Solutions of the sDiff(2)Toda equation with SU(2) Symmetry

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Abstract

We present the general solution to the Plebański equation for an $\mathfrak{h}$ space that admits Killing vectors for an entire SU(2) of symmetries, which is therefore also the general solution of the sDiff(2)Toda equation that allows these symmetries. Desiring these solutions as a bridge toward the future for yet more general solutions of the sDiff(2)Toda equation, we generalize the earlier work of Olivier, on the Atiyah-Hitchin metric, and re-formulate work of Babich and Korotkin, and Tod, on the Bianchi IX approach to a metric with an SU(2) of symmetries. We also give careful delineations of the conformal transformations required to ensure that a metric of Bianchi IX type has zero Ricci tensor, so that it is a self-dual, vacuum solution of the complex-valued version of Einstein's equations, as appropriate for the original Plebański equation.

PACS numbers: 04.20.Jb

I. Introduction

We have long been interested in the Plebański[1] formulation for an $\mathfrak{h}$-space, i.e., a 4-dimensional, complex manifold with an anti-self-dual conformal curvature tensor[2] that also satisfies the Einstein vacuum field equations. Any such space is determined by a solution to the Plebański heavenly equation, a constraining pde for a single function of 4 variables. By now many different, equivalent forms of that equation have been developed; however, the two most common are the original ones given by Plebański: the first form, for a function $u = u(p, \tilde{p}, q, \tilde{q})$, and the second form, for a function $v = v(x, y, p, q)$, either of which serve as potentials for the metric via their second derivatives:

$$u_{,p\tilde{p}}u_{,q\tilde{q}} - u_{,p\tilde{q}}u_{,q\tilde{p}} = 1 \quad \text{and} \quad \mathbf{g} = 2(u_{,p\tilde{p}}\,dp\,d\tilde{p} + u_{,p\tilde{q}}\,dp\,d\tilde{q} + u_{,q\tilde{p}}\,dq\,d\tilde{p} + u_{,q\tilde{q}}\,dq\,d\tilde{q})\,,$$

$$\text{or} \tag{1.1}$$

$$v_{,xx}v_{,yy} - v_{,xy}^2 + v_{,xp} + v_{,yq} = 0 \quad \text{and} \quad \mathbf{g} = 2dp(dx - v_{,yy}dp + v_{,xy}dq) + 2dq(dy + v_{,xy}dp - v_{,xx}dq)\,,$$

where partial derivatives are indicated by a subscript which begins with a comma. This approach already has a long history; nonetheless, there still seems to be considerable effort being made[3] to better understand the structure of the space of solutions, and any notion as to the behavior of the "general" solution is still far from being found.

Nonetheless, to make some progress on that problem, one looks to simplify the question. A standard approach is to simplify the question by looking for those metrics that admit a Killing vector. If the (necessarily skew-symmetric) covariant derivative of that Killing vector has an anti-self-dual part that vanishes, then the problem has indeed been completely resolved,[4] filling in some part of the solution space. In this case the constraining equation for u can be reduced to simply an appropriate solution of the 3-dimensional Laplace equation.[4] However, when that anti-self-dual part is non-zero, the Killing vector reduction instead gives a single pde for an unknown function $\Omega = \Omega(q, \tilde{q}, s)$ of only three remaining (complex) variables,[5] which we refer to as the sDiff(2) Toda equation[6] because of its symmetry properties. The equation, again along with the form of the metric[7] that it generates is the following, where we use φ as the coordinate along the flows generated by the Killing vector:

$$\begin{gathered}\Omega_{,q\tilde{q}} + \left(e^{\Omega}\right)_{,ss} = 0 \,, \qquad \mathbf{g} = V\,\boldsymbol{\gamma} + V^{-1}(d\varphi + \underset{\approx}{\omega})^2 \,, \\ V \equiv \tfrac{1}{2}\Omega_{,s} \,, \quad \boldsymbol{\gamma} \equiv ds^2 + 4\,e^{\Omega}\,dq\,d\tilde{q} \,, \qquad \underset{\approx}{\omega} \equiv \tfrac{i}{2}\{\Omega_{,q}dq - \Omega_{,\tilde{q}}d\tilde{q}\}\end{gathered} \tag{1.2}$$

This equation has been of interest in general relativity in various contexts, as well as some other fields of theoretical physics, for over twenty years. Many explicit solutions have been found[8] although few solutions of general type are known, especially for cases without any additional Killing vectors. Hoping to understand methods to find solutions without additional Killing vectors, there have, fairly recently, been several deliberate searches for so-called "non-invariant" solutions of the sDiff(2) Toda equation.[9]

However, it is true that the use of symmetries continues to be the most efficient method we have for solving nonlinear pde's. Fairly early in the studies of this equation, M.V. Saveliev[10] used it as a platform to begin his study of continuum Lie algebras,[11] and even presented a form which, in principle, gave the desired "general solution" of the equation in terms of some initial conditions. Unfortunately that form is very complicated and does not seem to be useful for obtaining manageable and interesting, specific new solutions. Various explications of the symmetries, and the generalized symmetries have been made.[12] It is hoped that the detailed characterization of those symmetries facilitates both their use for finding additional classes of general solutions as well as checking for noninvariant solutions. However, we propose a different approach in this article.

