Physics 570

Homework #8

Due Thursday, 29 March, 2007

Solutions

1. An observer is moving on a circular orbit around the central mass in the Schwarzschild manifold; i.e., her location is on a geodesic with a constant value of r and $\theta = \pi/2$.

a. First write out the entire set of geodesic equations, but surely already including the constants of the motion, A and B. Since r is a constant, we must have $du^{\hat{r}}/d\tau = 0$. Therefore use that equation and the one that says that the geodesic has $(\widetilde{u})^2 = -1$ to determine the values of A and B in terms of the constant radius r.

b. Now determine the locally-measured 3-velocity which this observer has, using the usual Schwarzschild coordinates, and then the Lorentz transformation that would go between the stationary, local observer's frame and that of the observer on the circular orbit. As well, describe a locally co-moving basis for our circularly-moving observer's tangent space, i.e., a frame so that she may make proper measurements of vectorial quantities. Denote her basis for tangent vectors by $\{\widetilde{f}_\alpha\}_1^4$, require as usual that they constitute an orthonormal frame and that $\widetilde{f}_4 = \widetilde{u}$, her 4-velocity. Demonstrate some reasonable, simple choice for her basis in terms of the original, orthonormal Schwarzschild frame vectors.

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Referring to the handout and insisting that the orbit be in the equatorial plane, we know that the worldline for such an observer would have 4-velocity

$$\widetilde{u} = \frac{B}{r}\,\widetilde{e}_{\hat{\varphi}} + \frac{A}{\sqrt{H}}\,\widetilde{e}_{\hat{t}} \ .$$

To determine B and A in terms of the (**constant**) radius, r of her orbit, we note two requirements that will provide the two equations needed to allow this: first, that the tangent vector should have (squared) length -1, and, secondly, that $du^{\hat{r}}/d\tau = 0$; this second requirement of course means we have to again refer to the handout to obtain the general form for that equation. These give us

$$(\widetilde{u})^2 = \left(\frac{B}{r}\right)^2 - \left(\frac{A}{\sqrt{H}}\right)^2 = -1 \ ,$$

$$0 = \frac{1}{\sqrt{H}}\frac{d}{d\tau}\left(\frac{dr/d\tau}{\sqrt{H}}\right) = \frac{B^2}{r^3} - \frac{1}{2}A^2\frac{dH/dr}{H^2} \quad \Longrightarrow \quad \left(\frac{A}{B}\right)^2 - \frac{2}{r^3}\frac{H^2}{H'} = 0 \ .$$

The solution of these two equations says that

$$B = \pm\sqrt{\frac{mr}{1-3m/r}}\ , \quad A = \frac{H}{\sqrt{1-3m/r}} = \frac{1-2m/r}{\sqrt{1-3m/r}}\ ,$$

$$\Longrightarrow \quad \beta \equiv B/r = \frac{\sqrt{m/r}}{\sqrt{1-3m/r}}\ , \qquad \gamma \equiv A/\sqrt{H} = \frac{\sqrt{1-2m/r}}{\sqrt{1-3m/r}}\ ,$$

$$\widetilde{u} \equiv \beta\,\widetilde{e}_{\hat{\varphi}} + \gamma\,\widetilde{e}_{\hat{t}}\ , \quad \beta^2 - \gamma^2 = -1 \ .$$

It is worth noting that these equations can all be integrated easily, since r is a constant; they give us

$$
\begin{aligned}
u^{\hat\varphi} = r\frac{d\varphi}{d\tau} = B/r \equiv \beta = \frac{\sqrt{m/r}}{\sqrt{1-3m/r}} \quad &\Longrightarrow \quad \varphi = \frac{B}{r^2}\tau = \frac{\sqrt{m/r^3}}{\sqrt{1-3m/r}}\tau \ , \\
u^{\hat t} = \sqrt{H}\frac{dt}{d\tau} = A/\sqrt{H} \equiv \gamma = \frac{\sqrt{H}}{\sqrt{1-3m/r}} \quad &\Longrightarrow \quad t = \frac{\tau}{\sqrt{1-3m/r}} \ , \\
\Longrightarrow \quad \frac{d\varphi}{dt} = \sqrt{m/r^3} \equiv \omega \quad &\Longleftrightarrow \quad mT^2 = 2\pi r^3 \ ,
\end{aligned}
$$

where, in both integrations we have chosen the constant of integration to make the formulae simpler; i.e., we have chosen $\tau_0 = 0$ and $\varphi(\tau_0) = 0$ as well as $t(\tau_0) = 0$. As well we see that the same "Kepler" law relating the square of the period and the cube of the radius applies here as well as in Newtonian physics.

