Physics 570

Homework #6

Due Thursday, 8 March, 2007

Solutions

1. Using the Second Bianchi Identities, verify that the Einstein tensor is divergenceless:

$$\nabla_{\underset{\sim}{e}_\mu} G^{\mu\nu} = 0 \; .$$

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We begin with the Cartan form of the second Bianchi identities, and quickly resolve it into a constraint on the covariant derivatives of the components, remembering that the simplest way to write out the action of the exterior derivative on the components of some tensor relative to a non-holonomic, orthonormal basis is to use covariant derivatives:

$$0 = D\Omega^\mu{}_\nu \equiv d\Omega^\mu{}_\nu + \underset{\sim}{\Gamma}^\mu{}_\eta \wedge \Omega^\eta{}_\nu - \underset{\sim}{\Gamma}^\eta{}_\nu \wedge \Omega^\mu{}_\eta = \tfrac{1}{2} R^\mu{}_{\nu\rho\sigma;\tau}\, \underset{\approx}{\omega}^\rho \wedge \underset{\approx}{\omega}^\sigma \wedge \underset{\approx}{\omega}^\tau$$

$$\Longleftrightarrow$$

$$R^\mu{}_{\nu\rho\sigma;\tau} + R^\mu{}_{\nu\sigma\tau;\rho} + R^\mu{}_{\nu\tau\rho;\sigma} = 0 \; ,$$

or in an alternative notation

$$\nabla_\tau R^\mu{}_{\nu\rho\sigma} + \nabla_\rho R^\mu{}_{\nu\sigma\tau} + \nabla_\sigma R^\mu{}_{\nu\tau\rho} = 0 \; ,$$

where we have used the skew symmetry of this tensor on the last pair of components to reduce the expected $6 = 3!$ number of components down to just these 3.
We now set $\mu = \rho$ and perform the sum, which gives us

$$\nabla_\tau \boldsymbol{\mathcal{R}}_{\nu\sigma} + \nabla_\mu R^\mu{}_{\nu\sigma\tau} - \nabla_\sigma \boldsymbol{\mathcal{R}}_{\nu\tau} = 0 \; .$$

Next we multiply by $g^{\nu\tau}$ and perform that sum:

$$\nabla_\nu \boldsymbol{\mathcal{R}}^\nu{}_\sigma + \nabla_\mu \boldsymbol{\mathcal{R}}^\mu{}_\sigma - \nabla_\sigma \boldsymbol{\mathcal{R}} = 0 \; ,$$

where of course we have been using the trace formulae for the curvature components:

$$\boldsymbol{\mathcal{R}}_{\nu\sigma} \equiv R^\mu{}_{\nu\mu\sigma} \; , \quad \text{the Ricci tensor;} \qquad \boldsymbol{\mathcal{R}} \equiv \boldsymbol{\mathcal{R}}^\mu{}_\mu \; , \quad \text{the Ricci scalar.}$$

At this point we notice that the first two terms in the last equation are the same, which gives us the first step in the desired result, namely the divergence of the Ricci tensor, which equals half of the gradient of the Ricci scalar:

$$\nabla_\nu \boldsymbol{\mathcal{R}}^\nu{}_\sigma = \tfrac{1}{2} \nabla_\sigma \boldsymbol{\mathcal{R}} \; .$$

We now simply recall the definition of the Einstein tensor, and calculate its divergence:

$$G^\mu{}_\nu \equiv \boldsymbol{\mathcal{R}}^\mu{}_\nu - \tfrac{1}{2}\delta^\mu_\nu \boldsymbol{\mathcal{R}} \quad \Longrightarrow \quad \nabla_\mu G^\mu{}_\nu = \nabla_\mu \boldsymbol{\mathcal{R}}^\mu{}_\nu - \tfrac{1}{2}\nabla_\mu \delta^\mu_\nu \boldsymbol{\mathcal{R}} = \tfrac{1}{2}\nabla_n u \boldsymbol{\mathcal{R}} - \tfrac{1}{2}\nabla_\nu \boldsymbol{\mathcal{R}} = 0 \; ,$$

as was desired.