There is one particular subset of all solutions that has truly received extensive study during the last 15 or so years; this is when the metric allows an entire SU(2) [or, equivalently, an SO(3)] of symmetries.[13] In this case Einstein's field equations are reduced to simply a system of ordinary differential equations, for functions of one remaining independent variable. At least the question of the solutions themselves has been completely answered for this case. On the other hand, the usual approach to this problem begins from an entirely different mechanism, built specifically on that large family of symmetries, and usually referred to as a solution of Bianchi type IX. Those solutions are formulated either via Schwarzian triangle functions[14] or Painlevé transcendents.[15] Because the mechanism used to obtain these solutions is quite different from one that would begin from Plebański's equation it is not of as much use as we would like, from the point of view of using it as a starting point to work backwards and find more general classes of solutions of the sDiff(2) Toda equation. Indeed other researchers have also thought about this question, and work of Olivier[16] and also of Tod[17] has shown many of the details of relationships of some of the cases with solutions of the sDiff(2) Toda equation; however, the essential purpose of this paper is to make more explicit that sort of information about functions $\Omega(q, \tilde{q}, s)$, constituting solutions of our equation, with the hope that this will help push forward that search for a much better understanding of the solution space. In particular we will begin with the most general form for a vacuum, Bianchi IX solution, in the form given by Babich and Korotkin,[18] who express their solutions in terms of elliptic theta functions. Then we will show how one may determine the corresponding variables for the sDiff(2) Toda equation. It is perhaps also worth noting here that other approaches to extending these solutions, by Ionaş,[19] and by Ohyama,[20] have emphasized the desirability of using elliptic theta functions for problems related to this.

II. The Bianchi IX Approach

We begin here with the important details of the formalism usually used for investigations of Bianchi IX metrics, using notation that follows Babich and Korotkin.[18] The assumed three Killing vectors give foliations of the 4-space, as orbits of those Killing vectors, that are, at least topologically, spheres, so that a very reasonable approach is to use the Pauli 1-forms, $\{\underset{\sim}{\sigma}_i\}_1^3$, as a Maurer-Cartan basis, for the assumed SU(2) symmetries. In addition we use a fourth coordinate normal to those surfaces, that we denote by μ. The general form of the Bianchi IX metric, with additional conformal factor, which may be resolved so that it has zero Ricci tensor and also **has** a conformal tensor which is either anti-self-dual or self-dual is usually stated as follows:

$$
\begin{gathered}
\mathbf{g} = F^2 w_1 w_2 w_3 \left\{ d\mu^2 + \frac{\underset{\sim}{\sigma}_1^2}{w_1^2} + \frac{\underset{\sim}{\sigma}_2^2}{w_2^2} + \frac{\underset{\sim}{\sigma}_3^2}{w_3^2} \right\} \quad , \\
d\underset{\sim}{\sigma}_1 = \underset{\sim}{\sigma}_2 \wedge \underset{\sim}{\sigma}_3 \; , \quad d\underset{\sim}{\sigma}_2 = \underset{\sim}{\sigma}_3 \wedge \underset{\sim}{\sigma}_1 \; , \quad d\underset{\sim}{\sigma}_3 = \underset{\sim}{\sigma}_1 \wedge \underset{\sim}{\sigma}_2 \; ,
\end{gathered}
\tag{2.1}
$$

where the three functions $\{w_i\}_1^3$ and the conformal factor F depend only on μ. The requirements concerning anti-self-duality or self-duality of the conformal tensor, and the vanishing of the Ricci scalar, are turned into (a system of ordinary) differential equations that must be satisfied by the three functions $\{w_i \mid i = 1, 2, 3\}$. The vanishing of the quantities labeled as $\mathcal{W}_{+i}$ below cause the self-dual part of the conformal tensor to vanish, causing the resultant space to have an anti-self-dual conformal tensor, while the vanishing of the other set, $\mathcal{W}_{-i}$ cause the anti-self-dual part to vanish, with the opposite conclusion, where we do notice that the one version may be changed into the other simply by a change of the sign of the independent variable, μ, its derivative being indicated below simply by a prime:

$$
\left.
\begin{aligned}
\mathcal{W}_{\pm i} &\equiv \mp {a_{\pm i}}' + a_{\pm i}(a_{\pm j} + a_{\pm k}) - a_{\pm j} a_{\pm k} \; , \\
2a_{\pm i} = Y_{\pm j} + Y_{\pm k} - Y_{\pm i} \; , \quad Y_{\pm i} &\equiv \pm \frac{w_i'}{w_i} + \frac{w_j w_k}{w_i} \equiv a_{\pm j} + a_{\pm k} \; ,
\end{aligned}
\right\} \; i, j, k = 1, 2, 3, \text{ cyclic,}
\tag{2.2}
$$

where we use the word “cyclic” with a fairly standard meaning, i.e., to mean that the indices $\{i, j, k\}$ should always take distinct values and in cyclic order, so that they imply each of the three possible sets of values $1, 2, 3$, and $2, 3, 1$, and $3, 1, 2$. A complete derivation of the provenance of these equations is given in Appendix I.

Since we have chosen, following historical precedent with the Plebański approach to these problems, to concern ourselves with the anti-self-dual solution, we will ask that the $\mathcal{W}_{+i}$ should vanish. With that choice we will write out more explicitly the coupled set of six equations which must be solved, and will suppress the +-subscript on what was called a_{+i} above since this is the only one with which we will be concerned:

$$
0 = \mathcal{W}_{+i} \quad \Longrightarrow \quad
\left.
\begin{aligned}
a_i' + a_j a_k &= a_i (a_j + a_k) \; , \\
w_i' + w_j w_k &= w_i (a_j + a_k) \; ,
\end{aligned}
\right\} \; i, j, k = 1, 2, 3, \text{ cyclic.}
\tag{2.3}
$$

