Comments on the Jacobi Elliptic Functions in the Complex Plane

Knowing that the function $u \equiv \mathrm{sn}(z;k)$, for $0 < k^2 < 1$, satisfies the differential equation

$$(u')^2 = (1-u^2)(1-k^2u^2) \equiv f(u)$$

allows us to constructively think about places where it might be real-valued, or, perhaps, purely imaginary, as opposed to the more general notion of being complex-valued. Plotting $f(u)$, we note that it has 4 zeroes, at ± 1 and $\pm 1/k$, and of course goes to large, positive values as $|u|$ increases beyond the largest, and smallest, zero.

For $-1 \le z \le +1$, the value of $(u')^2 \ge 0$, as befits something we intend to think of as, at least proportional to, a kinetic energy. On the other hand, when $|z|$ lies between 1 and $1/k$, $f(u) < 0$, and so cannot be interpreted as a kinetic energy. Lastly, when $|z|$ is larger than $1/k$, again $f(u) > 0$ and it can again be considered as a "physical region" for a kinetic-energy interpretation. Therefore, we should anticipate that the behavior of u, vis-a-vis the elliptic sine function, should be rather different in the three different "physical regions."

To understand this better, we should first note that the general solution of the differential equation given above is

$$u = \mathrm{sn}(z - z_0; k) \ .$$

The "constant of integration," z_0, helps define these different regions. This is analogous to the ode for ordinary simple harmonic motion, $(w')^2 = 1-w^2$, with general solution $w = \sin(z-z_0)$. The important difference in the elliptic case is that z_0 may well not be real.

The elliptic sine function is doubly periodic in the complex z plane. (In fact it and its derivative may be taken as an algebraic basis for the most general doubly-periodic, meromorphic functions in the plane. A meromorphic function is one that is single-valued, but may have isolated poles.) In general the ratio of those two periods must be complex, or they would not arrange a "tiling" of the complex plane; however, in many important cases we can arrange it so that one of them is real and the other one pure imaginary. We will only consider those cases here; it is customary to refer to the real quarter period by the symbol K, and the imaginary half period by the symbol iK'. In this case then all the values that the elliptic sine would have would be taken on in a socalled period rectangle. We may think of some sort of "most generic" period rectangle as the one centered at the origin, in the plane, having real width $4K$ and (imaginary) height (2iK'), and the entire plane as covered with identical such rectangles. Within this rectangle the elliptic sine has (simple) poles at iK' and $2K + iK'$.

We intend now to draw several straight (horizontal or vertical) lines across a period rectangle, and describe the values of $u = \mathrm{sn}(z;k)$ as z varies across each line. We will also do the same for $\mathrm{cn}(z;k)$ and $\mathrm{dn}(z;k)$:

$$\mathrm{cn}(z;k)^2 + \mathrm{sn}(z;k)^2 = 1 = \mathrm{dn}(z;k)^2 + k^2\,\mathrm{sn}(z;k)^2\ .$$

For the other Jacobi elliptic functions, we just (also) recall that the periods of $\mathrm{cn}(z;k)$ may be chosen to be $4K(k)$ and $2K(k)+2iK'(k)$, while the periods of $\mathrm{dn}(z;k)$ are $2K(k)$ and $4iK'(k)$. Recall the definitions of the periods:

$$\begin{gathered}\text{use } 0 \le k \le 1\,,\ \text{ and set }\ k' \equiv \sqrt{1-k^2}\,,\\ K(k) = K'(k') \equiv \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} \ge \pi/2\ ,\\ K'(k) \equiv K(k') = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-{k'}^2t^2)}} \ge \pi/2\ .\end{gathered}$$

More general comments related to the integrals above can compare these functions to the standard form of *elliptic integrals*, which are the incomplete, or indefinite, forms of the above definite integrals, which determine quarter periods. We may write the following relations between them:

$$\begin{gathered}u = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\phi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \mathrm{sn}^{-1}(\sin\phi;k) \equiv F(\phi;k)\ ,\\ \text{and also notice that }\ \phi = \sin^{-1}[\mathrm{sn}(u;k)] \equiv \mathrm{am}(u)\ .\end{gathered}$$

