Physics 570

Homework #5

Due Thursday, 22 February, 2007

Solutions

1. Using the fact that the covariant derivative of a vector is again a vector, determine the transformation of an affine connection from one basis set to another.
More precisely, let us define our transformation via the transformations on the reciprocal bases of 1-forms and tangent vectors as follows:

$$\underset{\sim}{\omega}'^{\alpha} \equiv X^{\alpha}{}_{\mu}\,\underset{\sim}{\omega}^{\mu}\,, \qquad \widetilde{e}'_{\beta} \equiv Y^{\nu}{}_{\beta}\,\widetilde{e}_{\nu}\,, \qquad Y^{\nu}{}_{\beta}\,X^{\beta}{}_{\mu} = \delta^{\nu}_{\mu}\,.$$

Then show that the affine connections relative to the two choices of basis sets are related as follows:

$$\underset{\sim}{\Gamma}'^{\alpha}{}_{\beta} = X^{\alpha}{}_{\mu}\,Y^{\nu}{}_{\beta}\,\underset{\sim}{\Gamma}^{\mu}{}_{\nu} + X^{\alpha}{}_{\mu}\,dY^{\mu}{}_{\beta}\,.$$

Now consider as an example the 2-dimensional, flat plane, and choose as the unprimed basis sets the usual Cartesian, coordinate basis vectors, i.e., $\widetilde{e}_a = \partial_{x^a}$, and as the primed basis set the orthonormal, non-holonomic basis vectors in polar coordinates $\widetilde{e}'_{\hat{r}} = \partial_r$ and $\widetilde{e}'_{\hat{\theta}} = \frac{1}{r}\partial_\theta$. Determine the 2×2 matrices X and Y and determine the Levi-Civita connection for the primed basis, using the known fact that the Levi-Civita connection in Cartesian coordinates vanishes in flat space.

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We begin by using what was given to determine the transformation properties of the components of a tangent vector, which one could just begin with, of course:

$$v^{\nu}\widetilde{e}_{\nu} = \widetilde{v} = v'^{\beta}\widetilde{e}'_{\beta} = Y^{\nu}{}_{\beta}v'^{\beta}\,\widetilde{e}_{\nu} \quad \Longrightarrow \quad v^{\nu} = Y^{\nu}{}_{\beta}v'^{\beta}\,.$$

Next we may just write down the component form for the covariant derivative (with not-yet-specified direction) of $\widetilde{v}$:

$$\begin{aligned}
\widetilde{e}'_{\alpha}\otimes\left(dv'^{\alpha} + v'^{\beta}\underset{\sim}{\Gamma}'^{\alpha}{}_{\beta}\right) = \nabla\widetilde{v} = \widetilde{e}_{\mu}\otimes\left(dv^{\mu} + v^{\nu}\underset{\sim}{\Gamma}^{\mu}{}_{\nu}\right) &= \widetilde{e}'_{\alpha}\otimes\left(X^{\alpha}{}_{\mu}dv^{\mu} + X^{\alpha}{}_{\mu}Y^{\nu}{}_{\beta}v'^{\beta}\underset{\sim}{\Gamma}^{\mu}{}_{\nu}\right)\\
&= \widetilde{e}'_{\alpha}\otimes\left(d(X^{\alpha}{}_{\mu}v^{\mu}) - v^{\mu}dX^{\alpha}{}_{\mu} + X^{\alpha}{}_{\mu}Y^{\nu}{}_{\beta}v'^{\beta}\underset{\sim}{\Gamma}^{\mu}{}_{\nu}\right)\\
&= \widetilde{e}'_{\alpha}\otimes\left(dv'^{\alpha} - Y^{\mu}{}_{\beta}v'^{\beta}dX^{\alpha}{}_{\mu} + X^{\alpha}{}_{\mu}Y^{\nu}{}_{\beta}v'^{\beta}\underset{\sim}{\Gamma}^{\mu}{}_{\nu}\right)\\
&= \widetilde{e}'_{\alpha}\otimes\left[dv'^{\alpha} + v'^{\beta}\left(X^{\alpha}{}_{\mu}Y^{\nu}{}_{\beta}\underset{\sim}{\Gamma}^{\mu}{}_{\nu} - Y^{\mu}{}_{\beta}dX^{\alpha}{}_{\mu}\right)\right]\,.
\end{aligned}$$