The general case of a solution of these equations, with no constraint on the conformal factor, $F = F(\mu)$, will have a non-zero, although traceless Ricci tensor. The equations for the three functions $\{a_i\}_{i=1}^3$ are often referred to as the Halphen system,[21] and there are several different known forms for the solution, involving Painlevé transcendents of type III,[17], Schwarzian triangle functions,[14] or complete elliptic integrals.[22] We will not often need to use the explicit solutions; nonetheless, we prefer the form given by Babich and Korotkin,[18] in terms of *theta functions.* The general solution for the three functions $\{a_i \mid i = 1, 2, 3\}$ is a 3-parameter one given in terms of the general Möbius transformation of the upper-half of the complex τ-plane when it acts on the following (generic) particular solution of the system. We replace the usual variable τ for the theta functions, which must have positive imaginary part, by $\tau \equiv i\mu$, so that we may treat μ as real-valued and positive, befitting its role in the form of our metric. The particular solution of the system is given by:

$$a_i = 2\frac{d}{d\mu}\log\vartheta_{5-i}\ ,\ i = 1, 2, 3\ ; \quad \text{where} \quad \begin{cases} \vartheta_2 \equiv \vartheta\left[{}^{1/2}_{0}\right](0\,|i\mu) = e^{-\frac{\pi}{4}\mu}\displaystyle\sum_{m=-\infty}^{+\infty} e^{-\pi m(m+1)\mu}\ , \\ \vartheta_3 \equiv \vartheta\left[{}^{0}_{0}\right](0\,|i\mu) = \displaystyle\sum_{m=-\infty}^{+\infty} e^{-\pi m^2\mu}\ , \\ \vartheta_4 \equiv \vartheta\left[{}^{0}_{1/2}\right](0\,|i\mu) = \displaystyle\sum_{m=-\infty}^{+\infty} (-1)^m\, e^{-\pi m^2\mu}\ , \end{cases} \tag{2.4}$$

The Möbius transformation sends $\tau \to (a\tau + b)/(c\tau + d)$, with $ad - bc = 1$. More details of the transformation and how it scales the functions a_i, are given in Appendix III. As well we describe the general theta functions, which also depend on a complex variable z. When those functions are evaluated at $z = 0$, as they are above, they are often referred to as *theta coefficients*, and are related to the complete elliptic integrals, $K(k)$ and $E(k)$.

The functions $w_i(\mu)$ are related to these others. For the case when the Ricci scalar vanishes they are related in a very simple way, involving the conformal factor. We are interested only in the pure vacuum case, so that the entire Ricci tensor vanishes. We will explain in detail in Appendix I that this can be done provided one chooses

$$F(\mu) = c_0(\mu + d_0)\ , \text{ and } \ w_i = a_i + \frac{d}{d\mu}\log F\ , \tag{2.5}$$

for arbitrary constant values c_0 and d_0 as discussed in the paper of Babich and Korotkin,[18] with reference to work of Tod.[25] It is also worth noting that the first solution of this sort, determined by Atiyah and Hitchin[22], and explained more fully by Olivier[16] as regards its relation to Plebański's equation, corresponds to the limit where $c_0 \to 0$ at the same that $c_0\, d_0 \to$ a non-zero constant, corresponding to a constant value for F, and therefore $w_i = a_i$.

III. The Matching Process

To create the desired mappings between the two sets of variables, we will need parametrizations of the Pauli 1-forms, using a set of Euler angles on the sphere, and identified in such a way that the coordinate angle φ represents the variable along our original Killing vector, the one required by the sDiff(2)Toda equation itself. We choose one such representation given by Tod:[17]

$$\underset{\sim}{\sigma}_1 + i\underset{\sim}{\sigma}_2 = e^{i\psi}(d\theta - i\sin\theta\, d\varphi)\ , \quad \underset{\sim}{\sigma}_3 = d\psi + \cos\theta\, d\varphi\ . \tag{3.1}$$

We begin the process of comparing the two forms of the metric, and establishing coordinate transformations by first considering that the terms involving $d\varphi^2$ are uniquely picked out, since it is our "obvious" Killing vector. Expanding out the metric in terms of these Euler angles, we have the following identification, as a first step:

$$\begin{aligned} \frac{1}{V} = \|\partial/\partial\varphi\|^2 \ &= F^2\{[\frac{w_1 w_3}{w_2}\cos^2\psi + \frac{w_2 w_3}{w_1}\sin^2\psi]\sin^2\theta + \frac{w_1 w_2}{w_3}\cos^2\theta\} \equiv \frac{F^2}{w_1 w_2 w_3}M\ , \\ M &\equiv \left[w_3^2(w_2^2\sin^2\psi + w_1^2\cos^2\psi)\sin^2\theta + w_1^2 w_2^2\cos^2\theta\right]\ , \end{aligned} \tag{3.2}$$

where we have defined the quantity M for convenience since it will appear in many different places in the discussion. As the 3-dimensional portion of our metric, $\boldsymbol{\gamma}$, does not depend on φ, we may next identify the 1-form $\underset{\sim}{\omega}$ in the Plebański form of the metric:

$$\frac{1}{V}\underset{\sim}{\omega} = F^2\Big\{\left(\frac{w_2 w_3}{w_1} - \frac{w_3 w_1}{w_2}\right)\sin\theta\sin\psi\cos\psi d\theta + \frac{w_1 w_2}{w_3}\cos\theta\, d\psi\Big\} \tag{3.3}$$

