e^{\lambda Q_z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_v & v\gamma_v \\ 0 & 0 & v\gamma_v & \gamma_v \end{pmatrix} .$$

A standard eigenvalue calculation gives the characteristic polynomial as

$$0 = (\lambda - 1)^2(\lambda - \gamma_v(1+v))(\lambda - \gamma_v(1-v)) = (\lambda - 1)^2 \left(\lambda - \sqrt{\frac{1+v}{1-v}}\right)\left(\lambda - \sqrt{\frac{1-v}{1+v}}\right) ,$$

which gives us the eigenvalues. The associated eigenvectors may then be found:

$$\lambda = 1 \;\Rightarrow\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

$$\lambda = 1 \;\Rightarrow\; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ,$$

$$\lambda = \sqrt{\frac{1+v}{1-v}} \;\Rightarrow\; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} ,$$

$$\lambda = \sqrt{\frac{1-v}{1+v}} \;\Rightarrow\; \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} .$$

An eigenvector is of course a vector which is left unchanged, modulo a scalar factor, when operated on by the matrix. These eigenvectors tell us, first that of course the vectors in the $\hat{x}$- and $\hat{y}$-directions are of course not affected by this boost in the $\hat{z}$-direction. Next we find that the only vectors with components in the $\hat{z}$-direction which are unchanged, modulo a factor, are just the two light rays which travel in the positive or negative $\hat{z}$-direction. Moreover we see that the factor which multiplies them is exactly the appropriate factor for the Doppler shift of their frequencies. On the other hand, while the above is correct, and the GRADER should consider it sufficient, let me visualize it, as well, in a different way. The two eigenvectors for the two non-trivial eigenvalues are both null vectors, i.e., tangent vectors to trajectories for light rays. Therefore, one might wonder whether it would be useful to have all the eigenvectors as light rays. Since the eigenvalue 1 is a multiple eigenvalue any linear combinations of the two eigenvectors chosen above would also satisfy the requirements that they be eigenvectors. One could then ask whether or not it is in fact possible to choose linear combinations of $\hat{x}$ and $\hat{y}$ that are of zero length, i.e., null vectors. Well, yes it is, BUT we have to use complex coefficients. I suggest that we instead choose as eigenvectors for the multiple eigenvalue $\lambda = +1$, the following complex null vectors:

$$\lambda = +1 \; , \qquad \begin{cases} \hat{x} + i\hat{y} \;\Longrightarrow\; \begin{pmatrix} 1 \\ +i \\ 0 \\ 0 \end{pmatrix} \; , \\ \hat{x} - i\hat{y} \;\Longrightarrow\; \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \; . \end{cases}$$

With these choices, each of the 4 eigenvectors is a null vector, albeit two of them are complex; as well, this last pair are orthogonal to the other pair. As a last **additional comment** I want to suppose that we choose to use these 4 eigenvectors, each divided by $\sqrt{2}$, as a basis for tangent vectors, instead of the (standard) one being used above. We may find the new metric tensor in this basis simply by calculating the metric in the following approach, where we label the 4 of them as $\{\xi_\alpha\}_1^4$:

$$g_{\alpha\beta} = \xi_\alpha \cdot \xi_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; .$$

From the results shown, you can see why I chose to divide each of them by $\sqrt{2}$. This is clearly a rather different form of metric, or a rather different form of basis vectors; however, such a "null basis set" will in fact be very useful in our later studies of gravitational waves, as they have eigen-directions associated with them which are all null—since gravitational waves, like electromagnetic waves, travel at speed $1 = c$.

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3. Suppose that a particular curve on a (special-relativistic) manifold is given by

$$x = x(\lambda) = A\cosh(\lambda/A) \; , \quad t = t(\lambda) = A\sinh(\lambda/A) \; ,$$

where λ is the parameter along the curve and A is some constant.
Show the curve on a Minkowski diagram. Calculate the 4-velocity and the 4-acceleration, and the squares of each of these 4-vectors and their (mutual) scalar product. Show that λ is the proper time of a particle which has this curve as its worldline—which, among other things, requires that you show that the curve has an everywhere timelike tangent vector. Also please determine the physical meaning of the constant A.