Comparing the first and last terms in the above string of equalities, we may pick out the relationship between the two desired quantities, i.e., the connection in the two frames:

$$\underset{\sim}{\Gamma}'^{\alpha}{}_{\beta} = X^{\alpha}{}_{\mu}Y^{\nu}{}_{\beta}\underset{\sim}{\Gamma}^{\mu}{}_{\nu} - Y^{\mu}{}_{\beta}dX^{\alpha}{}_{\mu}\,.$$

The second term on the right-hand side of this equation is usually referred to as an inhomogeneous term, since it does not involve the connection itself; the first term is of course linear in the connection and, all by itself, is exactly what the transformation of a tensor of this rank should be. It is the inhomogeneous term that makes the connection **other than a tensor.** However, we may also note that while this is indeed correct it, however, does not have the form that we were asked to show. Therefore, one more step is needed. Consider the inhomogeneous term separately, and we will perform an "integration by parts" on it:

$$-Y^\mu{}_\beta dX^\alpha{}_\mu = -d\left(Y^\mu{}_\beta X^\alpha{}_\mu\right) + X^\alpha{}_\mu dY^\mu{}_\beta = -d\delta^\alpha_\beta + X^\alpha{}_\mu dY^\mu{}_\beta = X^\alpha{}_\mu dY^\mu{}_\beta \ .$$

Inserting this equality into the transformation law now puts it into the form requested by the statement of the problem:

$$\underset{\sim}{\Gamma}'^\alpha{}_\beta = X^\alpha{}_\mu Y^\nu{}_\beta \underset{\sim}{\Gamma}^\mu{}_\nu + X^\alpha{}_\mu dY^\mu{}_\beta \ .$$

For the simple example, we take the transformation in two steps, constructing three different bases for tangent vectors:

i.) we label $\{\partial_{x^\mu}\}_1^2 = \{\partial_x, \partial_y\}$,

ii.) and then $\{\partial_{r^\sigma}\}_1^2 = \{\partial_r, \partial_\phi\}$,

iii.) and, lastly, $\{\widetilde{e}_\alpha\}_1^2 = \{\widetilde{e}_{\hat{r}} = \partial_r \, , \, \widetilde{e}_{\hat{\varphi}} = \frac{1}{r}\partial_\phi\}$.

Then the transformation matrix from the first set to the second set is given by the chain rule:

$$\frac{\partial}{\partial r^\sigma} = \frac{\partial x^\alpha}{\partial r^\sigma}\frac{\partial}{\partial x^\alpha} \qquad \Longrightarrow \qquad \frac{\partial x^\alpha}{\partial r^\sigma} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & +r\cos\phi \end{pmatrix} \ .$$

Next the transformation matrix from this second set to the, desired, third set can be made just by "eye":

$$\begin{pmatrix} \widetilde{e}_{\hat{r}} \\ \widetilde{e}_{\hat{\varphi}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\phi \end{pmatrix} \ .$$

Therefore the needed transformation matrix is just the product of these two matrices:

$$\widetilde{e}'_\beta \equiv Y^\nu{}_\beta \, \widetilde{e}_\nu \qquad \Longrightarrow \qquad Y^\nu{}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & +r\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & +\cos\phi \end{pmatrix} \ .$$

We next notice that this is an orthogonal matrix, i.e., it corresponds to a rotation by the angle ϕ in this 2-dimensional plane; therefore, its inverse is obtained simply by changing the sign of the angle. [We should, perhaps, also note that it should have been expected that the matrix in question would be a rotation, since *it is a transformation from one orthonormal basis,* $\{\partial_x, \partial_y\}$*, to another,* $\{\widetilde{e}_{\hat{r}}, \widetilde{e}_{\hat{\varphi}}\}$.]