The notation γ for the time component of $\widetilde{u}$ has been used intentionally of course, since this is the γ-factor for this 4-velocity, which, then, has ordinary 3-velocity $\vec{v} = (\beta/\gamma)\,\widetilde{e}_{\hat\varphi}$. Therefore, in order to move from "the frame at infinity," i.e., the standard orthonormal frame which came with the coordinates, to our observer's frame, we simply need to perform the appropriate Lorentz transformation between orthonormal frames, corresponding to this velocity. Since we are using $\{r, \theta, \varphi, t\}$ as our coordinates, and as labels for our basis forms, the matrix representing such a Lorentz transformation should also be labelled that way, so that

$$
\Lambda^\alpha{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta \\ 0 & 0 & -\beta & \gamma \end{pmatrix} \quad \Longrightarrow \quad u'^\alpha \equiv \Lambda^\alpha{}_\mu u^\mu = (\Lambda^\alpha{}_\mu) \begin{pmatrix} 0 \\ 0 \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .
$$

We now want to have a locally co-moving basis for our observer's tangent space. We begin by choosing $\widetilde{f}_4 \equiv \widetilde{u}$. Then the remainder must form an "orthonormal frame," i.e., a frame such that $\mathbf{g}(\widetilde{f}_\alpha, \widetilde{f}_\beta) = \eta_{\alpha\beta}$. We of course want all of these vectors parallel transported along $\widetilde{u}$; i.e., we need

$$
\nabla_{\widetilde{u}} \widetilde{f}_\alpha = 0 \ , \qquad \alpha = 1, 2, 3, 4 \ .
$$

Of course, that this is true for $\widetilde{f}_4 \equiv \widetilde{u}$ is apparent since it is a geodesic. On the other hand, for the others this means that we need to solve the transport equations. Let us recall, however, that IF we solve the parallel transport equations, then any scalar products that are arranged at a single point along the orbit will be satisfied always. Therefore, in particular—having already determined $\widetilde{f}_4$—we may now insist that the other 3 must be perpendicular to $\widetilde{u}$. Letting a take on the values from 1 to 3, this tells us that

$$
0 = \widetilde{f}_a \cdot \widetilde{u} = f_a^{\hat\varphi}\frac{B}{r} - f_a^{\hat t}\frac{A}{\sqrt{H}} \quad \Longrightarrow \quad f_a^{\hat t} = \frac{B}{A}\frac{\sqrt{H}}{r} f_a^{\hat\varphi} \ .
$$

Then we may simply copy down the explicit form of the parallel transport equations, using the fact that our geodesic has non-zero components only in the $\widetilde{e}_{\hat{\varphi}}$ and $\widetilde{e}_{\hat{t}}$ directions:

$$\begin{aligned}
0 = \omega^{\hat{r}}\left(\nabla_{\widetilde{u}}\widetilde{f}_a\right) &= \widetilde{u}(f_a^{\hat{r}}) - B\frac{\sqrt{H}}{r^2}f_a^{\hat{\varphi}} + A\frac{m/r^2}{H}f_a^{\hat{t}}\ ,\\
0 = \omega^{\hat{\theta}}\left(\nabla_{\widetilde{u}}\widetilde{f}_a\right) &= \widetilde{u}(f_a^{\hat{\theta}})\\
0 = \omega^{\hat{\varphi}}\left(\nabla_{\widetilde{u}}\widetilde{f}_a\right) &= \widetilde{u}(f_a^{\hat{\varphi}}) + B\frac{\sqrt{H}}{r^2}f_a^{\hat{r}}\\
0 = \omega^{\hat{t}}\left(\nabla_{\widetilde{u}}\widetilde{f}_a\right) &= \widetilde{u}(f_a^{\hat{t}}) + A\frac{m/r^2}{H}f_a^{\hat{r}}
\end{aligned}$$

