Physics 406

Some Notes relative to the Electromagnetic Potentials for a Moving Point Charge

To begin, I simply want to present a clean listing of the various electromagnetic field vectors and scalars associated with a single point charge, q, that is at the location $\vec{r}'(t)$, so that it has a non-zero value for both its velocity, $\vec{v}(t)$, and its acceleration, $\vec{a}(t)$. We suppose that these fields are being measured at some location $\vec{r}$ and at a specific time, t, so that the most important quantity involved is

the vector from the charge, considered as the source of the field, and the field point, where the measurement is being made, is given by the equation

$$\vec{\xi}(t) \equiv \vec{r} - \vec{r}'(t) \ . \tag{0}$$

We then have the following equations for the fields this charge generates, where the details of the notation are given in the text further along:

$$\begin{aligned}
\vec{E}(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\frac{q}{s^3}\left\{\left[1 - v^2/c^2\right]\vec{\xi} + \left[\vec{\xi}\right] \times \left(\vec{\xi} \times [\vec{a}]/c^2\right)\right\} \ , \\
\vec{B}(\vec{r},t) &= \frac{1}{c}[\hat{\xi}] \times \vec{E}(\vec{r},t) \\
&= \frac{\mu_0 q}{4\pi s^3}\left[(1 - v^2/c^2 + \vec{a}\cdot\vec{\xi}/c^2)\vec{v} \times \vec{\xi} + \frac{s}{c}\vec{a} \times \vec{\xi}\right] \ ,
\end{aligned}$$

$$\left.\begin{aligned}
V(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{1}{s} \ , \\
\vec{A}(\vec{r},t) &= \frac{1}{c^2}V(\vec{r},t)\,[\vec{v}] \\
&= \frac{\mu_0 q}{4\pi}\frac{[\vec{v}]}{s} \ ,
\end{aligned}\right\} \quad \Longleftrightarrow \quad \widetilde{A}(\widetilde{r}) = \widetilde{A}(\vec{r}, ct) = \frac{\mu_0 q}{4\pi}\frac{1}{[\gamma_v]s}\,[\widetilde{\eta}] \ ,$$

$$\vec{\xi}(t) = \left[\vec{\xi} - \xi\,\vec{v}/c\right] \ , \qquad s = s(t) \equiv [\xi - \vec{\xi}\cdot\vec{v}/c] = [\hat{\xi}]\cdot\vec{\xi} \ , \qquad \widetilde{\eta} = \begin{pmatrix}\gamma_v\vec{v} \\ \gamma_v c\end{pmatrix} = \gamma_v\begin{pmatrix}\vec{v} \\ c\end{pmatrix} \ .$$

Next I will point out the correspondence between the notation that I have been using in class and the notation used in Griffiths' text, relative to the Liénard-Wichert potentials, i.e., the vector and scalar potentials for a single, point charge, in arbitrary motion. We also need to compare, in the same vein, the forms given for the electric and magnetic fields. Later I will point out some of the differences in their derivations.

On p. 432-3, Griffiths gives forms for the scalar and vector electromagnetic potentials, which I copy here, except for the fact that, as you know, since I have no symbol in my typefont which creates the symbol he uses, often referred to as "squiggly r", I use instead the (vector) symbol $\vec{\xi}$, either of which symbol is defined as the vector between the field point and the source point, i.e.,

$$\text{"squiggly r vector"} \equiv \vec{r} - \vec{r}\,' \equiv \vec{\xi} \ .$$

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On the other hand, we have here an additional notational difficulty since Griffiths has decided to use the notation $\vec{w}(t)$, in Eq. (10.32) on p. 429, to indicate what he and I, and everybody else, has heretofore been calling the source point, namely $\vec{r}^{\,'}$, i.e., he has set

$$\text{Griffiths' notation:} \ \ \vec{r}\,'\Big|_{\text{for a single, point charge}} \equiv \vec{w}(t) \ . \tag{0a}$$

Therefore he writes, in his Eq. (10.34) the following

$$\text{Griffiths' notation:} \ \ \text{vector squiggly r} \ = \vec{r} - \vec{w}(t_r) \ . \tag{0b}$$

Since I am using $\vec{\xi}$ to denote the "vector squiggly r," this difference should just be denoted that way; however, since the difference in question is time dependent, it is of course different when evaluated at the field time, t, or at the retarded time, t_r. Griffiths has carefully explained that he is evaluating this quantity at the retarded time, on the bottom of p. 429, even though earlier it has not been that way. Therefore, I am following most other authors and using the following notation:

We have time-dependent quantities, such as $\vec{\xi}(t) = \vec{r} - \vec{r}\,'(t)$, and if they should be evaluated at the retarded time, I will enclose them within brackets:

$$\text{Finley's notation:} \qquad \begin{aligned} \vec{\xi} &= \vec{\xi}(t) \equiv \vec{r} - \vec{r}\,'(t) \ , \\ [\vec{\xi}] &= [\vec{\xi}(t)] \equiv \vec{\xi}(t)\Big|_{t=t_r} = \vec{r} - \vec{r}\,'(t_r) \ . \end{aligned} \tag{0c}$$

To repeat, the meaning of the brackets is that all quantities contained within them are functions of time, and are to be evaluated **NOT** at the time t, but, rather, at the retarded time, t_r.

The retarded time, t_r, is that particular time, earlier than the current time, t, i.e., the time at which the measurement is being made, which allows for information traveling at the speed of light to pass from the source point

to the field point; therefore, it is a function of both those (3-dimensional) points. The equation below simply states the English language above in mathematical symbols, using the standard phrase for something moving at constant velocity, i.e., time is distance divided by speed:

$$c(t - t_r) = [|\vec{r} - \vec{r}\,'|] \equiv \left[|\vec{\xi}|\right] \equiv [\xi] = \xi(t_r)\ . \tag{1}$$

Do notice that this is a rather complicated equation since t_r appears on the left-hand side, and also inside $[\vec{r}\,']$ on the right hand side so that t_r appears on both sides of the equation.

With these notational changes, his equations (10.39) and (10.40) are as follows, where the coordinates $\vec{r}$ and t tell us the location, and time, at which we (or someone) are measuring the potential due to this charge:

$$\begin{aligned} \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\frac{qc}{[(\xi c - \vec{\xi}\cdot\vec{v})]}[\vec{v}]\ , \\ V(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\frac{qc}{[(\xi c - \vec{\xi}\cdot\vec{v})]}\ . \end{aligned} \tag{2}$$

However, I also note that Griffiths could, just as easily, have decided to divide the top and bottom of each of the two equations by c, which would give us the following, which will be easier to compare with my own notation:

$$\begin{aligned} \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\frac{q}{[(\xi - \vec{\xi}\cdot\vec{v}/c)]}[\vec{v}]\ , \\ V(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\frac{q}{[(\xi - \vec{\xi}\cdot\vec{v}/c)]}\ . \end{aligned} \tag{3}$$

Remember here that the quantity $\vec{v}$ denotes the velocity of the point charge; i.e.,

in Griffiths' notation, $\vec{v}(t) \equiv \left(\frac{\partial\,\vec{w}(t)}{\partial t}\right)(t) = \left(\frac{\partial\,\vec{r}\,'(t)}{\partial t}\right)(t) \implies [\vec{v}] = \left(\frac{\partial\,\vec{w}(t)}{\partial t}\right)(t)\Big|_{t=t_r}\ .$ (4)

Now I remind you of the final forms that I gave you for these potentials, as derived in class, for a particle of charge q and moving with a velocity $\vec{v}(t)$: (They were of course derived by alternative means, using special relativity, which I will discuss a little more a bit later.) Those equations were the following:

$$\left.\begin{aligned} \vec{A}(\vec{r},t) &= \frac{\mu_0 q}{4\pi}\frac{[\vec{v}]}{s}\ , \\ V(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{1}{s}\ , \end{aligned}\right\} \quad ; \qquad s \equiv [\xi - \vec{\xi}\cdot\vec{v}/c]\ . \tag{5}$$

Comparing Eqs.(3) with Eqs.(5), we see that in fact they agree completely, remembering the comparisons we have already made:

a. I'm using $\vec{\xi}$ for his "squiggly r" symbol, both of which mean $\vec{r} - \vec{r}\,'$.

b. Lastly, he seems more or less to neglect the reminder that it is all being evaluated at the retarded time, which he does tell you in the English, while I'm putting the symbols in brackets, to provide a very immediate reminder of that fact.