With these forms we may now calculate $\frac{1}{V}(d\varphi + \underset{\sim}{\omega})^2$ and remove that term from the Bianchi IX form of the metric, providing us with the form of $\boldsymbol{\gamma}$ in those coordinates:

$$\begin{aligned} \boldsymbol{\gamma} = F^4\Big\{&M\, d\mu^2 + [w_3^2\sin^2\theta + (w_1^2\sin^2\psi + w_2^2\cos^2\psi)\cos^2\theta]\, d\theta^2 \\ &+ 2(w_1^2 - w_2^2)\sin\psi\cos\psi\sin\theta\cos\theta\, d\theta\, d\psi + (w_1^2\cos^2\psi + w_2^2\sin^2\psi)\sin^2\theta\, d\psi^2\Big\} \\ &= \gamma_{\mu\mu}d\mu^2 + \gamma_{\theta\theta}d\theta^2 + 2\gamma_{\theta\psi}d\theta\, d\psi + \gamma_{\psi\psi}d\psi^2 = ds^2 + 4\, e^{\Omega}\, dq\, d\tilde{q}\ . \end{aligned} \tag{3.4}$$

Our next step is to determine the factor ds^2, which is related to the 1-form $\underset{\sim}{\omega}$ as follows, beginning from Eqs. (1.2):

$$\begin{aligned} 2V =&\Omega_{,s}\ , \quad \underset{\sim}{\omega} \equiv \tfrac{i}{2}\{\Omega_{,q}dq - \Omega_{,\tilde{q}}d\tilde{q}\} \implies -i\, d\underset{\sim}{\omega} = V_{,q}ds\wedge dq - V_{,\tilde{q}}ds\wedge d\tilde{q} + \Omega_{,q\tilde{q}}d\tilde{q}\wedge dq\ , \\ \Rightarrow \underset{\gamma}{*}d\underset{\sim}{\omega} &= \frac{(e^{\Omega})_{,ss}}{e^{\Omega}}ds + V_{,q}dq + V_{,\tilde{q}}d\tilde{q} = V_{,q}dq + V_{,\tilde{q}}d\tilde{q} + (V_{,s} + 2V^2)ds = V^2 d(2s - 1/V)\ , \end{aligned} \tag{3.5}$$

where we have used the fact that Ω must satisfy its constraining pde, and we note that the Hodge dual in question is the one generated by the 3-metric $\boldsymbol{\gamma}$ in the Plebański form, which says

$$\underset{\gamma}{*}(dq \wedge ds) = i dq \ , \quad \underset{\gamma}{*}(ds \wedge d\tilde{q}) = i d\tilde{q} \ , \quad \underset{\gamma}{*}(dq \wedge d\tilde{q}) = -\tfrac{i}{2} e^{-\Omega} ds \ . \tag{3.6}$$

As we have $\underset{\sim}{\omega}$ and $1/V$ in terms of the Euler angles, we may solve this for the desired $d(2s)$:

$$d(2s) = V^{-2} \underset{\gamma}{*} \underset{\sim}{\omega} + d(1/V) = d(1/V) + \underset{\gamma}{*} \left[\frac{1}{V} d\left(\frac{1}{V} \underset{\sim}{\omega} \right) + \frac{1}{V} \underset{\sim}{\omega} \wedge d\left(\frac{1}{V} \right) \right] \ , \tag{3.7}$$

where it is re-expressed in this second form since it then involves only those quantities that we already know. Of course, in order to do that we must first determine the dual mapping, but now relative to $\boldsymbol{\gamma}$, in terms of these coordinates—rather more complicated than the other ones because of its off-diagonal terms. We give the results below of the duals of the three basis 2-forms, sufficient to determine what is wanted. This calculation is straightforward, if rather lengthy, and is described in more detail in Appendix II, with the following result:

$$\begin{aligned} ds = F^2 \Big\{ & [w_3(w_1 \cos^2\psi + w_2 \sin^2\psi) \sin^2\theta + w_1 w_2 \cos^2\theta] d\mu \\ & + (w_3 - w_1 \sin^2\psi - w_2 \cos^2\psi) \sin\theta \cos\theta \, d\theta + (w_2 - w_1) \sin\psi \cos\psi \sin^2\theta \, d\psi \Big\} \\ \equiv {} & \mathcal{F} \, d\mu + \mathcal{G} \, d\theta + \mathcal{H} \, d\psi \ . \end{aligned} \tag{3.8}$$

We need to square this and subtract appropriately from $\boldsymbol{\gamma}$ so as to determine the remainder of the transformation equations, which will be done shortly. Nonetheless, we will first note here that this equation can be explicitly integrated, to give this Plebański coordinate as a function of the ones used in the Bianchi IX approach:

$$s = s(\mu, \theta, \psi) = \tfrac{1}{2} F^2 [w_1 + w_2 + (w_3 - w_1 \sin^2\psi - w_2 \cos^2\psi) \sin^2\theta] \ . \tag{3.9}$$

Returning now to the forms given in Eqs. (3.4), we want to have $\boldsymbol{\gamma} - ds^2$ in the form $e^{\Omega}(2dq)(2d\tilde{q})$; i.e., we want a pair of coordinates q and $\tilde{q}$, and a function Ω such that this equation would be satisfied. However, what we actually have is the difference of those two second-rank tensors, which we want to describe in the form above. Therefore, it is first useful to factor that difference into a pair of (complex-valued) 1-forms, which we call $\underset{\sim}{e}_+$ and its "conjugate," $\underset{\sim}{e}_-$, such that

$$\underset{\sim}{e}_+ \underset{\sim}{e}_- \equiv \boldsymbol{\gamma} - (ds)^2 \ . \tag{3.10}$$