We first notice that the $\omega^{\hat{\theta}}$ equation is easily integrated, simply saying that $f_a^{\hat{\theta}}$ is constant with respect to τ. A simple choice is to choose it either 0 or 1; we take it as 1 for one of our triad members, along with the other components 0, while we choose it as 0 for the others, which will ensure their perpendicularity:

$$f_3^{\hat{\theta}} = 0 = f_1^{\hat{\theta}}\ , \qquad f_2^{\hat{\theta}} = 1 \text{ and } \qquad \widetilde{f}_2 = \widetilde{e}_{\hat{\theta}}\ .$$

Having now a correct form for $a = 2$, we still need to resolve these equations for the two remaining triad members, $a = 1$ and $a = 3$. Therefore, we insert the equation above that gives $f_a^{\hat{t}}$ in terms of $f_a^{\hat{\varphi}}$, and write out the result:

$$\begin{aligned}
\frac{d}{d\tau}f_a^{\hat{r}} = \frac{\omega}{\gamma}f_a^{\hat{\varphi}}\ , &\quad \frac{d}{d\tau}f_a^{\hat{\varphi}} = -\omega\gamma\, f_a^{\hat{r}}\\
\Longrightarrow &\qquad \frac{d^2}{d\tau^2}f_a^{\hat{\varphi}} = -\omega^2 f_a^{\hat{\varphi}}\ .
\end{aligned}$$

The last (second-order) equation is easily integrated, since $\omega^2 = m/r^3$ is a constant, giving us

$$f_a^{\hat{\varphi}} = C_a\cos(\omega\tau + \delta_a)\ .$$

Inserting this back into the other two equations, in order, we acquire

$$f_a^{\hat{r}} = \frac{C_a}{\gamma}\sin(\omega\tau + \delta_a)\ , \quad f_a^{\hat{t}} = \frac{\beta}{\gamma}C_a\cos(\omega\tau + \delta_a)\ .$$

Having all the components for these two, we may now calculate their scalar product:

$$\begin{aligned}
\eta_{ab} = \widetilde{f}_a\cdot\widetilde{f}_b = \frac{1}{\gamma^2}C_aC_b\Big\{&\gamma^2\cos(\omega\tau+\delta_a)\cos(\omega\tau+\delta_b) + \sin(\omega\tau+\delta_a)\sin(\omega\tau+\delta_b)\\
&- \beta^2\cos(\omega\tau+\delta_a)\cos(\omega\tau+\delta_b)\Big\}\\
= \frac{C_aC_b}{\gamma^2}&\left\{\cos(\omega\tau+\delta_a)\cos(\omega\tau+\delta_b) + \sin(\omega\tau+\delta_a)\sin(\omega\tau+\delta_b)\right\}\\
= \frac{C_aC_b}{\gamma^2}&\cos(\delta_b - \delta_a)\ ,
\end{aligned}$$

A simple approach to ensure their orthonormality is to choose $\delta_3 = 0$, $\delta_1 = +\pi/2$, and $C_3 = C_1 = \gamma$, which gives us finally our desired tetrad for our orbiting observer:

$$\begin{aligned}
\widetilde{f}_3 &= (\gamma\,\widetilde{e}_{\hat\varphi} + \beta\,\widetilde{e}_{\hat t})\cos(\omega\tau) + \sin(\omega\tau)\,\widetilde{e}_{\hat r}\ , \quad \widetilde{f}_4 = \beta\,\widetilde{e}_{\hat\varphi} + \gamma\,\widetilde{e}_{\hat t}\ ,\\
\widetilde{f}_1 &= -\,(\gamma\,\widetilde{e}_{\hat\varphi} + \beta\,\widetilde{e}_{\hat t})\sin(\omega\tau) + \cos(\omega\tau)\,\widetilde{e}_{\hat r}\ , \quad \widetilde{f}_2 = \widetilde{e}_{\hat\theta}\ ,
\end{aligned}$$

