Flows: mostly in 2 variables

1. Notes relative to Steve's Presentation on Flows

We consider the behavior of the solutions to the **autonomous** flow equations, at least mostly in 2 dimensions, of the generic, first-order form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ w \\ \vdots \end{pmatrix} \equiv \frac{d}{dt}\zeta = G(\zeta) \equiv \begin{pmatrix} X(x,y,z,w,\ldots \\ Y(x,y,z,w,\ldots \\ Z(x,y,z,w,\ldots \\ W(x,y,z,w,\ldots \\ \vdots \end{pmatrix} , \tag{1.1}$$

where ζ and G are vector quantities, as shown. In general there will be singular points, i.e., points at which all components of G vanish. Denote the generic such point as ζ_0, and expand G in a Taylor series about that point, assuming that the first derivatives do not vanish there, and setting $\mathbf{z} \equiv \zeta - \zeta_0$, which gives us

$$\frac{d}{dt}\mathbf{z} = A\,\mathbf{z} + \ \ldots \ , \qquad A \equiv \frac{\partial G}{\partial \zeta_0} \ . \tag{1.2}$$

This gives us a linearized form of the equation, so we may investigate the motion in the near neighborhood of the singular point.

> **See the discussion on p. 400-410 of Chapter 7 of José, and also his discussion on p. 415-417 of the plane with coordinates given by the trace, Tr, and the determinant, Δ, of the matrix A, for the 2-dimensional case.** A similar discussion is on p. 196ff of Lichtenberg, with especial concentration on the details for the 2-dimensional case on p. 200-208. See also p. 657-663, Section 16.3.5 of Richards, for quite excellent pictures. Lastly, one may also see Percival and Richards, for nice, slow discussions, also including the 1-dimensional cases.

The behavior of the system in this near neighborhood may be described in terms of the values of the eigenvalues and eigenvectors of the matrix A. The resulting motions may be divided into several types. In the case of 2 variables, they are (as taken directly from Richards, p 663):

1. Eigenvalues real and different, which is the same as requiring that $(Tr)^2 > 4\Delta$:
   - a.) **Stable node**: eigenvalues real and negative,
   - b.) **Unstable node**: eigenvalues real and positive,
   - c.) **Saddle**: eigenvalues real and of different sign.
2. Eigenvalues complex, and therefore complex conjugates; i.e., $(Tr)^2 < 4\Delta$:
   - a.) **Stable spiral**: real part of eigenvalues negative,
   - b.) **Unstable spiral**: real part of eigenvalues positive,
   - c.) **Center**: real part of eigenvalues zero; i.e., they are pure imaginary.
3. Eigenvalues the same, i.e., $(Tr)^2 = 4\Delta$:
   - a.) **Stable star**: A diagonalizable, and Tr negative,
   - b.) **Unstable star**: A diagonalizable, and Tr positive,

c.) **Stable improper node**: A not diagonalizable and Tr negative,

d.) **Unstable improper node**: A not diagonalizable and Tr positive.

A very important theorem is the Linearization Theorem due to Hartman and Grobman, which tells us that

> Every fixed point of a nonlinear system that is **not a center** has some neighborhood for which there is a continuous, invertible mapping between the dynamical motions predicted by the linearization analysis and the true motion.

There are many special cases in which one may say more about the behavior near centers, which are of course a very important case in physical systems. One interesting one is, for instance, the one that says that if $G^x(x, y)$ is even in x while $G^y(x, y)$ is odd in x, then the full, nonlinear problem will have a center wherever the linearized problem has a center.

The van der Pol oscillator was discussed with examples for different values of β:

$$\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0 \; . \tag{1.3}$$

The equation was originally derived, in 1926, for the oscillations of the grid voltage in a vacuum tode with a (nonlinear) triode valve, important for understanding the behavior of early radios. There are discussions on p 392-3 of Ch. 7 of José, for example. It was being used as an example of a system that had a *limit cycle* among its possible behaviors. A limit cycle is an *isolated, periodic orbit*, where by isolated we imply that all nearby orbits either spiral towards it or away from it, as opposed to the periodic orbits in, say, a simple pendulum, which have other similar orbits quite near it, but which never come "toward" it or "away" from it. There is still no clear understanding of just how one determines whether a given system has one or more limit cycles. The Poincaré-Bendixson Theorem, for systems with 2 variables, is a relevant criterion, but often somewhat difficult to apply:

> If $\mathcal{D}$ is a closed, bounded region of phase space, and a motion is such that it eventually remains in that region, i.e., for all times after some time,
> then the motion is either a closed path, approaches a closed path as the time becomes infinite, or it approaches a fixed point.