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We begin by calculating the tangent vector for the curve, assuming that it is indeed a timelike curve, so that we are allowed to use the proper time along that curve, τ, as the parameter. This gives us

$$\frac{dx}{d\tau} = \sinh(\lambda/A)\,\frac{d\lambda}{d\tau} \; , \quad \frac{dt}{d\tau} = \cosh(\lambda/A)\,\frac{d\lambda}{d\tau} \; .$$

Assuming, reasonably, that this is indeed the usual 4-velocity along this curve, we know that its square must equal -1, so we calculate

$$-1 = (\widetilde{u})^2 = \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\right)^2 = \left(\sinh^2(\lambda/A) - \cosh^2(\lambda/A)\right)\left(\frac{d\lambda}{d\tau}\right)^2 = -\left(\frac{d\lambda}{d\tau}\right)^2 .$$

From this we calculate that surely $\lambda = \pm\tau + b$, where b is some constant. We can always arrange our clocks, were we to want to, to make $b = 0$. As for the sign, we notice that t increases as λ increases; surely, as well, t increases as τ increases. Therefore, we must choose the positive sign, and we have indeed shown that λ is the same as τ, as desired. We shall go ahead from now on and use τ instead of λ.

Now we must calculate the acceleration 4-vector which is just the τ-derivative of the 4-velocity above:

$$a^x = \frac{d^2x}{d\tau^2} = \frac{1}{A}\cosh(\tau/A) , \quad a^t = \frac{d^2t}{d\tau^2} = \frac{1}{A}\sinh(\tau/A) \quad \Longrightarrow \quad (\widetilde{a})^2 = \left(\frac{1}{A}\right)^2 .$$

Since we have calculated already that the value of the scalar invariant for the acceleration, i.e., $(\widetilde{a})^2$, is just the acceleration in the instantaneous rest frame, it follows that A is the inverse of that instantaneous-rest-frame acceleration.

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4. Using the definitions of the $\eta^{\alpha\beta\gamma\delta}$ tensor constructed from the volume 4-form in 4-dimensional spacetime and/or the Levi-Civita symbol in 4-dimensional spacetime, use spherical, polar coordinates $\{r, \theta\, \varphi, t\}$ and determine explicitly the Hodge duals of the 1-forms

$$dr\ , \quad d\theta\ , \quad d\varphi\ , \quad dt\ .$$

Then determine the necessary values of the scalars α, β, and γ such that the following 2-forms are equal to their own duals:

$$dr \wedge d\theta + i\alpha d\varphi \wedge dt\ , \quad dr \wedge d\varphi + i\beta d\theta \wedge dt\ , \quad d\theta \wedge d\varphi + i\gamma dr \wedge dt\ .$$

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To determine these duals we must first note that we are using the coordinate basis for 1-forms $\{dr, d\theta, d\varphi, dt\}$, so that the metric (or line element) may be written

$$\begin{aligned} ds^2 =& dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 - dt^2 = g_{\mu\nu} dr^\mu\, dr^\nu \\ \Longrightarrow& \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \\ \Longrightarrow& \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/r^2 & 0 & 0 \\ 0 & 0 & 1/(r^2\sin^2\theta) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \end{aligned}$$

We also need to determine the volume form, and the associated skew-symmetric tensor which is proportional to the Levi-Civita symbol:

$$\underset{\sim}{V} = dx \wedge dy \wedge dz \wedge dt = r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi \wedge dt \, .$$

But we know that the wedge product of all 4 basis elements gives us the volume form multiplied by the desired coefficient:

$$dr \wedge d\theta \wedge d\varphi \wedge dt \equiv \eta^{1234} \underset{\sim}{V} \, .$$

Therefore it follows that

$$\eta^{1234} = \frac{1}{r^2 \sin\theta} = \frac{-1}{\eta_{1234}} \, .$$

We are now equipped to determine Hodge duals, where we will use the generic symbol r^μ to indicate the 4 coordinates currently in use, i.e., $\{r, \theta, \varphi, t\}$. We begin with $*dr$:

$$*dr = \frac{i^{1\cdot 3+1}}{1!3!} g^{1\beta} \eta_{\beta\alpha\gamma\delta} dr^\alpha \wedge dr^\gamma \wedge dr^\delta = -r^2 \sin\theta \, d\theta \wedge d\varphi \wedge dt \, .$$

The calculations of the duals of the other 1-forms is similar, although involving somewhat more calculation since in those cases the appropriate component of the inverse metric is not simply +1:

$$\begin{aligned}
*d\theta &= \tfrac{1}{6!} g^{2\beta} \eta_{\beta\alpha\gamma\delta} dr^\alpha \wedge dr^\gamma \wedge dr^\delta = -\frac{1}{r^2} (r^2 \sin\theta) \, d\varphi \wedge dr \wedge dt = -\sin\theta \, d\varphi \wedge dr \wedge dt \, , \\
*d\varphi &= \tfrac{1}{6!} g^{3\beta} \eta_{\beta\alpha\gamma\delta} dr^\alpha \wedge dr^\gamma \wedge dr^\delta = -\frac{1}{r^2 \sin^2\theta} (r^2 \sin\theta) \, dr \wedge d\theta \wedge dt = -\frac{1}{\sin\theta} \, dr \wedge d\theta \wedge dt \, , \\
*dt &= \tfrac{1}{6!} g^{4\beta} \eta_{\beta\alpha\gamma\delta} dr^\alpha \wedge dr^\gamma \wedge dr^\delta = -r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi \, .
\end{aligned}$$