2. The Schwarzschild (vacuum) metric has 4 Killing vectors, namely ∂_t and those 3 that pertain to a metric with spherical symmetry, namely

$$\begin{aligned}\widetilde{K}_1 &= \partial_\varphi = r\sin\theta\,\widetilde{e}_{\hat\varphi}\\ \widetilde{K}_2 &= \cos\varphi\,\partial_\theta - \cot\theta\,\sin\varphi\,\partial_\varphi = r[\cos\varphi\,\widetilde{e}_{\hat\theta} - \cos\theta\,\sin\varphi\,\widetilde{e}_{\hat\varphi}]\,,\\ \widetilde{K}_3 &= -\sin\varphi\,\partial_\theta - \cot\theta\,\cos\varphi\,\partial_\varphi = -r[\sin\varphi\,\widetilde{e}_{\hat\theta} + \cos\theta\,\cos\varphi\,\widetilde{e}_{\hat\varphi}]\,.\end{aligned}$$

a. Please show that each of these satisfies Killing's equations. You may decide whether it is easier to show that these do indeed satisfy Killing's equations in one of several modes: i.e., in coordinate basis or orthonormal basis, and with ordinary derivatives or covariant derivatives.

b. Please also determine their commutator algebra; i.e., determine the 3 independent commutators they possess, and show that they, themselves, span the set of these 3 commutators.

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a. I will look at the three purported Killing vectors separately, using throughout a coordinate basis and ordinary derivatives, so that the required constraint is the following:

$$\underset{\widetilde{K}}{\mathcal{L}}\, g_{\alpha\beta} = \widetilde{K}(g_{\alpha\beta}) + K^\delta{}_{,\alpha} g_{\delta\beta} + K^\delta{}_{,\beta} g_{\alpha\delta}\,.$$

I should perhaps point out that my experience has told me that this approach is usually the one requiring least computation for explicit computation of Killing vectors, while one of the others is more valuable for calculations involved more theoretical questions.

i.) For $\widetilde{K}_1 \equiv \partial_\varphi$, we have simply $K_1^\alpha = \delta^\alpha_\varphi$. Then those terms involving partial derivatives of the components of the Killing vector are all zero since the components are constants; secondly those terms involving derivatives of the metric components are also zero since none of them depend on φ. Therefore the satisfaction of Killing's equations for this vector is very simple.

ii.) For $\widetilde{K}_2$, we divide the situation into two parts: those involving derivatives of the metric components, labeled $X_{\alpha\beta}$, and those involving derivatives of the Killing components, labeled $W_{\alpha\beta}$, both quantities of course being symmetric in their indices.
Therefore, first we consider

$$X_{\alpha\beta} = \widetilde{K}_2(g_{\alpha\beta}) = \cos\varphi\,\partial_\theta\,(g_{\alpha\beta}) = \delta^\varphi_\alpha\delta^\varphi_\beta \cos\varphi\partial_\theta(r^2\sin^2\theta) = 2r^2\sin\theta\cos\theta\cos\varphi\,\delta^\varphi_\alpha\delta^\varphi_\beta\,.$$

Then we must compute the other portion, which will have non-trivial components only when the indices α and β are chosen from θ and φ:

$$\begin{aligned}W_{\theta\theta} &= 2\left(\partial_\theta K_2^\lambda\right) g_{\lambda\theta} = 2\left(\partial_\theta K_2^\theta\right) g_{\theta\theta} = 2\left(\partial_\theta\cos\varphi\right) r^2 = 0\,,\\ W_{\theta\varphi} &= \left(\partial_\theta K_2^\lambda\right) g_{\lambda\varphi} + \left(\partial_\varphi K_2^\lambda\right) g_{\theta\lambda} = -\left(\partial_\theta \cot\theta\sin\varphi\right) r^2\sin^2\theta + \left(\partial_\varphi\cos\varphi\right) r^2\\ &\qquad = \csc^2\theta\sin\varphi\, r^2\sin^2\theta - \sin\varphi\, r^2 = 0\,,\\ W_{\varphi\varphi} &= 2\left(\partial_\varphi K_2^\lambda\right) g_{\lambda\varphi} = -2\left(\partial_\varphi\cot\theta\sin\varphi\right) r^2\sin^2\theta = -2r^2\cos\varphi\sin\theta\cos\theta\,.\end{aligned}$$

We see that all the desired constraints have been met with the possible exception of those with indices φ, φ. However, we must then add together $X_{\varphi\varphi} + W_{\varphi\varphi}$, which do exactly cancel, so that all the Killing equations are indeed satisfied for this vector field.