$$\beta = \sqrt{\frac{m/r}{1-3m/r}}\ , \quad \gamma = \sqrt{\frac{1-2m/r}{1-3m/r}}\ , \quad \gamma^2 - \beta^2 = +1\ , \qquad \omega^2 = m/r^3\ ,$$

where the choice of positive sign for δ_1 has arranged it so that $\widetilde{f}_1$ starts out, at $\tau = 0$, pointing in the purely radial direction, and $\widetilde{f}_3(\tau = 0)$ is the most obvious choice for a direction normal to the 4-velocity. Notice of course that her orthonormal triad of spatial basis vectors is rotating all the time, as a function of her proper time. This is only reasonable since she is on a rotating orbit.

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2. The constant-time, equatorial plane of the Schwarzschild metric is a 2-dimensional, curved surface. Please determine an embedding of it in flat, 3-dimensional space. That is, take the metric in the 3-dimensional space as $dz^2 + dr^2 + r^2 d\varphi^2$ and determine $z = z(r, \varphi)$ so that the so-determined surface in that 3-space has the same metric as does that plane in the Schwarzschild manifold.

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On the constant-time, equatorial plane of the Schwarzschild manifold, the 2-dimensional metric is simply

$$\mathbf{g}_2 = \frac{dr^2}{1-2m/r} + r^2 d\varphi^2\ .$$

To embed this is a flat, 3-dimensional space we need to have a surface in that space, where $z = z(r)$. We choose for z not to also depend on the angle, φ, since the entire system should have that angle as a generator for a Killing vector, i.e., there should be such a symmetry.
Therefore, we may now suppose that on that surface, in the 3-dimensional space, we have the projection

$$\mathbf{g}_3 = dz^2 + dr^2 + r^2 d\varphi^2 \underset{z=z(r)}{\longrightarrow} \left[\left(\frac{dz}{dr}\right)^2 + 1\right] dr^2 + r^2 d\varphi^2 = \mathbf{g}_2\ ,$$

which simply gives us a straightforward differential equation to solve for z, namely

$$\begin{aligned}
\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1-2m/r} \qquad &\Longrightarrow \quad z = \int dr\,\sqrt{\frac{2m/r}{1-2m/r}}\\
&\Longrightarrow \quad z = \sqrt{8m(r-2m)}
\end{aligned}$$

or, perhaps easier to visualize as the paraboloid $r = 2m + z^2/8m$.

Remembering to rotate this last form around the z-axis, through all values of the angle φ, shows us a paraboloid that has a throat of width $2m$ at $z = 0$, and two large open expanses, more or less locally flat, for large positive, and large negative, values of z. As we understand from having already looked at the manifold in Kruskal coordinates, no values of r less than $2m$ are present on this spacelike, constant time surface.

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3. Let's begin preparations for considerations of rotating, compact sources of gravitational potential. A proper study of them will require that we utilize oblate spheroidal coordinates, since this sort of oblateness is what rotation causes for an originally spherically-symmetric fluid body. Of course that also means we have to study rotation a bit.

a. Therefore, we will first look at oblate spheroidal coordinates in flat space. We begin with the usual form of the Minkowski-space metric in cylindrical coordinates,

$$\mathbf{g} = d\rho^2 + \rho^2\, d\varphi^2 + dz^2 - dt^2 \ ,$$

and transform to new (oblate, spheroidal) coordinates $\{\sigma, \theta, \phi\}$ via

$$\rho = \sqrt{\sigma^2 + a^2}\, \sin\theta \ , \quad z \equiv \sigma\, \cos\theta \ , \quad \phi = \varphi \ ,$$

where $a > 0$ is a positive constant. In these new coordinates, please show that the metric has the form

$$\mathbf{g} = (\sigma^2 + a^2\cos^2\theta)\left\{\frac{d\sigma^2}{\sigma^2 + a^2} + d\theta^2\right\} + (\sigma^2 + a^2)\sin^2\theta\, d\phi^2 - dt^2 \ .$$