We now must insert all this into the transformation equation, noting of course that the connection 1-forms are zero for the Cartesian basis set, which gives us just:

$$\underset{\sim}{\Gamma}'^\alpha{}_\beta = X^\alpha{}_\mu \, dY^\mu{}_\beta = \begin{pmatrix} \cos\phi & +\sin\phi \\ -\sin\phi & +\cos\phi \end{pmatrix} d \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & +\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & +\sin\phi \\ -\sin\phi & +\cos\phi \end{pmatrix} \begin{pmatrix} -\sin\phi & -\cos\phi \\ +\cos\phi & -\sin\phi \end{pmatrix} d\phi \ ,$$

where, since α and β only take on 2 distinct values, we are presenting the result as a 2×2 matrix, with 1-forms as elements. Lastly, matrix multiplication gives us the following:

$$\underset{\sim}{\Gamma}'^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} d\phi \quad \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}} = -d\phi = -\frac{1}{r}\,\underset{\sim}{\omega}^{\hat{\varphi}} = -\underset{\sim}{\Gamma}_{\hat{\varphi}\hat{r}}\ , \quad \underset{\sim}{\Gamma}_{\hat{r}\hat{r}} = 0 = \underset{\sim}{\Gamma}_{\hat{\varphi}\hat{\varphi}}\ .$$

To **verify** that this is correct, we should now begin directly with the dual basis for 1-forms, namely $\underset{\sim}{\omega}^{\hat{r}} = dr$ and $\underset{\sim}{\omega}^{\hat{\varphi}} \equiv r\,d\phi$ and use the First Structure Equations to directly determine the connections:

$$\begin{aligned} 0 = ddr = d\underset{\sim}{\omega}^{\hat{r}} &= \underset{\sim}{\omega}^{\hat{\varphi}} \wedge \underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}} \quad \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}} \propto \underset{\sim}{\omega}^{\hat{\varphi}}\ , \\ \frac{1}{r}\underset{\sim}{\omega}^{\hat{r}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} = dr \wedge d\phi = d\underset{\sim}{\omega}^{\hat{\varphi}} &= \underset{\sim}{\omega}^{\hat{r}} \wedge \underset{\sim}{\Gamma}_{\hat{\varphi}\hat{r}} \quad \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{\varphi}\hat{r}} = \frac{1}{r}\underset{\sim}{\omega}^{\hat{\varphi}} = -\underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}}\ , \end{aligned}$$

where we have lowered the indices as desired since the metric is simply the identity. The two calculations are indeed consistent!

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2. Consider the metric on a 3-sphere, i.e., a 3-dimensional sphere in 4-space. Use the generalization of the usual angular coordinates that is appropriate for this "higher-dimensional" sphere, namely $\{\psi, \theta, \varphi\} \equiv \{\lambda^i\}_1^3$, and the usual metric is then given by the following:

$$\mathbf{g} = ds^2 = (d\psi)^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2) \equiv (\underset{\sim}{\omega}^1)^2 + (\underset{\sim}{\omega}^2)^2 + (\underset{\sim}{\omega}^3)^3\ ,$$

where the set $\{\underset{\sim}{\omega}^i\}_1^3$ is a non-holonomic, orthonormal frame for the problem, at least in some "good" neighborhood of some generic point.

a. Please consider Cartan's First Structure equations, for a metric-compatible, torsion-free connection:

$$d\underset{\sim}{\omega}^i = \underset{\sim}{\omega}^j \wedge \underset{\sim}{\Gamma}^i{}_j\ ,$$

and use these to determine the 3 independent 1-forms that determine the connection, i.e.,

$$\underset{\sim}{\Gamma}_{12}\ , \quad \underset{\sim}{\Gamma}_{23}\ , \quad \underset{\sim}{\Gamma}_{31}\ .$$

Use the usual "guess" method to obtain them.

b. Determine the 3 independent 2-forms that determine the curvature, as given by the Second Structure equations:

$$\underset{\sim}{\Omega}^i{}_j \equiv d\underset{\sim}{\Gamma}^i{}_j + \underset{\sim}{\Gamma}^i{}_k \wedge \underset{\sim}{\Gamma}^k{}_j\ .$$

