Physics 570

Homework #7

Due Thursday, 22 March, 2007

NOTE: Problems 2, 3, and 4 will require some sort of computer calculations.

1. Begin with the usual, flat Minkowski spacetime and transform to a new set of coordinates, $\{p, q, \theta, \varphi\}$, via the following equations:

$$t + r \equiv \tan p \, , \quad t - r \equiv \tan q \, .$$

Show that the metric of Minkowski spacetime may now be written in the (conformal) form:

$$\mathbf{g} = \phi^2 \left\{ -4\, dp\, dq + \sin^2(p-q)[d\theta^2 + \sin^2\theta\, d\varphi^2] \right\} \, .$$

The to-be-determined function, ϕ^2 is a function of p and q which is never negative. What is it? In order to cover all of Minkowski spacetime, what are the ranges of the new coordinates p and q. Create a two-dimensional graph showing these ranges, but supressing the angular coordinates. On this graph show a few lines of constant t, a few lines of constant r, and a few light rays starting at the origin.

2. A (test) particle with non-zero mass falls radially toward the horizon of a Schwarzschild black hole of mass m. The geodesic it follows has $A = 0.95$.

a. At what value of r is at rest? At that value of r, what is its physical distance from $r = 2m$?

b. How much proper time is required for it to fall from $r = 3m$ to $r = 2m$? How much coordinate time is required?

c. How much proper time is required for it to fall from the horizon at $r = 2m$ to the singularity at the center at $r = 0$? In order to compute this quantity show that a solution to the differential equation determining $r = r(\tau)$ may be given via the following equations:

$$r = \tfrac{1}{2}R(1 + \cos\eta) \, , \quad \tau = \tfrac{1}{2}R\sqrt{\frac{R}{2m}}(\eta + \sin\eta) \, ,$$

for some particular constant R. What is the meaning of R, and what is its value in terms of the "energy constant," A?

d. What is its ordinary 3-velocity as a function of its radial coordinate r?

e. As it passes the radius $r = 2.01m$, it emits a radio signal back toward observers who have safely decided to wait at the earth. What is the redshift of the signal when it is received at the earth? Do not forget to include the Doppler shift because of its earth-measured speed at the time of emission.

3. The perihelion distance of Mercury may be taken as 46.00 million kilometers, while the aphelion distance is 69.82 million kilometers. Please use this data to determine accurate values for the integration constants, A, and B [or $b = (B/m)^2$]. Then perform a numerical integration of the equation of motion to obtain an accurate value for the perihelion shift, using the values of $H(r)$ and $J(r)$ that are given to us by Einstein's equations. You will need the mass of the sun in kilometers, which you should look up in one of your texts, or calculate yourself.

4. Consider the orbit of a (massive) test particle in the Schwarzschild metric with a value of $b \equiv (B/m)^2 = 14$, and the following two separate cases: Please show that if $A = 0.96$ then the orbit is an elliptic one, while in the case $A = 0.97$ the orbit does not have a perihelion but, rather, eventually falls below the horizon and into the singularity.

5. Please begin with the usual non-holonomic basis set for a metric which is spherically symmetric and static, but this time let us not assume the vacuum solution, but instead let us suppose that there is an electric field in the region exterior to the source, caused by an electric charge, q, at the location of that source. The solution to Einstein's equations for the matter tensor corresponding to such an electric field is given by the following metric:

$$\mathbf{g} = J\,dr^2 + r^2\,d\Omega^2 - H\,dt^2\ , \qquad J^{-1} = H = 1 - \frac{2m}{r} + \frac{q^2}{r^2}\ .$$

a. Please choose the standard orthonormal basis for such metrics, and then use general form of the connections and curvature, and Einstein tensor, given in the handout on such metrics to write out the independent components of the Einstein tensor, at least, explicitly.

b. Now we want to know what is the electric field caused by the central charge, q. To do this, it seems reasonable that, in an orthonormal basis, the field of a central charge would be of the form $\underset{\sim}{F} = E(r)\,\underset{\sim}{\omega}^{\hat{r}} \wedge \underset{\sim}{\omega}^{\hat{t}}$. Therefore assume that this is the case, and solve Maxwell's equations, on this manifold, to determine the explicit form for $E(r)$, in terms, presumably, of r and q. Do make sure that the so-constructed $E(r)$ satisfies ALL of Maxwell's equations, where I mean those equations to say that $d\underset{\sim}{F} = 0 = d * \underset{\sim}{F}$. Lastly, what is the value of the 1-form $\underset{\sim}{A}$ that is the 4-vector potential for this field?