Then choose an orthonormal basis for this metric, in these coordinates. Although I am sure you could indeed calculate the connection 1-forms and the curvature 2-forms for this metric yourself, INSTEAD, please use the software package GRTensorII to determine the connection 1-forms in these coordinates, and also the curvature 2-forms. [These last quantities should of course all be zero!] This will be a very good time to begin practicing using this software, or other software for the same purpose, if you desire. This is because the upcoming discussions of the Kerr metric (rotating star) will probably be beyond your hand-written ability, or desire, to calculate the connections, so we need to begin practicing with something simpler. [Please append to your homework a printout of your session with GRTensorII, where this is done, or the other program if you choose to do it that way.]

b. Give a description, in cylindrical coordinates, of the surfaces of constant σ, including some representative graphs. IF you don't want to make 3-dimensional graphs, because of the axial symmetry of cylindrical coordinates, it is sufficient to draw them in the plane which constitutes a particular value of ϕ and also that value plus π, so that one has a particular plane section. On such a section, the surface in question is reduced to simply a curve; a useful parameter along that curve is the other variable, θ.

c. Likewise, now look at points of constant θ, taking σ as the parameter along the planar sections of that surface, to give nice descriptions of what they "look like." Compare these sections to the similar sections one would have if we were dealing with ordinary spherical coordinates, instead of these "squashed" ones.

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a. We begin with the usual form of the Minkowski-space metric in cylindrical coordinates,

$$\mathbf{g} = d\rho^2 + \rho^2\, d\varphi^2 + dz^2 - dt^2 \ ,$$

and transform to new (oblate, spheroidal) coordinates $\{\sigma, \theta, \phi\}$ via

$$\rho = \sqrt{\sigma^2 + a^2}\ \sin\theta \ , \quad z \equiv \sigma\ \cos\theta \ , \quad \phi = \varphi \ ,$$

where $a > 0$ is a positive constant, and then determine the associated metric in these coordinates. We use the "chain rule" on these new coordinates:

$$\begin{aligned} d\rho = \frac{\partial\rho}{\partial\sigma}d\sigma + \frac{\partial\rho}{\partial\theta}d\theta + \frac{\partial\rho}{\partial\phi}d\phi &= \frac{\sigma}{\sqrt{\sigma^2+a^2}}\sin\theta\, d\sigma + \sqrt{\sigma^2+a^2}\ \cos\theta\, d\theta \ , \\ dz &= \cos\theta\, d\sigma - \sigma\sin\theta\, d\theta \ , \\ d\phi &= d\varphi \ . \end{aligned}$$

Therefore we find that

$$\begin{aligned} \mathbf{g} &= \left\{ \frac{\sigma^2\sin^2\theta}{\sigma^2+a^2} + \cos^2\theta \right\} d\sigma^2 + (\sigma^2 + a^2\cos^2\theta)\, d\theta^2 + (\sigma^2+a^2)\sin^2\theta\, d\phi^2 \\ &= (\sigma^2 + a^2\cos^2\theta)\left\{ \frac{d\sigma^2}{\sigma^2+a^2} + d\theta^2 \right\} + (\sigma^2+a^2)\sin^2\theta\, d\phi^2 - dt^2 \ . \end{aligned}$$

A reasonable orthonormal basis for this metric is given by

$$\begin{gathered} \underset{\approx}{\omega}^{\hat\sigma} \equiv \sqrt{\frac{\Sigma}{\Delta}}\, d\sigma \ , \quad \underset{\approx}{\omega}^{\hat\theta} \equiv \sqrt{\Sigma}\, d\theta \ , \quad \underset{\approx}{\omega}^{\hat\varphi} \equiv \sqrt{\Delta}\ \sin\theta\, d\phi \ , \quad \underset{\approx}{\omega}^{\hat t} \equiv dt \ , \\ \Delta \equiv \sigma^2 + a^2 \ , \quad \Sigma \equiv \sigma^2 + a^2\cos^2\theta \ . \end{gathered}$$