The usual way to use this is to determine some region $\mathcal{D}$ which has no interior fixed points, often a region which is topologically "annular" and to choose it so that all orbits either enter or leave through only one or the other of the boundaries.

There is also Liénard's Theorem, relevant to the second-order nonlinear equation of the form

$$\ddot{x} + \beta\,\Gamma(x)\,\dot{x} + f(x) = 0 \; , \tag{1.4}$$

which states that there is a unique limit cycle in an annular region surrounding the origin when we have the conditions satisfied that Γ is nonlinear and an odd function of x, while f is an even function, while both are of order C^1, i.e., continuously differentiable.

The version of the Poincaré-Bendixson Theorem given in José, on p. 400, states that a necessary but not necessarily sufficient condition for closed orbits in some finite, **simply connected** region, $\mathcal{D}$, again in simply 2 variables, is that the quantity we could think of as *"the divergence"* of the flow,

$$\nabla \cdot \vec{G} \equiv X_{,x} + Y_{,y} \tag{1.5}$$

must either change sign within $\mathcal{D}$, or be always everywhere zero within. This version was improved upon by Dulac who stated the theorem in a negative way:

If there exists a C^1 function, $\rho = \rho(x, y)$, such that $\nabla \cdot (\rho \vec{v})$ has constant sign in a simply connection region $\mathcal{D}$, then there are no periodic orbits within that region.

This entire structure leads us, however, to the notion of **the Poincaré index** of any closed curve, and, thereby, of fixed points. One begins by noting that at any ordinary point in the phase plane we may define the angle, ϕ between the vector to the point in question and the $\hat{x}$-axis as

$$\tan\phi \equiv Y/X \qquad \Longrightarrow \qquad d\phi = \frac{X\,dY - Y\,dX}{X^2 + Y^2}\ . \tag{1.6a}$$

We may then acquire some closed curve, Γ, in our 2-dimensional space and integrate ϕ all the way around it, and then normalize that relative to 2π:

$$I_\Gamma \equiv \frac{\Delta\phi}{2\pi} = \frac{1}{2\pi}\oint_\Gamma d\phi = \frac{1}{2\pi}\oint_\Gamma \frac{X\,dY - Y\,dX}{X^2 + Y^2}\ , \tag{1.6b}$$

where, because the denominator of the integrand would vanish were the curve Γ to pass through a fixed point, we must choose Γ to ensure that this does not occur. Since the vector in question returns to its same value when it is traced all the way around the curve, it must be that this quantity is an integer, possibly zero of course. It is not particularly difficult to calculate the value of the integral if Γ is a periodic orbit; the value is $+1$. [If one worries about such things, it is straightforward to check carefully that this means that $d\phi$ is an example of a closed 1-form that is **not exact**; i.e., $d(d\phi) = 0$, but there is no globally-defined continuous function ϕ, so that it is not an exact differential. If it were, then the value of this integral would indeed be zero.] Since the integrand is certainly locally continuous, and since the value of the integral is an integer, it is clear that small deformations of the curve will not change its value. More precisely, if two curves can be continuously deformed one into the other, without passing through any fixed points, then the value of the integral will be the same for both. Therefore, we may now define the index of a fixed point to be that appropriate to some closed curve which contains only that point. It is of course obvious that if there are no fixed points within the curve, then the value must be zero, because we could shrink the curve to where it went nowhere. We may therefore suppose that some curve contains some set of N fixed points, and say that

$$I_\Gamma = \sum_{k=1}^{N} I_k\ , \quad I_k \text{ is the index of the } k\text{-th fixed point.} \tag{1.6c}$$

It requires then just some reasonably simple algebra to check out the following values for fixed points:

a.) the index of a periodic orbit is $+1$;

b.) the index of a node, an improper node, a spiral, a star, or a center is also $+1$;

c.) the index of a saddle is -1.

These allow us to make general statements about systems. Those statements are most valuable when the system depends, also, upon one or more parameters, a; i.e., when $G = G(\zeta; a)$. Since the solutions depend continuously on those parameters, it follows, for instance, that as a changes (1) a fixed point may not simply just disappear, and (2) a center may not become a saddle.