We may then find a function e^{Ω} so that the separate 1-forms that make up that product may each be integrated. Therefore, we define a complex-valued scalar function, $f = f(\mu, \theta, \psi)$, to be chosen so that

$$\left.\begin{aligned} d(e^{-f}\underset{\sim}{e}_{+}) = 0 \, , \\ d(e^{-\overline{f}}\underset{\sim}{e}_{-}) = 0 \, , \end{aligned}\right\} \quad e^{f} e^{\overline{f}} = e^{\Omega} \, , \quad \Longrightarrow \quad \left\{\begin{aligned} \underset{\sim}{e}_{+} &= e^{f}\, 2dq \, , \\ \underset{\sim}{e}_{-} &= e^{\overline{f}}\, 2d\tilde{q} \, . \end{aligned}\right. \tag{3.11}$$

where q and $\tilde{q}$, are the desired new coordinates. We begin with the desired factorization into the product of two 1-forms:

$$\begin{aligned} \underset{\sim}{e}_{+} =& F^2 \sin(\theta) \left\{ \left[w_3 (w_1 \cos^2\psi + w_2 \sin^2\psi) - w_1 w_2 \right] \cos\theta + i w_3 (w_2 - w_1) \sin\psi \cos\psi \right\} \, d\mu \\ & - F^2 \left\{ (w_1 \sin^2\psi + w_2 \cos^2\psi) \cos^2\theta + w_3 \sin^2\theta - i (w_2 - w_1) \sin\psi \cos\psi \cos\theta \right\} \, d\theta \\ & \quad + F^2 \sin\theta \left\{ (w_2 - w_1) \sin\psi \cos\psi \cos\theta - i (w_1 \cos^2\psi + w_2 \sin^2\psi) \right\} \, d\psi \\ & \equiv \mathcal{A}\, d\mu + \mathcal{T}\, d\theta + \mathcal{L}\, d\psi \, , \end{aligned} \tag{3.12}$$

while the other one, $\underset{\sim}{e}_{-}$, is obtained from $\underset{\sim}{e}_{+}$ simply by changing all the i's above to $-i$'s, i.e., treating it as if all our variables are real-valued and they should be complex conjugates of each other. It is a lengthy but straightforward calculation to verify that this pair does indeed accomplish the desired factorization scheme. The necessary "integrating factor," i.e. the function e^{f}, is not uniquely determined; nonetheless the choice that we find acceptable is given, conveniently, in terms of its square as follows:

$$e^{2f} \equiv 2F^2 \left[(w_1 - w_2)(\sin\psi + i \cos\psi \cos\theta)^2 + (w_3 - w_1) \sin^2\theta \right] \, , \tag{3.13}$$

with the conjugate $e^{2\overline{f}}$ again obtained simply by changing the i above to $-i$. It is true that this is the square of the desired factor, rather than the factor itself; however, because of its complex nature it is better to display it in this form rather than insisting on just which square root is appropriate. It is then again straightforward algebra to show the following:

$$e^{3f} d(e^{-f}\underset{\sim}{e}_{+}) = e^{2f}\, d\underset{\sim}{e}_{+} - \tfrac{1}{2}\, de^{2f} \wedge \underset{\sim}{e}_{+} = 0 \, , \tag{3.14}$$

which **guarantees** the existence of q, and also $\tilde{q}$, so that we may now write explicitly their differentials.

While the necessary forms have indeed already been written out explicitly above, in Eqs. (3.12), the integrations that need to be performed, to determine the explicit form of q and $\tilde{q}$, are much more easily performed in a slightly different set of coordinates. We will choose a new set, $\{\mu, \theta, p\}$, replacing ψ in favor of $p \equiv i(\psi + \pi/2) + \log\tan(\theta/2)$, which gives $dp = i\, d\psi + d\theta/\sin(\theta)$. This will allow us to determine all the dependence on θ quite explicitly and easily, leaving differential equations to be solved only in terms of μ and p. We must then re-express $\underset{\sim}{e}_+$ in terms of $\{\mu, \theta, p\}$:

$$\begin{aligned} e^f\, d(2q) &= \underset{\sim}{e}_+ = \mathcal{A}\, d\mu + \mathcal{B}\, d\theta + \mathcal{C}\, dp \,, \qquad\qquad u \equiv \cosh p \,; \\ &\mathcal{A} \equiv F^2 \sin(\theta) \Big\{ w_3(w_1 - w_2) u \left[u\cos\theta + \sqrt{u^2-1}\right] + w_2(w_3 - w_1)\cos\theta \Big\} \,, \\ &\mathcal{B} \equiv -F^2 \sin^2\theta \left[(w_1-w_2)u^2 + (w_3 - w_1)\right] \quad = \quad -\tfrac{1}{2} e^{2f} \equiv -\tfrac{1}{2} P^2 \sin^2\theta \,, \\ &\mathcal{C} \equiv F^2 \sin(\theta) \left[(w_1 - w_2) u \left(u + \sqrt{u^2-1}\cos\theta\right) - w_1 \right] \,, \end{aligned} \tag{3.15}$$

where we have given a simple symbol for the oft-repeated quantity $u \equiv \cosh p$, and also given a separate name, $P^2(\mu, p)$, to that portion of e^{2f} independent of θ. The compatibility of these pde's for $2q$ has already been shown, i.e., we know that such a q exists. The integration procedure is explained in Appendix III, with the following result:

$$2q = \tfrac{1}{2} P \cos\theta + \mathbf{N}(\mu, p) \,, \quad P \sin\theta\, 2dq = \underset{\sim}{e}_+ \,, \qquad P^2 = 2F^2 \left[(w_1 - w_2)u^2 + (w_3 - w_1)\right] \,, \tag{3.16}$$

where $\mathbf{N} = \mathbf{N}(\mu, p)$ is shown, in that appendix, to have several, equivalent, forms in terms of elliptic integrals:

$$\begin{aligned} \sqrt{\pi}\, \mathbf{N}(\mu, p) &= c_0 \left[\tfrac{1}{2}(\mu + d_0) \frac{d}{dz} \log \vartheta_4(z \mid i\mu) + \pi z \right] \Big|_{z = \frac{1}{2} F(u,k)/K(k)} \\ &= c_0 (\mu + d_0) \left[K(k) E(u,k) - E(k) F(u,k)\right] + \tfrac{\pi}{2} c_0 \frac{F(u,k)}{K(k)} \,, \quad u \equiv \cosh p \,. \end{aligned} \tag{3.17}$$