To determine the duals of the 2-forms we first find it simpler to just determine the duals of all 6 2-forms:

$$\begin{aligned}
*(dr \wedge d\theta) &= \frac{i^{2\cdot 2+1}}{2!} g^{1\sigma} g^{2\tau} \eta_{\sigma\tau\gamma\delta} dr^\gamma \wedge dr^\delta = \frac{i}{2} \frac{1}{r^2} \eta_{12\gamma\delta} dr^\gamma \wedge dr^\delta = -i \sin\theta \, d\varphi \wedge dt \, , \\
*(dr \wedge d\varphi) &= \frac{i}{2} g^{1\sigma} g^{3\tau} \eta_{\sigma\tau\gamma\delta} dr^\gamma \wedge dr^\delta = \frac{i}{2} \frac{1}{r^2 \sin^2\theta} \eta_{13\gamma\delta} dr^\gamma \wedge dr^\delta = -i \frac{1}{\sin\theta} dt \wedge d\theta \, , \\
*(dr \wedge dt) &= \frac{i}{2} g^{1\sigma} g^{4\tau} \eta_{\sigma\tau\gamma\delta} dr^\gamma \wedge dr^\delta = -\frac{i}{2} \eta_{14\gamma\delta} dr^\gamma \wedge dr^\delta = +i r^2 \sin\theta \, d\theta \wedge d\varphi \, , \\
*(d\theta \wedge d\varphi) &= \frac{i}{2} g^{2\sigma} g^{3\tau} \eta_{\sigma\tau\gamma\delta} dr^\gamma \wedge dr^\delta = \frac{i}{2} \frac{1}{r^2} \frac{1}{r^2 \sin^2\theta} \eta_{23\gamma\delta} dr^\gamma \wedge dr^\delta == -i \frac{1}{r^2 \sin\theta} \, dr \wedge dt \, , \\
*(d\theta \wedge dt) &= \frac{i}{2} g^{2\sigma} g^{4\tau} \eta_{\sigma\tau\gamma\delta} dr^\gamma \wedge dr^\delta = -\frac{i}{2} \frac{1}{r^2} \eta_{24\gamma\delta} dr^\gamma \wedge dr^\delta = +i \sin\theta \, d\varphi \wedge dr \, , \\
*(d\varphi \wedge dt) &= \frac{i}{2} g^{3\sigma} g^{4\tau} \eta_{\sigma\tau\gamma\delta} dr^\gamma \wedge dr^\delta = -\frac{i}{2} \frac{1}{r^2 \sin^2\theta} \eta_{34\gamma\delta} dr^\gamma \wedge dr^\delta = i \frac{1}{\sin\theta} \, dr \wedge d\theta \, .
\end{aligned}$$

Beginning with the first one, this tells us that

$$*(dr \wedge d\theta + i\alpha d\varphi \wedge dt) = -i \sin\theta d\varphi \wedge dt - \frac{\alpha}{\sin\theta} dr \wedge d\theta \, .$$

Choosing $\alpha = -\sin\theta$ tells us that

$$*(dr \wedge d\theta - i\sin\theta d\varphi \wedge dt) = dr \wedge d\theta - i\sin\theta d\varphi \wedge dt \ .$$

Similarly we find that

$$*(dr \wedge d\varphi + i\beta d\theta \wedge dt) = -\frac{i}{\sin\theta} dt \wedge d\theta - \beta \sin\theta d\varphi \wedge dr \ ,$$

from which we see self-duality if we choose $\beta = 1/\sin\theta$; i.e, we find that

$$*(\sin\theta dr \wedge d\varphi + id\theta \wedge dt) = \sin\theta dr \wedge d\varphi + id\theta \wedge dt \ .$$

Lastly considering

$$*(d\theta \wedge d\varphi + i\gamma dr \wedge dt) = -\frac{i}{r^2\sin\theta} dr \wedge dt - \gamma r^2 \sin\theta d\theta \wedge d\varphi \ ,$$

so that self-duality asks for $\gamma = -1/(r^2\sin\theta)$, which tells us that

$$*(r^2\sin\theta d\theta \wedge d\varphi - idr \wedge dt) = r^2\sin\theta d\theta \wedge d\varphi - idr \wedge dt \ .$$

The above results determine the 3-dimensional self-dual subset of the 6 basis vectors of 2-forms. The other 3 linearly-independent ones may be chosen so as to all be anti-self-dual, i.e, they become their own negatives under the operation of duality.