iii.) Now we must repeat the same sort of calculation for $\widetilde{K}_3$. As it is reasonably similar to the previous one we again see that the indices that could create non-zero entries must be either θ or φ, and the separation above into the two parts of the calculation is still valid, so that we obtain the following:

$$X_{\alpha\beta} = \widetilde{K}_3(g_{\alpha\beta}) = -\sin\varphi\,\partial_\theta(g_{\alpha\beta}) = -2r^2\sin\theta\cos\theta\sin\varphi\,\delta^\varphi_\alpha\delta^\varphi_\beta\ .$$

And then for the other set of terms:

$$\begin{aligned}
W_{\theta\theta} &= 2\left(\partial_\theta K_3^\lambda\right)g_{\lambda\theta} = 2\left(\partial_\theta K_3^\theta\right)g_{\theta\theta} = -2\left(\partial_\theta\sin\varphi\right)r^2 = 0\ ,\\
W_{\theta\varphi} &= \left(\partial_\theta K_3^\lambda\right)g_{\lambda\varphi} + \left(\partial_\varphi K_3^\lambda\right)g_{\theta\lambda} = -\left(\partial_\theta\cot\theta\cos\varphi\right)r^2\sin^2\theta - \left(\partial_\varphi\sin\varphi\right)r^2\\
&\qquad\qquad = \csc^2\theta\cos\varphi\,r^2\sin^2\theta - \cos\varphi\,r^2 = 0\ ,\\
W_{\varphi\varphi} &= 2\left(\partial_\varphi K_2^\lambda\right)g_{\lambda\varphi} = -2\left(\partial_\varphi\cot\theta\cos\varphi\right)r^2\sin^2\theta = +2r^2\sin\varphi\,\sin\theta\cos\theta\ .
\end{aligned}$$

As before we see that all quantities vanish immediately, and separately, except for the Lie derivative of $g_{\varphi\varphi}$, and that in that case the two portions are both non-zero, BUT they cancel.

b. There are three independent commutators, which we now calculate:

$$\begin{aligned}
[\widetilde{K}_1\,,\,\widetilde{K}_2] &= \partial_\varphi\left(\cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi\right) = -\sin\varphi\,\partial_\theta - \cot\theta\cos\varphi\,\partial_\varphi = \widetilde{K}_3\ ,\\
[\widetilde{K}_3\,,\,\widetilde{K}_1] &= -\partial_\varphi\left(-\sin\varphi\,\partial_\theta - \cot\theta\cos\varphi\,\partial_\varphi\right) = \cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi = \widetilde{K}_2\ ,\\
[\widetilde{K}_2\,,\,\widetilde{K}_3] &= \left[\cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi\,,\,-\sin\varphi\,\partial_\theta - \cot\theta\cos\varphi\,\partial_\varphi\right]\\
&= \cos\varphi\left[\csc^2\theta\cos\varphi\,\partial_\varphi\right] - \cot\theta\sin\varphi\,\left[-\cos\varphi\,\partial_\theta + \cot\theta\sin\varphi\,\partial_\varphi\right]\\
&\qquad + \sin\varphi\,\left[\csc^2\theta\sin\varphi\,\partial_\varphi\right] + \cot\theta\cos\varphi\,\left[-\sin\varphi\,\partial_\theta - \cot\theta\cos\varphi\,\partial_\varphi\right]\\
&= \left\{\csc^2\theta\cos^2\varphi - \cot^2\theta\sin^2\varphi + \csc^2\theta\sin^2\varphi - \cot^2\theta\cos^2\varphi\right\}\partial_\varphi = \partial_\varphi = \widetilde{K}_1\ .
\end{aligned}$$

This is in fact, using cyclic order in the indices, the correct commutation relations for the generators of the rotation group, $\boldsymbol{SO}(3,\mathbb{R})$, as we wanted.

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3. Let us try to choose new coordinates that make the Schwarzschild metric look somewhat more like the usual sort of isotropic coordinates by considering the following pair of coordinate transformations.

a. First define a new coordinate p such that

$$r = p(1 + m/2p)^2\ .$$

Next, perform the coordinate transformation on the metric to rewrite it in terms of the coordinates (p, θ, φ, t). Then define isotropic coordinates

$$x \equiv p \sin\theta \cos\varphi \ , \quad y \equiv p \sin\theta \sin\varphi \ , \quad z \equiv p \cos\theta \ ,$$

and then transform the metric a second time, showing that in these coordinates it has the form

$$\mathbf{g} = (1 + m/2p)^4 (dx^2 + dy^2 + dz^2) - [(1 - m/2p)/(1 + m/2p)]^2 dt^2 \ , \qquad p^2 \equiv x^2 + y^2 + z^2 \ .$$

b. In the usual Schwarzschild coordinates the angular coordinates behave as expected, while so do the radial and timelike coordinates, i.e., r varies from 0 to $+\infty$, while t varies from $-\infty$ to $+\infty$. What are the corresponding ranges for the coordinates $\{p, \theta, \varphi, t\}$ and how do they map to and from the more usual Schwarzschild coordinate ranges? ....