IV. Passing to the sDiff(2)Toda equation

We have determined the desired coordinate transformation, in the direction

$$\left. \begin{aligned} s &= s(\mu, \theta, \psi) \,, \\ q &= q[\mu, \theta, p(\psi)] \,, \\ \tilde{q} &= \tilde{q}[\mu, \theta, p(\psi)] \,, \end{aligned} \right\} \qquad \text{and also} \;\; \Omega = \Omega(\mu, \theta, \psi) = \tfrac{1}{2} \log \left(e^{2f} e^{2\overline{f}} \right) \,. \tag{4.1}$$

However, what we wanted was $\Omega = \Omega(s, q, \tilde{q})$, but the equations given are (seriously) transcendental, so that they cannot be explicitly solved to provide $\Omega = \Omega(s, q, \tilde{q})$. Instead we may look at this set of equations as a parametric approach to that question: all the different functions named in Eqs. (4.1) give the desired functions in terms of "parameters," μ, θ, and p [or $p(\psi)$]. (We do note that while the integration for q was much simpler in terms of the complex-valued variable p, the reality conditions for the more general process are simpler to follow if we retreat back to the use of ψ instead.) In order to proceed from there, to show that the function Ω determined in this way does actually satisfy the sDiff(2)Toda equation, given in Eqs. (1.2), we only need to determine the appropriate partial derivatives, as, for instance, in the following simple example:

$$\frac{\partial}{\partial s}\Omega = \left(\frac{\partial\mu}{\partial s}\frac{\partial}{\partial\mu} + \frac{\partial\theta}{\partial s}\frac{\partial}{\partial\theta} + \frac{\partial\psi}{\partial s}\frac{\partial}{\partial\psi}\right)\Omega\,. \tag{4.2}$$

Of course we also do not have the partial derivatives in the directions given just above, such as $\frac{\partial\mu}{\partial s}$. However, the 1-forms that form the basis for our 3-space, in Eqs. (3.12) for dq and $d\tilde{q}$ and Eq. (3.8) for ds, along with the form for e^{2f} given in Eq. (3.13), give us the entries for the Jacobian matrix for the coordinate transformation between these two sets of coordinates, in the opposite direction:

$$J \equiv \frac{\partial(q,\tilde{q},s)}{\partial(\mu,\theta,\psi)} = \begin{matrix} \\ q \\ \tilde{q} \\ s \end{matrix} \begin{matrix} \quad\mu \qquad\quad \theta \qquad\quad \psi \\ \begin{pmatrix} \frac{1}{2}e^{-f}\mathcal{A} & \frac{1}{2}e^{-f}\mathcal{T} & \frac{1}{2}e^{-f}\mathcal{L} \\ \frac{1}{2}e^{-\overline{f}}\overline{\mathcal{A}} & \frac{1}{2}e^{-\overline{f}}\overline{\mathcal{T}} & \frac{1}{2}e^{-\overline{f}}\overline{\mathcal{L}} \\ \mathcal{F} & \mathcal{G} & \mathcal{H} \end{pmatrix} \end{matrix}\,. \tag{4.3}$$

The inverted partial derivatives we need are simply the entries of the inverse matrix to this one. As a first step toward determining them in some convenient way, we note that it is even hopeful that when the rather complicated values for all these quantities are inserted into this matrix, Maple finds that its determinant is rather simple:

$$\det(J) = -\tfrac{i}{2}e^{-f}e^{-\overline{f}}\sin\theta F^6\,M = -\tfrac{i}{2}\,e^{-\Omega}\sqrt{\det\gamma}\,, \tag{4.4}$$

where we have used Eq. (A2.3b) to insert the determinant of the 3-metric. Then the inverse of the Jacobian matrix itself, determined by the computer algebra program Maple, is given here:

$$J^{-1} \;\equiv \frac{\partial(\mu,\theta,\psi)}{\partial(q,\overline{q},s)} = \begin{matrix} \\ \mu \\ \theta \\ \psi \end{matrix} \begin{matrix} \;q \qquad \overline{q} \qquad s \\ \begin{pmatrix} e^{f}\alpha & e^{\overline{f}}\overline{\alpha} & \zeta \\ e^{f}\tau & e^{\overline{f}}\overline{\tau} & \eta \\ e^{f}\lambda & e^{\overline{f}}\overline{\lambda} & \kappa \end{pmatrix} \end{matrix} \frac{1}{M\,F^2}\,, \tag{4.5}$$

with their values given below, and we have recalled M from Eq. (3.2):

