Physics 570

Homework #4

Due Thursday, 15 February, 2007

1. On a 2-dimensional manifold, using coordinates $\{x, y\}$, we are given the local vector field $\widetilde{V}$, with components, relative to the obvious coordinate basis, given by

$$V^x = xy\,, \quad V^y = -y^2\,.$$

a. Please determine the curve $\Gamma(\eta)$ for which this vector field is the tangent vector; normalize things so that for $\eta = 0$ the curve is at the point on the manifold which has coordinates $x = 2$, $y = 1$, and produce a sketch of this curve, and a few other adjacent curves, i.e., ones that pass through nearby points.

b. Next do the same thing for the vector field $\widetilde{C}$, which has components, again relative to the coordinate basis, given by

$$C^x = y\,,\ C^y = -x\,.$$

2. Because the earth has a gravitational field, the equivalence principle tells us that we may replace considerations of that field by thinking of the earth as at the center of some non-flat manifold which may well be described in spherical coordinates so that it would have a singularity at $r = 0$. [Of course this is only actually true exterior to the physical earth, so that this singularity of the coordinates does not really occur.]

A good approximation to the metric for that curved space, outside the surface of the earth, ignoring its rotation, is provided by

$$\mathbf{g} = ds^2 = \frac{1}{J^2}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) - J^2\,dt^2\,, \quad J \equiv \sqrt{1+2\Phi}\,, \Phi \equiv -\frac{M}{r}\,,$$

where one can use the equivalence principle to interpret Φ as the potential for the earth's gravitational field.

a. Using this metric, what is the value of the proper time elapsed on a stationary clock at a distance R from the earth's center, as a function of the coordinate time t? If we compare two such clocks, at different distances from the earth's center, which one runs faster?

b. Please write down a (non-holonomic) basis for 1-forms for which the metric is simply

$$ds^2 = (\underset{\sim}{\omega}^{\hat{r}})^2 + (\underset{\sim}{\omega}^{\hat{\theta}})^2 + (\underset{\sim}{\omega}^{\hat{\varphi}})^2 - (\underset{\sim}{\omega}^{\hat{t}})^2\,.$$

Then use the exterior differentials of your chosen basis set to determine the connection 1-forms, $\underset{\sim}{\Gamma}^{\mu}{}_{\nu}$, remembering that they are determined via

$$d\underset{\sim}{\omega}^{\mu} \equiv \underset{\sim}{\omega}^{\nu} \wedge \underset{\sim}{\Gamma}^{\mu}{}_{\nu}\,.$$

As those 1-forms are simply proportional to the gradients of the coordinates, and the metric coefficients are constant, the "guess" method should work.

c. In terms of the 4-velocity $\widetilde{u}$, use these connection 1-forms to write out the four geodesic equations,

$$\frac{du^\gamma}{d\tau} + u^\alpha \Gamma^\gamma{}_{\alpha\beta} u^\beta = 0 \ ,$$

for a timelike observer moving under the action of no outside forces. Now apply those equations to the situation of an observer who simply orbits the earth at the equator, i.e., moves so that r and $\theta = \pi/2$ are constant. Therefore one has the initial conditions for the pde's that $u^{\hat{r}}\Big|_{\tau=0} = 0$ and $u^{\hat{\theta}}\Big|_{\tau=0} = 0$. First show that the equations allow these initial conditions to be maintained at all later proper times. Next use the remaining equations to determine the angular frequency of rotation of our observer in his circular orbit. Are you surprised by the result?

3. Take $\underset{\approx}{\alpha}$ as an arbitrary 2-form; therefore, in terms of a basis of 1-forms, $\{\underset{\approx}{\omega}^\mu\}_1^4$, it may be written as

$$\underset{\approx}{\alpha} = \tfrac{1}{2}\alpha_{\mu\nu}\underset{\approx}{\omega}^\mu \wedge \underset{\approx}{\omega}^\nu \ .$$

Let the reciprocal basis of tangent vectors be labeled as $\{\widetilde{e}_\lambda\}_1^4$. Show in considerable detail the calculation that shows that the components of $\underset{\approx}{\alpha}$ are determined by

$$\alpha_{\mu\nu} = \underset{\approx}{\alpha}(\widetilde{e}_\mu, \widetilde{e}_\nu) \ .$$

4. The Brinkman metric is a solution of the Einstein vacuum field equations that describes plane gravitational waves. In terms of coordinates $\{a, b, u, v\}$ and one arbitrary function of one variable, $h = h(u)$, it may be written as follows:

$$\mathbf{g} \equiv ds^2 = 2\, da\, db + a^2\, h(u)\, du^2 - du\, dv \ .$$

We will also use the following basis for 1-forms:

$$\underset{\approx}{\omega}^a \equiv da \ , \quad \underset{\approx}{\omega}^b \equiv db \ , \quad \underset{\approx}{\omega}^u \equiv du \ , \quad \underset{\approx}{\omega}^v \equiv \tfrac{1}{2}(a^2\, h\, du - dv) \ , \quad \mathbf{g} \equiv g_{\alpha\beta}\, \underset{\approx}{\omega}^\alpha \otimes \underset{\approx}{\omega}^\beta \ .$$

a. First write out the 4×4 matrix that presents the quantities $g_{\alpha\beta}$, just defined above, i.e., the components of the given metric, relative to the given basis of 1-forms.

b. Then write down definitions for the basis of tangent vectors, $\{\widetilde{e}_\beta\}_1^4$, that is reciprocal to this basis for 1-forms.

c. An arbitrary tangent vector may be described by giving explicitly its components with respect to that reciprocal basis:

$$\widetilde{w} = w^\alpha\, \widetilde{e}_\alpha \equiv \rho\, \widetilde{e}_a + \sigma\, \widetilde{e}_b + \psi\, \widetilde{e}_u + \phi\, \widetilde{e}_v \ .$$

Please determine its associated 1-form

$$\underset{\approx}{w} \equiv w_\beta\, \underset{\approx}{\omega}^\beta \equiv (g_{\beta\alpha} w^\alpha)\, \underset{\approx}{\omega}^\beta \ .$$

5. Consider the following 4×4 matrix:

$$L \equiv \begin{pmatrix} -1 & 0 & a & -a \\ 0 & 1 & 0 & 0 \\ -a & 0 & -1 + a^2/2 & -a^2/2 \\ a & 0 & -a^2/2 & 1 + a^2/2 \end{pmatrix} .$$

a. Please show that L is in fact a special, orthochronous Lorentz transformation, for all real values of the constant parameter a. However, also give arguments that it is neither a pure rotation, nor a pure Lorentz boost.

b. Even though the matrix L is in the connection part of the Lorentz group that contains the identity, it is nevertheless a Lorentz transformation that is sufficiently far from the identity that it may NOT be written as a single exponential. I will not ask you to show such a non-existence statement; however, do show that it can be written as a product of a rotation $R(\theta; \hat{y})$ multiplied (on the right) by a pure boost. What is the direction of the velocity associated with this boost?