Next I want to follow along with Griffiths text, where he begins, on p. 435 in Section 10.3.2, to calculate the electric and magnetic fields associated with these potentials. At this point he introduces two other useful vectors, namely $\vec{u} \equiv c\,\hat{\xi} - \vec{v}$, in Eq. (10.64), and $\vec{R}$ which he describes as the vector from the present location of the particle to $\vec{r}$, the field point. Then he writes the equation for it, just above Eq. (10.68), as $\vec{R} \equiv \vec{r} - \vec{v}t$, which is somewhat misleading. More on this in a moment! Using his definition, above, we calculate two quantities that appear in his Eq.(10.65), for the electric field, remembering that his symbol "squiggly r" is now retarded, i.e., it is the same as my notation $[\vec{\xi}]$:

$$\text{Griffiths' notation:} \quad \begin{cases} [\xi]\vec{u} = c[(\xi)(\hat{\xi} - \vec{v}/c)] = c[\vec{\xi} - \xi\vec{v}/c]\ , \\ \left[\vec{\xi}\cdot\vec{u}\right] = c\left[\vec{\xi}\cdot(\hat{\xi} - \vec{v}/c)\right] = c\left[\xi - \vec{\xi}\cdot\vec{v}/c\right]\ . \end{cases} \tag{6}$$

The scalar quantity in the second line is just c multiplied by what I have earlier been calling $[s]$, as stated in Eq.(5), which is the correct quantity to appear in the denominator of fields, being the appropriate generalization of the retarded distance for moving charges. The first quantity, $[\xi]\vec{u}$, is somewhat more interesting. During the time interval between the retarded time and the current time, or field event time, the particle has continued to move. If we assume that it continued to move with the last velocity for which we have any information, i.e., at the velocity $[\vec{v}]$, then it will now be at the location

$$\vec{r}\,'(t) = \vec{r}\,'(t_r) + (t - t_r)\vec{v}(t_r) = [\vec{r}\,' + (t - t_r)\vec{v}] = [\vec{r}\,' + \xi\vec{v}/c]\ , \tag{7a}$$

where I have used Eq.(1) above to replace $t - t_r$ by its equivalent value, ξ/c. I may then calculate what we could refer to as the vector to the field point, $\vec{r}$, from the (possible) current position of the point charge as

$$\vec{\xi}(t) = \vec{r} - \vec{r}\,'(t) = \vec{r} - [\vec{r}\,' + \xi\vec{v}/c] = \left[\vec{\xi} - \xi\vec{v}/c\right]\ . \tag{7b}$$

This is clearly just the quantity $[\xi]\vec{u}$ calculated above in Eq.(6), divided by c; i.e., $[\xi\vec{u}] = c\vec{\xi}$, where the right-hand side does not have brackets, i.e., it is evaluated at the field event time, t.

We do therefore interpret it as c times the vector from the "present" position of the particle to the field point, which I have simply referred to as $\vec{\xi}$. This is of course evaluated at the present time, rather than the retarded time, and is therefore presented without the brackets around it that would indicate that it is was evaluated "back then." At this point I can note that his vector $\vec{R}$, on p. 439, is really the same vector, but only evaluated under very special circumstances, where he has set the retarded time to 0, and also the retarded position—which one may of course do by special choices of coordinates, but, nonetheless, it is presented in a somewhat misleading way.

If we now return to Griffiths' equation for the electric field, as in his Eq.(10.65), dividing the numerator and denominator by c^3, we may rewrite it as

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{s^3}\left\{\left[1 - v^2/c^2\right]\vec{\xi} + \left[\vec{\xi}\right] \times \left(\vec{\xi} \times [\vec{a}]/c^2\right)\right\} , \tag{8}$$

which is the same as the expression that I derived in class.

We should now spend a few minutes considering the various different ways in which one may write out equivalent expressions for the very important quantity $s \equiv [\xi - \vec{\xi}\cdot\vec{v}/c]$, noting that for velocities small with respect to c, it is just $[\xi]$, i.e., the **distance** between the retarded position of the charge and the field point. To do this we consider some figure that shows the retarded position of the charge, the field point, and the vector, $[\vec{\xi}]$, from that retarded position to the field point, the retarded velocity of the charge, $[\vec{u}]$, and the vector $\vec{\xi}$ from the current position of the charge (assuming it has been travelling at constant velocity since the retarded position) to the field point. Let the angle between $[\vec{\xi}]$ and $\vec{\xi}$ be called α, and the angle between $[\vec{w}]$ and $\vec{\xi}$ be called θ. We may then write down some useful relationships:

$$\vec{\xi} = [\vec{\xi}] - (t - t_r)[\vec{v}] = [\vec{\xi} - \xi\vec{v}/c] ,$$
$$\xi\cos\alpha = \vec{\xi}\cdot[\hat{\xi}] = \left[(\vec{\xi} - \xi[\vec{v}]/c)\cdot\hat{\xi}\right] = \left[\xi - \vec{\xi}\cdot\vec{v}/c\right] = s .$$