$$
\begin{aligned}
M &= w_3^2(w_1^2\cos^2\psi + w_2^2\sin^2\psi)\sin^2\theta + w_1^2w_2^2\cos^2\theta \ , \\
e^{-f}MF^2\frac{\partial\mu}{\partial q} = \alpha \ &\equiv \sin\theta\Big\{\Big[w_3(w_1\cos^2\psi + w_2\sin^2\psi) - w_1w_2\Big]\cos\theta \\
&\qquad - iw_3(w_2-w_1)\sin\psi\cos\psi\Big\} \\
e^{-f}MF^2\frac{\partial\theta}{\partial q} = \tau \ &\equiv -w_3(w_2^2\sin^2\psi + w_1^2\cos^2\psi)\sin^2\theta - w_1w_2(w_1\cos^2\psi + w_2\sin^2\psi)\cos^2\theta \\
&\qquad + iw_1w_2(w_2-w_1)\sin\psi\cos\psi\cos\theta \ , \\
e^{-f}MF^2\frac{\partial\psi}{\partial q} = \lambda \ &= \Big\{ -(w_2-w_1)[w_1w_2\cos^2\theta + w_3(w_1+w_2-w_3)\sin^2\theta]\sin\psi\cos\psi\cos\theta \\
&\quad + i[w_1w_2(w_2\cos^2\psi + w_1\sin^2\psi)\cos^2\theta + w_3^2(w_1\cos^2\psi + w_2\sin^2\psi)\sin^2\theta]\Big\}/\sin\theta \ , \\
MF^2\frac{\partial\mu}{\partial s} = \zeta \ &= w_1w_2\cos^2\theta + w_2(w_1\cos^2\psi + w_2\sin^2\psi)\sin^2\theta \ , \\
MF^2\frac{\partial\theta}{\partial s} = \eta \ &= [w_2^2(w_1-w_3)\sin^2\psi + w_1^2(w_2-w_3)\cos^2\psi]\sin\theta\cos\theta \ , \\
MF^2\frac{\partial\psi}{\partial s} = \kappa \ &= (w_2-w_1)[w_3^2\sin^2\theta + (w_1w_3 + w_2w_3 - w_1w_2)\cos^2\theta]\sin\psi\cos\psi \ ,
\end{aligned} \tag{4.6}
$$

where the (not-displayed) partial derivatives with respect to $\tilde{q}$ are just the complex conjugates of those with respect to q, where all the various functions we have are **considered to be** real. An interesting result that comes from this calculation is a close relationship between those partial derivatives that involve μ, although we have not been able to determine any useful result from it:

$$
MF^2\,\frac{\partial\mu}{\partial s} = \frac{1}{F^2}\frac{\partial s}{\partial\mu} \ , \qquad MF^2\,e^{-f}\,\frac{\partial\mu}{\partial q} = \frac{1}{F^2}e^{\overline{f}}\frac{\partial\tilde{q}}{\partial\mu} \ , \tag{4.7}
$$

along with the conjugate of the second equation as well. On the other hand, with these derivatives in hand from the inversion of the Jacobian matrix, we may now explicitly evaluate $\Omega_{,s}$, which as noted in Eq. (1.2) should equal $2V$. We first define a name, R, for the quantity that we actually have in hand, from Eq. (3.13) and also Eq. (3.15), the product of e^{2f} and $e^{2\overline{f}}$,

$$
\begin{aligned}
R \equiv e^{2\Omega} = e^{2f}e^{2\overline{f}} = 4F^4\Big\{ &\big[(w_1-w_2)(\sin\psi + i\cos\psi\cos\theta)^2 + (w_3-w_1)\sin^2\theta\big] \\
&\big[(w_1-w_2)(\sin\psi - i\cos\psi\cos\theta)^2 + (w_3-w_1)\sin^2\theta\big]\Big\} \qquad (4.8a) \\
= \big[2F^2\sin^2\theta\,\big|(w_1-w_2)u^2 &- (w_1-w_3)\big|\big]^2 = \big[\sin^2\theta\,\big|P^2\big|\big]^2 \ .
\end{aligned}
$$

We may then use the chain rule for $\partial/\partial s$, as written out in Eq. (4.2), and verify, again via Maple, the (required) relationship that is given in Eq. (1.2):

$$
\frac{\partial}{\partial s}\Omega = \frac{1}{2R}\frac{\partial}{\partial s}R = 2\frac{w_1w_2w_3}{MF^2} = 2V(\mu,\theta,\psi) \qquad \Longrightarrow \qquad \frac{\partial}{\partial s}R = 4RV \ . \tag{4.8b}
$$

Next we use these forms to determine a form for one of the desired second derivatives in the sDiff(2) Toda equation, namely the second s-derivative of e^{Ω}:

$$
\begin{aligned}
\partial_s^2 R &= 2\partial_s(e^{\Omega}\partial_s e^{\Omega}) = 2[e^{\Omega}\partial_s^2 e^{\Omega} + (\partial_s e^{\Omega})^2] \\
\Longrightarrow \quad \partial_s^2 e^{\Omega} &= \frac{1}{2e^{\Omega}}\left[\partial_s^2 R - \frac{(\partial_s R)^2}{2R}\right] = 2e^{\Omega}V^2\left(2 - \frac{\partial}{\partial s}\frac{1}{V}\right) .
\end{aligned}
\tag{4.9}
$$

It is true that the last formulation above for our second s-derivative still has an external factor e^{Ω}; however, when we obtain the desired formulation—in terms of the coordinates $\{\mu, \theta, \psi\}$—for the other second derivative, there will be another such factor, so that they will eventually factor out and not cause any difficulty. However, there is the serious difficulty that all of the partial derivatives involving q or $\tilde{q}$ contain e^f and $e^{\overline{f}}$, so that it is not immediately obvious how to extract them in the right format. They can in fact be extracted in the desired form, but we will have to consider not only the product $R \equiv (e^{2f})(e^{2\overline{f}})$ that we have been considering but also the quotient, $S \equiv (e^{2f})/(e^{2\overline{f}})$, which will allow us to take care of factors of f and $\overline{f}$ separately:

$$
\begin{aligned}
&\log R = 2(f + \overline{f}) , \qquad \log S = 2(f - \overline{f}) , \\
\Longrightarrow \quad & 2f + \overline{f} = \frac{1}{4}(3\log R + \log S) , \quad f + 2\overline{f} = \frac{1}{4}(3\log R - \log S) .
\end{aligned}
\tag{4.10}
$$