We begin by simply transforming the original Schwarzschild metric, for which we need the following two calculations:

$$dr = \left[1 - \left(\frac{M}{2p}\right)^2\right] dp \ , \quad 1 - \frac{2M}{r} = \left(\frac{1 - \frac{M}{2p}}{1 + \frac{M}{2p}}\right)^2 \ .$$

These give us

$$\begin{aligned} \mathbf{g} &= \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 \, d\Omega^2 - \left(1 - \frac{2M}{r}\right) dt^2 \\ &= \left(\frac{1 + \frac{M}{2p}}{1 - \frac{M}{2p}}\right)^2 \left[1 - \left(\frac{M}{2p}\right)^2\right]^2 dp^2 + p^2 \left(1 + \frac{M}{2p}\right)^4 d\Omega^2 - \left(\frac{1 - \frac{M}{2p}}{1 + \frac{M}{2p}}\right)^2 dt^2 \\ &= \left(1 + \frac{M}{2p}\right)^4 \left\{dp^2 + p^2 \, d\Omega^2\right\} - \left(\frac{1 - \frac{M}{2p}}{1 + \frac{M}{2p}}\right)^2 dt^2 \ . \end{aligned}$$

This is the form of the metric in variables $\{p, \theta, \varphi, t\}$. However, as the spatial portion is proportional to just $dp^2 + p^2 d\theta^2 + p^2 \sin^2\theta \, d\varphi^2$, we know that the (usual) transformation given, back to isotropic spherical coordinates, replaces that portion by $dx^2 + dy^2 + dz^2$, which gives us this last form of the metric:

$$\mathbf{g} = \left(1 + \frac{M}{2p}\right)^4 \left\{dx^2 + dy^2 + dz^2\right\} - \left(\frac{1 - \frac{M}{2p}}{1 + \frac{M}{2p}}\right)^2 dt^2 \ .$$

We note that there is a singularity in this metric, but of course only in the dt^2 portion, and it occurs at $p = M/2$.

b. To answer the range question, there is surely no change in the ranges for the angular coordinates, as nothing was done to them. As well, there is no change in the range for the time coordinate; again nothing was done to it, and the metric is static, i.e., depends on time in no way. However, for the change from r to p, we may first note that insertion of the singular value for p, i.e., setting $p = M/2$, causes r to have the value $2M$, so that the singularities are indeed mapped to each other!

However, there is a more serious difference. If we graph r as a function of p we find that it has a minimum at $p = p_0 = M/2$, and increases monotonically on either side. In more detail, this means that as p increases from p_0 toward infinity, r also increases, and does so in a linear way fairly soon; however, as p decreases, through positive values, from p_0, r again increases from $r = 2M$ up toward infinity again. The conclusion is that ONLY that portion of the Schwarzschild manifold for values of $r \geq 2M$ is covered by these coordinates. That portion of the Schwarzschild manifold that has smaller values of r does NOT have such an isotropic set of coordinates. [I should note that this is not all that surprising, I hope, since g_{rr} is negative for smaller values of r, so that ∂_r is no longer spacelike, but, rather, timelike.

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4. Using the timelike geodesic equations in the usual Schwarzschild coordinates, determine the minimum radii that a stable elliptical orbit may have. Give a good definition of stable in this context.

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Recall that the effective potential energy for orbital motion in the plane, in the Schwarzschild metric, is the following, where I have put $\mu = 1$ as is appropriate for a massive object:

$$V(r) = -\mu\frac{m}{r} + \frac{B^2}{2r^2} - \frac{mB^2}{r^3} = -\frac{1}{x} + \frac{b}{2x^2} - \frac{b}{x^3}\,, \qquad x \equiv \frac{r}{m}\,,\; b \equiv \left(\frac{B}{m}\right)^2\,,$$

where B is the constant that determines the angular motion via the conservation equation

$$r\,u^{\hat\varphi} = r^2\,\frac{d\varphi}{d\tau} = B\,.$$

a. An elliptic orbit is one where the radial motion oscillates back and forth around some minimum of the potential, coming to a stop—in the radial direction—at both ends of that oscillation. In order for this to happen there must be such a minimum for it to oscillate around. Therefore we first differentiate the potential and ask for its roots, to determine a minimum. These are determined by

$$x_\pm = \tfrac{1}{2}b[1 \pm \sqrt{1 - 12/b}] \quad\Longrightarrow\quad V(x_\pm) = \frac{1}{2x_\pm}\left(\frac{4 - x_\pm}{x_\pm - 3}\right)\,; \quad b \geq 12\,.$$