The second elliptic integral is

$$E(\phi;k) \equiv \int_0^{\phi} d\alpha\,\sqrt{1-k^2\sin^2\alpha} = \int_0^{\sin\phi} dt\,\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$
$$= \int_0^{\mathrm{sn}^{-1}(\sin\phi;k)} dw\ \mathrm{dn}^2\,w = k'^2 + k\int_0^{\mathrm{sn}^{-1}(\sin\phi;k)} dw\ \mathrm{cn}^2\,w\ .$$

There is also an elliptic integral of the third kind, $\Pi(\phi;k,n)$, which depends on an additional parameter n, and which is more complicated so that we will not actually describe it here. Other little facts worth remembering concerning these functions are

a. **odd- and even-ness**

$$\mathrm{sn}(-z;k) = -\,\mathrm{sn}(z;k);\quad \mathrm{cn}(-z;k) = +\,\mathrm{cn}(z;k);\quad \mathrm{dn}(-z;k) = +\,\mathrm{dn}(z;k);$$

b. **values different by real half-periods**, $2K(k)$:

$$\mathrm{sn}(z+2K;k) = -\,\mathrm{sn}(z;k);\quad \mathrm{cn}(z+2K;k) = -\,\mathrm{cn}(z;k);\quad \mathrm{dn}(z+2K;k) = +\,\mathrm{dn}(z;k).$$

A. Following along the real axis, $x = z \in (-2K, +2K]$, with values given at each of the following points, $-2K$ to $-K$ to 0 to $+K$ to $2K$:

a. $\mathrm{sn}(x;k)$ is odd in x, and varies from 0 to -1 to 0 to $+1$ to 0;

b. $\mathrm{cn}(x;k)$ is even in x, and varies from -1 to 0 to $+1$ to 0 to -1;

c. $\mathrm{dn}(x;k)$ is even in x, and varies from $+1$ to $+k'$ to $+1$ to $+k'$ to $+1$.

B. Following along a line parallel to the real axis, but with imaginary part $= +iK'$, i.e., for $z = x + iK' \in (-2K + iK', +2K + iK']$. Recall that $z = \pm iK'(k)$ is a pole for all three functions; the residues are $\pm 1/k$, $\mp i/k$ and $\mp i$, for $\mathrm{sn}(z;k)$, $\mathrm{cn}(z;k)$, and $\mathrm{dn}(z;k)$, respectively:

a. $\mathrm{sn}(x+iK';k) = 1/[k\ \mathrm{sn}(x;k)]$ is odd in x and varies from $-\infty$ to $-1/k$ to $-\infty$, has a discontinuity at 0, coming back in at $+\infty$, to $+1/k$ to $+\infty$;

b. $\mathrm{cn}(x+iK';k) = \mathrm{dn}(x;k)/[ik\ \mathrm{sn}(x;k)]$ **is pure imaginary**; its imaginary part is odd in x and varies from $+\infty$ to k'/k to $+\infty$, has a discontinuity at 0, comes back at $-\infty$ to $-k'/k$ to $-\infty$;

c. $\mathrm{dn}(x+iK';k) = -i\,\mathrm{cn}(x;k)/\,\mathrm{sn}(x;k)$ **is pure imaginary**; its imaginary part is odd in x and varies from $-\infty$ to 0 to $+\infty$, has a discontinuity at 0, comes back at $-\infty$ to 0 to $+\infty$.

C. Following along the imaginary axis, we write $z = iy \in (-iK', +iK')$ and consider the points $-iK'$, to 0, to $+iK'$. Recall that $2K + iK'$ is a pole for all three; the residues are $-1/k$, $+i/(k)$ and $+i$ for $\mathrm{sn}(z;k)$, $\mathrm{cn}(z;k)$, and $\mathrm{dn}(z;k)$, respectively:

a. $\mathrm{sn}(iy;k) = i\,\mathrm{sn}(y;k')/\,\mathrm{cn}(y;k')$ **is pure imaginary** ; its imaginary part is odd in y and varies from $-\infty$ to 0 to $+\infty$;

b. $\mathrm{cn}(iy;k) = 1/cn(y;k')$ is even in y and varies from $+\infty$ to $+1$ to $+\infty$;

c. $\mathrm{dn}(iy;k) = dn(y;k')/cn(y;k')$ is even in y and varies from $+\infty$ to $+1$ to $+\infty$.