The software package GRTensor was used to determine the connection 1-forms in these coordinates, and also the curvature 2-forms. An html-version of the Maple spreadsheet is available from a nearby weblink, with that session with GRTensor. (ALSO there is a link there where you may download the actual Maple worksheet—by right-clicking on the link and using the save file command.) It told me that the connection 1-forms were

$$\begin{gathered} \underset{\approx}{\Gamma}^{\hat\sigma}{}_{\hat\theta} = -\frac{1}{\Sigma^{3/2}}\left\{ \tfrac{1}{2}a^2\sin(2\theta)\,\underset{\approx}{\omega}^{\hat\sigma} + \sqrt{\Delta}\ \sigma\,\underset{\approx}{\omega}^{\hat\theta} \right\} \ , \\ \underset{\approx}{\Gamma}^{\hat\sigma}{}_{\hat\varphi} = -\frac{\sigma}{\sqrt{\Sigma\Delta}}\,\underset{\approx}{\omega}^{\hat\varphi} \ , \qquad \underset{\approx}{\Gamma}^{\hat\theta}{}_{\hat\varphi} = -\frac{\cot\theta}{\sqrt{\Sigma}}\,\underset{\approx}{\omega}^{\hat\varphi} \ . \end{gathered}$$

and that all the components of the Riemann curvature tensor were zero (as expected).

b. If we now pick a fixed value of σ, we may consider the original equations, which give $\rho = \rho(\sigma, \theta)$ and $z = z(\sigma, \theta)$, as parameterized equations for a graph, in the ρ, z-plane, where θ functions as the parameter along the *curves of constant* σ. [This ρ, z-plane is of course a constant ϕ slice through three-dimensional space, so that it is simply a copy, say, of half of the y, z-plane, rotated by some angle. Because it is only half a plane, it is more visual to actually look at that plane adjoined by the similar plane that has been rotated by π so that we get an entire 2-dimensional plane to look at. Having done that, we find that for most values of σ these curves are ellipses, with foci at $\pm a$ and semi-major axis, $a = \sqrt{\sigma^2 + a^2}$ and semi-minor axis $b = \sigma$. As the parameter, θ, varies from 0 to 2π, the entire (closed) curve is traced out:

$$\frac{\rho^2}{\sigma^2 + a^2} + \frac{z^2}{\sigma^2} = 1 \ .$$

As one then rotates this curve through all possible angles ϕ, following along with the cylindrical symmetry of the problem, these ellipses are rotated into closed ellipsoids. On the other hand, the ellipses all have foci at $\pm a$, so that when $\sigma = 0$, the ellipse degenerates to just the straight line—along the ρ axis—between $+a$ and $-a$. The 2- and 3-dimensional figures are included at the end of the attached Maple session, that is in html-format.

c. If, on the other hand, we now look at points of constant θ, taking σ as the parameter along the curves, we acquire a hyperbola—a single branch of it for $\rho > 0$—of the following form:

$$\frac{\rho^2}{a^2 \sin^2 \theta} - \frac{z^2}{a^2 \cos^2 \theta} = 1 \ .$$

Again as one rotates through all allowed value of ϕ, these become hyperboloids of one sheet. The typical such hyperbola intersects the ρ-axis at $a \sin \theta$ and is symmetric above and below. On the other hand, for $\theta = \pi/2$ one gets just the straight line along the ρ-axis that begins at $\rho = a$ and goes off to $\rho = +\infty$, while for $\theta = 0$, we get the entire z-axis. Again, the 2- and 3-dimensional figures are included at the end of the attached Maple session, that is in html-format.

Do notice that these surfaces are distinctly different from those that are more familiar to us, from spherical coordinates. For the case of spherical coordinates the surfaces of constant r are simply spheres, instead of the squashed (or oblate) ellipsoids that σ gives us. As well, in the spherical case the surfaces of constant θ are simply cones pointing up and down, respectively, that are joined at the origin, while here these cones "open up," instead of being joined at the origin, and become our hyperboloids of revolution, of one sheet.

It is worth comparing this, although it was not requested, to the Kerr metric itself. If we rewrite the complete form of the Kerr metric, but for $m = 0$, so that it is really just a description of a flat spacetime, with an associated scaling constant, a, then it takes the form

$$ds^2 = [r^2 + a^2 \cos^2 \theta] \left\{ \frac{dr^2}{r^2 + a^2} + d\theta^2 \right\} + (r^2 + a^2) \sin^2 \theta \, d\varphi^2 - dt^2 \ .$$

We easily see that the flat-space version of the Kerr metric is the same as the metric we have been looking at via the oblate spheroidal coordinates, identify the Kerr symbol r with the earlier symbol σ. This of course helps us begin an interpretation of the meaning of the symbols in the Kerr metric, with this oblate sort of shape, as described in some detail above.