However, the law of sines tells us that the ratios of the sine of an angle to the length of the opposite side is the same for all 3 such choices for a triangle, so that we may write the following:

$$\begin{aligned} & \frac{[\xi]}{\sin\theta} = \frac{[\xi v/c]}{\sin\alpha} \\ \Longrightarrow\quad & v\sin\theta = c\sin\alpha \\ & \Longrightarrow\quad 1 - (v/c)^2\sin^2\theta = 1 - \sin^2\alpha = \cos^2\alpha \\ & \Longrightarrow\quad s = \xi\cos\alpha = \xi\sqrt{1 - (v/c)^2\sin^2\theta} . \end{aligned} \tag{9}$$

I note that the last statement is basically the solution to Problem 10.14 of Griffiths, where he is using $\vec{R}$ to mean $\vec{\xi}$.

Lastly I note that we also considered these quantities in a very special Lorentz frame, which was co-moving along with the charge, i.e., moving with velocity $[\vec{v}]$. We are interested in $\vec{\xi} = \vec{r} - \vec{r}\,'$, as measured in that frame as compared with the resting inertial frame, where the velocity of the charge is indeed measured as $\vec{v}$. We need an associated time component to do that; however, in the co-moving, or rest, frame, the retarded time is the same as the measurement time, so that it is $\Delta t' = t' - t'_r = t' - t' = 0$, where we have labeled physical quantities in the co-moving frame with primes. We recall the Lorentz transformation equations, written in the form

$$\begin{aligned} \xi'_{\parallel} &= \gamma_v(\xi_{\parallel} - (v/c)(c\Delta t')) = \gamma_v \Delta\xi_{\parallel} \;, \\ \xi'_{\perp} &= \xi_{\perp} \;, \\ c\Delta t' &= \gamma_v(c\Delta t - (\vec{v}/c)\cdot\vec{\xi}) \;. \end{aligned} \tag{10}$$

This gives the following for the transformation of the (desired) magnitude of the displacement:

$$\begin{aligned} (\xi')^2 &= (\xi'_{\perp})^2 + (\xi'_{\parallel})^2 = (\xi_{\perp})^2 + \gamma_v^2(\xi_{\parallel})^2 = \gamma_v^2\{(\xi_{\parallel})^2 + (\xi_{\perp})^2\} + (1-\gamma_v^2)(\xi_{\perp})^2 \\ &= \gamma_v^2(\xi)^2 - \gamma_v^2(v^2/c^2)(\xi_{\perp})^2 = \gamma_v^2\xi^2\{1 - (v^2/c^2)\sin^2\theta\} \;, \\ &\Longrightarrow \quad \xi' = \gamma_v\xi^2\sqrt{1-(v^2/c^2)\sin^2\theta} = \gamma_v\, s \;. \end{aligned} \tag{11}$$

This last equality shows us the place where we originally began the derivation for these potentials. Namely, in the co-moving frame, the magnetic vector potential must surely be zero, because the charge is at rest, while the scalar potential is just given by Coulomb's Law; therefore, we may write

$$\vec{A}' = \vec{0} \;, \quad V' = kq/\xi' \quad \Longrightarrow \quad \widetilde{A}' = \begin{pmatrix} \vec{A}' \\ V'/c \end{pmatrix} = \begin{pmatrix} \vec{0} \\ kq/c\xi' \end{pmatrix} \;, \tag{12}$$

where we have used the identification shown to create a 4-vector from the two potentials, and have evaluated it in the co-moving frame for the point charge. I then recalled the general form for the 4-velocity, $\eta^\mu = \gamma_v(\vec{v}, c)^T$. Therefore, in the co-moving frame it simply looks like $(\vec{0}, c)^T$, clearly proportional to our 4-potential; of course in addition we know that the current density for a point charge should be just

$$\begin{pmatrix} \vec{J} \\ c\rho \end{pmatrix} = q \begin{pmatrix} \vec{v} \\ c \end{pmatrix} \delta^3(\vec{r} - \vec{r}\,') \;,$$

and that the time dependence should be at the retarded time; therefore, we decided that in general the 4-potential should look like

$$\begin{pmatrix} \vec{A} \\ V/c \end{pmatrix} = \widetilde{A} = \frac{\mu_0}{4\pi}\frac{q}{\xi'}\,[\widetilde{\eta}] = \frac{\mu_0}{4\pi}\frac{q}{[\gamma_v]\,s}\,[\widetilde{\eta}] = \frac{\mu_0}{4\pi}\frac{q}{s}\begin{pmatrix} [\vec{v}] \\ c \end{pmatrix} \;. \tag{13}$$

From this form, the formulae given at the first of this set of notes becomes straightforward.