For $b = 12$ these two points coincide at $x_\pm = 6$, while as b increases from this value x_+ increases as well, more or less linearly with b, it being the location of the minimum of V, which is always negative for all of these values, although decreasing as well as x_+ increases. On the other hand, x_- is the location of a maximum for the potential, and this location decreases, fairly slowly, to $x_- = 3$ as b approaches infinity. The value of the potential at this maximum is of course negative at $b = 12$, where it coincides with the minimum, but increases slowly as b increases, so that it becomes zero at $b = 16$, where we have $x_- = 4$, and then onward to positive infinity as b increases to infinity and x_- decreases to 3. Since the value of the potential at infinity is zero, a radial oscillation must correspond to some value of the "non-relativistic energy," $\frac{1}{2}(A^2 - 1)$, which is negative. Therefore its near turn-around point, i.e., its perihelion, will occur when that negative value is again taken on by the potential, somewhere smaller than the location of the minimum, i.e., smaller than $x_+ \geq 6$. The hunt for this location divides into two different sorts. In the event that the potential rises to non-negative values, as it approaches its maximum, then the lower bound of possible perihelion points will be the location of the larger value of the two roots of the potential itself. These two roots are given by

$$x_{0\pm} = \tfrac{1}{4}b[1 \pm \sqrt{1 - 16/b}] \ , \quad b \geq 16 \ ,$$

where it is the larger one that we want. Obviously the smallest possible value of that root is when $b = 16$ (or $B = 4m$), which puts $r_+ = 4m$. This must therefore be the lower limit on possible values for the perihelion point of an elliptic orbit that can arbitrarily large, but negative values of its energy. An ellipse with the right energy—very near to zero, but still negative—for this small a perihelion would have a very large aphelion, almost to infinity!

On the other hand, when the maximum value of the potential is also negative, then there are of course also allowed elliptic orbits, where in this case the lower bound on values for the perihelion point would be the location of that maximum. This occurs when $12 < b < 16$, in which case x_- varies upward from 4 (at $b = 16$) all the way to 6 when $b = 12$. Therefore this class of elliptic orbits, all required to have energy less than $V(x_-)$, does not give us any yet smaller values for the perihelion point.

To discuss the smallest possible value for a maximum turn-around point, i.e., an aphelion, we first note that the aphelion point must clearly be larger than the point at which the minimum is, around which the oscillation is occurring. However, the smallest value of x for which there is a minimum has already been noted as $x = 6$, or $r = 6m$. Therefore, this is surely the lower bound on such values.

b. The discussion of stability is a little bit trickier; there are two different routes for an elliptical orbit to become unstable:

i.) it could escape to infinity and never return, or

ii.) it could fall into the center, where of course there is a singularity so that it would never return from there either.

Since the value of the potential at infinity is zero, the only elliptic orbits unstable against small perturbations that might send it off to infinity are those with "energy" negative but very near to zero, so that this small perturbation sends it into a parabolic trajectory, with energy exactly zero, that never returns. However, these are really only marginally unstable, since their aphelion was already very, very large, and the change in the orbit is slight, although certainly large in the sense that it will never return instead of returning many times, but after very long waits.

However, there are those elliptical orbits for $12 \geq b < 16$ where the value of the potential at the maximum is negative, and they are at, or even very slightly below, that negative maximum value. They have aphelia that are not so very far away: only at the point where the potential will again have that value as r heads off to infinity. However, if given only very slightly more energy, they will then have no perihelia, and will move inward to $r = 0$, never to return. Therefore, by the usual (standard) definition of instability, we say that those orbits are unstable that have exactly the energy at that maximum value, and therefore the location x_- as their perihelia. When they have just infinitesimally more energy they will bypass that location on their way inward, and will accelerate rapidly inward to $r = 0$. Such elliptical orbits are allowed, for the correct value of A, only when b is between 12 and 16.