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4. Let us re-consider the spherically-symmetric, static metric one last time, but consider different input to Einstein's equations, involving no matter, i.e., vacuum, but there is a non-zero cosmological constant, Λ. Using the values for the Einstein tensor, in terms of $J(r)$ and $H(r)$, re-solve the Einstein equations when $\Lambda \neq 0$, but we still maintain the desire that there is a central mass, of constant value M. Are there still horizons in this new manifold, in these coordinates? If so, where are they?

. . . .

We first go to the handout for spherically-symmetric spacetimes, and find the values of the Einstein tensor in terms of J and H, and set $G_{\mu\nu} = \Lambda\, g_{\mu\nu}$:

$$
\begin{aligned}
G_{\hat{r}\hat{r}} &= \frac{H'}{rJH} - \frac{1-1/J}{r^2} = \Lambda \ , \\
G_{\hat{t}\hat{t}} &= \frac{J'}{rJ^2} + \frac{1-1/J}{r^2} = -\Lambda \ , \\
G_{\hat{\theta}\hat{\theta}} &= \frac{1}{2\sqrt{JH}}\left(\frac{H'}{\sqrt{JH}}\right)' + \frac{J'/J - H'/H}{2rJ} = \Lambda \ .
\end{aligned}
$$

Adding together the first two of these equations, we obtain $H'/H + J'/J = 0$, which tells us that the product JH must be constant. Using the usual normalizations, we set that constant equal to 1. Returning to the equation generated by $G_{\hat{r}\hat{r}}$, this allows us to write it in the form

$$
\begin{aligned}
\frac{H'}{r} - \frac{1-H}{r^2} = \Lambda \implies &\ [r(H-1)]' = \Lambda\, r^2 \\
\implies &\quad H = 1 + c_1/r + \tfrac{1}{3}\Lambda r^2 \ ,
\end{aligned}
$$

where c_1 is another constant of integration. Again we choose to model this on the Schwarzschild situation, approximating Newtonian gravity, and normalize c_1 to have the value $-2M$, which gives us

$$
\mathbf{g} = (1/H)\, dr^2 + r^2\, d\Omega^2 - H\, dt^2 \ , \qquad H \equiv 1 - (2M/r) + \tfrac{1}{3}\Lambda\, r^2 \ .
$$

The standard horizons occur when the coefficient of dt^2 vanishes. However, as H is now cubic in r, we should expect to have at least one root, and perhaps more, although in some cases that root may be negative, and therefore not necessarily a reasonable value for r. We first consider the case when Λ is positive, which means the cubic has no extrema, so that there is only one real root. As well, H is positive for sufficiently large values of r, while it is negative for sufficiently small values, so that this one real root is indeed always positive, so that it does create a horizon.

Of course for a zero value of Λ the root is at $2m$; as Λ becomes more and more positive, that single root becomes smaller and smaller, approaching 0. This situation is then rather similar to the case when there was no cosmological constant: positive values for Λ retain the same situation with respect to horizons, simply moving it inward somewhat.

On the other hand, for negative values of that constant, we find that there are real extrema only for $\Lambda > -(4/27)/(2M)^2$. In this case there are two real, positive roots of the cubic, corresponding to two horizons. For very small, negative values of Λ, these two real roots are at a value of r just larger than $2m$ and another one which is very, very large. As the value of Λ decreases, i.e., becomes more negative, these two roots approach one another, becoming equal, at $r = 3m$ for $\Lambda = -(4/27)(2M)^2$. As already noted, for values even more negative than this, there are no real roots. Therefore, a summary here is that one has, for negative values of Λ, either two or no horizons. In the case with no horizons, there is still a singularity at $r = 0$, which is now approachable, although such large (negative) values are rather unlikely physically. For smaller, negative values there are now two horizons, one which is quite similar to the one with no such constant, at values slightly larger than $2m$, while the other one is a "cosmological horizon," which exists at very large values of r, with "all of us on Earth," say, in between them.