Be sure and remember that these forms are skew-symmetric on their indices when both are at the same level, i.e.,

$$\underset{\sim}{\Gamma}_{jk} = -\underset{\sim}{\Gamma}_{kj}\ , \qquad \text{and} \qquad \underset{\sim}{\Omega}_{jk} = -\underset{\sim}{\Omega}_{kj}\ .$$

As these are 2-forms, it is also valuable to write out their components explicitly, which we write as

$$\underset{\sim}{\Omega}_{jk} = \frac{1}{2} R_{jklm}\, \underset{\sim}{\omega}^l \wedge \underset{\sim}{\omega}^m \ .$$

As there are only three independent values for the pairs (jk) and/or (lm), the quantities R_{jklm} may be displayed as a 3×3 matrix. Please give such a presentation.
Also please confirm Eq. (3.191) from Carroll's text as applied to this example, noting that $R \equiv g^{jl} g^{km} R_{jklm}$, i.e., it's the full-trace of the curvature tensor.
As a motivation for the problem, it is worth noting that this is the metric for "constant-time" slices of the standard expanding-universe in the case where the universe is closed, i.e., has at each moment the symmetry appropriate to a 3-sphere. [It is also a local metric for the manifold for the group SU(3).]

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a. The orthonormal frame is given by

$$\underset{\sim}{\omega}^{\hat{\psi}} = d\psi \ , \quad \underset{\sim}{\omega}^{\hat{\theta}} = \sin\psi\, d\theta \ , \quad \underset{\sim}{\omega}^{\hat{\varphi}} = \sin\psi\, \sin\theta\, d\varphi \ .$$

We use the first Structure equations, $d\underset{\sim}{\omega}^i = \underset{\sim}{\omega}^j \wedge \underset{\sim}{\Gamma}^i{}_j$, to determine the connection 1-forms:

$$\begin{aligned}
d\underset{\sim}{\omega}^{\hat{\psi}} &= 0 = \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\Gamma}_{\hat{\psi}\hat{\theta}} + \underset{\sim}{\omega}^{\hat{\varphi}} \wedge \underset{\sim}{\Gamma}_{\hat{\psi}\hat{\varphi}} \quad \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{\psi}\hat{\theta}} \propto \underset{\sim}{\omega}^{\hat{\theta}} \ , \quad \underset{\sim}{\Gamma}_{\hat{\psi}\hat{\varphi}} \propto \underset{\sim}{\omega}^{\hat{\varphi}} \ , \\
d\underset{\sim}{\omega}^{\hat{\theta}} &= \cot\psi\, \underset{\sim}{\omega}^{\hat{\psi}} \wedge \underset{\sim}{\omega}^{\hat{\theta}} \quad \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{\psi}\hat{\theta}} = -\cot\psi\, \underset{\sim}{\omega}^{\hat{\theta}} = -\underset{\sim}{\Gamma}_{\hat{\theta}\hat{\psi}} \ , \quad \underset{\sim}{\Gamma}_{\hat{\theta}\hat{\varphi}} \propto \underset{\sim}{\omega}^{\hat{\varphi}} \ , \\
d\underset{\sim}{\omega}^{\hat{\varphi}} &= \cot\psi\, \underset{\sim}{\omega}^{\hat{\psi}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} + \cot\theta \csc\psi\, \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} \\
&\qquad \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{\psi}\hat{\varphi}} = -\cot\psi\, \underset{\sim}{\omega}^{\hat{\varphi}} = -\underset{\sim}{\Gamma}_{\hat{\varphi}\hat{\psi}} \ , \quad \underset{\sim}{\Gamma}_{\hat{\theta}\hat{\varphi}} = -\csc\psi \cot\theta\, \underset{\sim}{\omega}^{\hat{\varphi}} = -\underset{\sim}{\Gamma}_{\hat{\varphi}\hat{\theta}} \ ,
\end{aligned}$$

where again a useful summary may now be presented, remembering that the various 1-forms are skew-symmetric in the indices shown:

$$\underset{\sim}{\Gamma}_{\hat{\psi}\hat{\theta}} = -\cot\psi\, \underset{\sim}{\omega}^{\hat{\theta}} \ , \quad \underset{\sim}{\Gamma}_{\hat{\psi}\hat{\varphi}} = -\cot\psi\, \underset{\sim}{\omega}^{\hat{\varphi}} \ , \quad \underset{\sim}{\Gamma}_{\hat{\theta}\hat{\varphi}} = -\csc\psi \cot\theta\, \underset{\sim}{\omega}^{\hat{\varphi}} \ .$$

b. We determine the 3 independent curvature 2-forms using the second Structure equations, and slightly more algebra than previously:

$$\begin{aligned}
\underset{\sim}{\Omega}^{\hat{\psi}}{}_{\hat{\theta}} &= d\underset{\sim}{\Gamma}^{\hat{\psi}}{}_{\hat{\theta}} + \underset{\sim}{\Gamma}^{\hat{\psi}}{}_{\hat{\varphi}} \wedge \underset{\sim}{\Gamma}^{\hat{\varphi}}{}_{\hat{\theta}} = d(-\cos\psi\, d\theta) + (-\cot\psi\, \underset{\sim}{\omega}^{\hat{\varphi}}) \wedge (\csc\psi \cot\theta\, \underset{\sim}{\omega}^{\hat{\varphi}}) \\
&= \sin\psi\, d\psi \wedge d\theta = \underset{\sim}{\omega}^{\hat{\psi}} \wedge \underset{\sim}{\omega}^{\hat{\theta}} \ , \\
\underset{\sim}{\Omega}^{\hat{\psi}}{}_{\hat{\varphi}} &= d\underset{\sim}{\Gamma}^{\hat{\psi}}{}_{\hat{\varphi}} + \underset{\sim}{\Gamma}^{\hat{\psi}}{}_{\hat{\theta}} \wedge \underset{\sim}{\Gamma}^{\hat{\theta}}{}_{\hat{\varphi}} = d(-\cos\psi\, \sin\theta\, d\varphi) + (-\cot\psi\, \underset{\sim}{\omega}^{\hat{\theta}}) \wedge (-\csc\psi\, \cot\theta\, \underset{\sim}{\omega}^{\hat{\varphi}}) \\
&= \sin\psi\, \sin\theta\, d\psi \wedge d\varphi - \cos\psi\, \cos\theta\, d\theta \wedge d\varphi + \cot\psi\, \csc\psi\, \cot\theta\, \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} \\
&= \underset{\sim}{\omega}^{\hat{\psi}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} \ , \\
\underset{\sim}{\Omega}^{\hat{\theta}}{}_{\hat{\varphi}} &= d\underset{\sim}{\Gamma}^{\hat{\theta}}{}_{\hat{\varphi}} + \underset{\sim}{\Gamma}^{\hat{\theta}}{}_{\hat{\psi}} \wedge \underset{\sim}{\Gamma}^{\hat{\psi}}{}_{\hat{\varphi}} = d(-\cos\theta\, d\varphi) + (\cot\psi\, \underset{\sim}{\omega}^{\hat{\theta}}) \wedge (-\cot\psi\, \underset{\sim}{\omega}^{\hat{\varphi}}) \\
&= \sin\theta\, d\theta \wedge d\varphi - \cot^2\psi\, \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} = (\csc^2\psi - \cot^2\psi) \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} \\
&= \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}} \ .
\end{aligned}$$

That all three of these curvature forms have "the same structure" tells us that all 3 of the angles involved are being treated equally, as is appropriate for the surface of a sphere, of whatever number of dimensions.

Since there are only 3 independent pairs of directions in a 3-dimensional manifold, we may present the components of the curvature tensor, R_{abcd}, as the entries in a 3×3 matrix, where we label the rows, i.e., the pair of indices a, b, and also the columns, i.e., the pair of indices c, d, in the order $\{(\hat{\psi},\hat{\theta}),(\hat{\psi},\hat{\varphi}),(\hat{\theta},\hat{\varphi})\}$:

$$R_{abcd} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

which is not unexpected for a sphere **of radius 1**, in however-many dimensions.