D. Following a line parallel to the imaginary axis, with real part as $K(k)$, we set $z = iy + K(k)$, and again take $y = -K'(k)$, 0, and $+K'(k)$:

a. $\mathrm{sn}(iy+K;k) = 1/dn(y;k')$ is even in y and varies from $1/k'$ to $+1$ to $1/k'$;

b. $\mathrm{cn}(iy+K;k) = -k'\,\mathrm{sn}(y;k')/\,\mathrm{dn}(y;k')$ **is pure imaginary**; its imaginary part is odd in y and varies from $+k'/k$ to 0 to $-k'/k$.

c. $\mathrm{dn}(iy+K;k) = k'\ \mathrm{cn}(y;k')/[\mathrm{dn}(y;k')]$ is even in y and varies from 0 to $+1$ to 0.

A last important statement is that these functions have addition theorems:

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v}\ ,$$
$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v - \mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,u\,\mathrm{dn}\,v}{1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v}\ ,$$
$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v - k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{cn}\,v}{1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v}\ ,$$

for the Weierstrass elliptic function(s)

The function $\wp(z)$ is a meromorphic function of complex z, with periods 2ω and $2\omega'$, the ratio of which is not real. We know that it has a double pole at the origin, and, of course, at every point related to that by the double-period relations.

$$[-\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\,\wp(z) - g_3 = 4[\wp(z)-e_1][\wp(z)-e_2][\wp(z)-e_3] \equiv g[\wp(z)]\ ,$$

where its behavior is determined in large part by its parameters and/or roots, which are related as follows:

$$
\begin{gathered}
e_1 + e_2 + e_3 = 0 \ , \\
2({e_1}^2 + {e_2}^2 + {e_3}^2) = -4(e_1 e_2 + e_2 e_3 + e_1 e_3) = g_2; \\
4 e_1 e_2 e_3 = \tfrac{4}{3}({e_1}^3 + {e_2}^3 + {e_3}^3) = g_3; \\
\Delta \equiv \{4(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)\}^2 = {g_2}^3 - 27\, {g_3}^2 \ .
\end{gathered}
$$

The function $\wp'(z)$ is a functionally-independent elliptic function, with the same periods and a third order pole at the origin.

$$
\begin{gathered}
\wp(\omega_i) = e_i \ , \qquad \wp'(\omega_i) = 0 \ , \\
\wp(z) = z^{-2} + \tfrac{g_2}{20} z^2 + \tfrac{g_3}{28} z^4 + O(z^6) \ .
\end{gathered}
$$

The discriminant, Δ, is very helpful in understanding the further properties, so that one may divide things into two reasonably different cases, valid so long as the parameters, g_2 and g_3, are real. If $g_3 < 0$, then one may use $\wp(z; g_2, g_3) = -\wp(iz; g_2, -g_3)$ to change the (independent) variable and have the new $g_3 > 0$; therefore, we always pre-suppose that we consider only the case where $g_3 > 0$.

$\Delta > 0$. All three roots are real, and ω real, ω' pure imaginary, and $|\omega'| > \omega$.

As z varies along the boundary of the half-period rectangle that lies in the first quadrant, i.e., as it goes from 0 to ω to $\omega_2 \equiv \omega + \omega'$ to ω' and then along the imaginary axis back down to 0, the values of $\wp(z)$ are always real, varying from $+\infty$ to e_1 to $e_2 < e_1$, to $e_3 < e_2$, to $-\infty$.

$\Delta < 0$. Only e_2 is real, and $\omega_2 \equiv \omega' + \omega$ real, ${\omega'}_2 \equiv \omega' - \omega$ pure imaginary, and $|{\omega'}_2| \leq \omega_2$.

As z varies from 0 to ω_2 to ${\omega'}_2 + \omega_2 = 2\omega'$, $\wp(z)$ varies through real values from $+\infty$ to e_2 and onward to $-\infty$.

The most useful immediate source for further properties of these functions is Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chs. 16-18.

For rather more detailed and careful elaborations of the properties, for instance, see E.T. Whittaker, *A Course of Modern Analysis*.