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3. Continuing in the vein where one does interesting problems that do not require too much algebra, consider the metric referred to as the Poincare hyperboloid, a 2-dimensional surface which we take to have coordinates ξ and η, and a metric given by

$$\mathbf{g} = \left(\frac{a}{\eta}\right)^2 (d\xi^2 + d\eta^2) \equiv (\underset{\sim}{\omega}^{\xi})^2 + (\underset{\sim}{\omega}^{\eta})^2 ,$$

where of course the last equality defines a non-holonomic, orthonormal basis for this manifold. Please again determine the metric-compatible, torsion-free connection and its associated curvature. As this is only a 2-dimensional manifold, the curvature should be determined by one, non-zero quantity, $R_{\xi\eta\xi\eta}$. Show that this quantity is $-1/a^2$.

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This is a rather brief calculation, beginning with the basis 1-forms:

$$\underset{\sim}{\omega}^{\xi} \equiv a\frac{d\xi}{\eta} , \quad \underset{\sim}{\omega}^{\eta} \equiv a\frac{d\eta}{\eta} ,$$

$$\Longrightarrow \begin{cases} d\underset{\sim}{\omega}^{\xi} = a\dfrac{d\xi\wedge d\eta}{\eta^2} = \underset{\sim}{\omega}^{\eta} \wedge \underset{\sim}{\Gamma}_{\xi\eta} , \\ d\underset{\sim}{\omega}^{\eta} = 0 = \underset{\sim}{\omega}^{\xi} \wedge \underset{\sim}{\Gamma}_{\eta\xi} , \end{cases}$$

This allows to quickly infer the following:

$$\underset{\sim}{\Gamma}_{\xi\eta} = -\frac{d\xi}{\eta} = -\frac{1}{a}\underset{\sim}{\omega}^{\xi} ,$$

$$\underset{\sim}{\Omega}_{\xi\eta} = d\underset{\sim}{\Gamma}_{\xi\eta} = \frac{1}{\eta^2}\, d\eta\wedge d\xi = -\frac{1}{a^2}\underset{\sim}{\omega}^{\xi}\wedge\underset{\sim}{\omega}^{\eta} ,$$

$$\Longrightarrow \qquad R_{\xi\eta\xi\eta} = -\frac{1}{a^2} .$$

Here the "curvature" is negative, as appropriate for a hyperboloid. This is a simple 2-dimensional reduction of the more interesting such cosmological space, in 3 spatial dimensions.

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4. Please return to the Brinkman metric, from the last problem set. Using the non-holonomic basis set from that problem, please determine the affine connection 1-forms and the curvature 2-forms.

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We first just recall the holonomic basis set, and its associated metric:

$$\mathbf{g} \equiv ds^2 = 2\,da\,db + a^2\,h(u)\,du^2 - du\,dv\ ;$$
$$\underset{\sim}{\omega}^a \equiv da\ ,\quad \underset{\sim}{\omega}^b \equiv db\ ,\quad \underset{\sim}{\omega}^u \equiv du\ ,\quad \underset{\sim}{\omega}^v \equiv \tfrac{1}{2}(a^2\,h\,du - dv)\ ,\quad \mathbf{g} \equiv g_{\alpha\beta}\,\underset{\sim}{\omega}^\alpha \otimes \underset{\sim}{\omega}^\beta\ ,$$
$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\ .$$

We then proceed to use the First Structure Equations to calculate the connections:

$$\begin{aligned}
0 = d\underset{\sim}{\omega}^a &= \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\Gamma}^a{}_\mu = \underset{\sim}{\omega}^a \wedge \underset{\sim}{\Gamma}_{ba} + \underset{\sim}{\omega}^u \wedge \underset{\sim}{\Gamma}_{bu} + \underset{\sim}{\omega}^v \wedge \underset{\sim}{\Gamma}_{bv} \\
&\Longrightarrow \quad \underset{\sim}{\Gamma}_{ba} \propto \underset{\sim}{\omega}^a\ ,\quad \underset{\sim}{\Gamma}_{bu} \propto \underset{\sim}{\omega}^u\ ,\quad \underset{\sim}{\Gamma}_{bv} \propto \underset{\sim}{\omega}^v\ ,\\
0 = d\underset{\sim}{\omega}^b &= \quad \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\Gamma}^b{}_\mu = \underset{\sim}{\omega}^b \wedge \underset{\sim}{\Gamma}_{ab} + \underset{\sim}{\omega}^u \wedge \underset{\sim}{\Gamma}_{au} + \underset{\sim}{\omega}^v \wedge \underset{\sim}{\Gamma}_{av} \\
&\Longrightarrow \quad \underset{\sim}{\Gamma}_{ab} \propto \underset{\sim}{\omega}^b\ ,\quad \underset{\sim}{\Gamma}_{au} \propto \underset{\sim}{\omega}^u\ ,\quad \underset{\sim}{\Gamma}_{av} \propto \underset{\sim}{\omega}^v\ ,\\
0 = d\underset{\sim}{\omega}^u &= \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\Gamma}^u{}_\mu = \underset{\sim}{\omega}^a \wedge \underset{\sim}{\Gamma}_{va} + \underset{\sim}{\omega}^b \wedge \underset{\sim}{\Gamma}_{vb} + \underset{\sim}{\omega}^u \wedge \underset{\sim}{\Gamma}_{vu} \\
&\Longrightarrow \quad \underset{\sim}{\Gamma}_{va} \propto \underset{\sim}{\omega}^a\ ,\quad \underset{\sim}{\Gamma}_{vb} \propto \underset{\sim}{\omega}^b\ ,\quad \underset{\sim}{\Gamma}_{vu} \propto \underset{\sim}{\omega}^u\ ,\\
ah\,\underset{\sim}{\omega}^a \wedge \underset{\sim}{\omega}^u = d\underset{\sim}{\omega}^v &= \underset{\sim}{\omega}^a \wedge \underset{\sim}{\Gamma}_{ua} + \underset{\sim}{\omega}^b \wedge \underset{\sim}{\Gamma}_{ub} + \underset{\sim}{\omega}^v \wedge \underset{\sim}{\Gamma}_{uv}\ ,\\
&\Longrightarrow \quad \underset{\sim}{\Gamma}_{ua} = ah\,\underset{\sim}{\omega}^u\ ,\quad \underset{\sim}{\Gamma}_{ub} \propto \underset{\sim}{\omega}^b\ ,\quad \underset{\sim}{\Gamma}_{uv} \propto \underset{\sim}{\omega}^v\ .
\end{aligned}$$

The various required proportionalities above indicate that, modulo the standard skew-symmetry, there is only one non-zero connection 1-form:

$$\underset{\sim}{\Gamma}_{ua} = ah\,\underset{\sim}{\omega}^u = -\underset{\sim}{\Gamma}_{au}\ .$$

From that we may very easily calculate the only non-zero curvature 2-form:

$$\underset{\sim}{\Omega}_{ua} = d\underset{\sim}{\Gamma}_{ua} = h\,\underset{\sim}{\omega}^a \wedge \underset{\sim}{\omega}^u = -\underset{\sim}{\Omega}_{au}\ ,$$

noting of course that, in the definition of the curvature forms, the quadratic products of connections that occur there must all vanish with only one non-zero connection form.
This says there is only one non-zero Riemann curvature tensor component, again modulo standard symmetries:

$$R_{uaua} = -h(u) = R_{auau} = -R_{uaau} = -R_{auua}\ .$$

A metric with such properties is classified as Petrov Type N, appropriate for gravitational waves, as we will see later on.

The associated Ricci tensor is of course zero—i.e., this is a vacuum metric—as we may calculate. We first note that of course the indices in the Ricci tensor may only be u and/or

$$\mathcal{R}_{ua} = g^{\lambda\eta} R_{\lambda u \eta a} = g^{au} R_{auua} = 0 \ ,$$
$$\mathcal{R}_{uu} = g^{\lambda\eta} R_{\lambda u \eta u} = g^{aa} R_{auau} = 0 \ .$$