We begin by keeping explicit track of the factors of e^f that appear, writing the following:

$$
\partial_q = \frac{e^f}{MF^2}\left(\alpha\partial_\mu + \tau\partial_\theta + \lambda\partial_\psi\right) \equiv e^f Q , \tag{4.10}
$$

along with its conjugate form for $\partial/\partial\tilde{q}$. There are then two apparently distinct ways in which the desired second derivative may be written, which, of course, must be identical since ordinary partial derivatives commute. The first ordering is the following:

$$
\begin{aligned}
\partial_{\tilde{q}}\partial_q\Omega &= \tfrac{1}{2}e^{\overline{f}}\,\widetilde{Q}\left(e^f Q\log R\right) = \tfrac{1}{2}e^{\overline{f}}\,\widetilde{Q}\left(e^f\frac{QR}{R}\right) = \tfrac{1}{2}e^{\overline{f}}\,\widetilde{Q}(e^{-f-2\overline{f}}QR) \\
&= \tfrac{1}{2}e^{-\Omega}\{\widetilde{Q}QR - (QR)\widetilde{Q}(f+2\overline{f})\} = \tfrac{1}{2}e^{\Omega}\left\{\frac{\widetilde{Q}QR}{R} - \tfrac{3}{4}\frac{QR}{R}\frac{\widetilde{Q}R}{R} + \tfrac{1}{4}\frac{QR}{R}\frac{\widetilde{Q}S}{S}\right\} ,
\end{aligned}
\tag{4.11a}
$$

which shows that indeed we can re-write our entire equation in such a way as to only need the product R, and the quotient S. The other order for the initial partial derivatives gives the following quantity, which appears to be different:

$$
\begin{aligned}
\partial_q\partial_{\tilde{q}}\Omega &= \tfrac{1}{2}\partial_q\partial_{\tilde{q}}\log(e^{2\Omega}) = \tfrac{1}{2}e^f Q\left(e^{\overline{f}}\widetilde{Q}\log R\right) = \tfrac{1}{2}e^f Q\left(e^{\overline{f}}\frac{\widetilde{Q}R}{R}\right) \\
&= \tfrac{1}{2}e^f Q(e^{-2f-\overline{f}}\widetilde{Q}R) = \tfrac{1}{2}e^{\Omega}\left\{\frac{Q\widetilde{Q}R}{R} - \tfrac{3}{4}\frac{QR}{R}\frac{\widetilde{Q}R}{R} - \tfrac{1}{4}\frac{QS}{S}\frac{\widetilde{Q}R}{R}\right\} .
\end{aligned}
\tag{4.11b}
$$

Since these two expressions must actually be the same we may do two useful things with them. As a first check on the somewhat complicated algebra, we may first insist that their difference is zero; namely we must have the following equality, which is simply the statement that the partial derivatives themselves commute:

$$\frac{4}{R}[Q,\widetilde{Q}]R = \frac{QS}{S}\frac{\widetilde{Q}R}{R} + \frac{\widetilde{Q}S}{S}\frac{QR}{R}\ . \tag{4.12}$$

The calculation (in Maple) involves quite a large number of terms on each side; however, they are in fact equal, verifying that all the algebra is correct. At this point then we finally want to determine the desired other second derivative, which takes its most symmetric form via half the sum of the two expressions given above in Eqs. (4.11a-b);

$$\partial_q\partial_{\tilde{q}}\Omega = \tfrac{1}{4}e^{\Omega}\left\{\frac{Q\widetilde{Q}+\widetilde{Q}Q)R}{R} - \frac{3}{2}\frac{\widetilde{Q}R}{R}\frac{QR}{R} + \frac{1}{4}\frac{\widetilde{Q}S}{S}\frac{QR}{R} - \frac{1}{4}\frac{QS}{S}\frac{\widetilde{Q}R}{R}\right\}\ . \tag{4.13}$$

As the expression for $\partial_s^2 e^{\Omega}$ given in Eq. (4.9) also has an overall factor of e^{Ω}, we may add that expression to this one, and divide out that overall factor, reducing the verification of the sDiff(2) Toda equation to the question as to whether or not that sum vanishes. It is a straightforward, if perhaps somewhat lengthy calculation, performed in Maple, to show that this sum does in fact vanish, which was the necessary and sufficient condition to guarantee that the parametric presentation we have obtained is in fact a solution of the sDiff(2) Toda equation.

V. Conclusions

The equations we have developed give the general solution to the Plebański equation, $\Omega = \Omega(s,q,\tilde{q})$, when the manifold is (locally) required to have SU(2) symmetry. Second derivatives of Ω determine the components of the metric; therefore Ω does not depend on the fourth coordinate for the manifold, φ, since variation of it has been chosen for the direction of the explicit Killing vector. The solution is determined parametrically in terms of an additional set of coordinates for the problem, $\{\mu,\theta,\psi\}$, so that in fact we have our three desired coordinates as functions of them. While the presentation of $\{s,q,\tilde{q}\}$ as functions of these original coordinates, $\{\mu,\theta,\psi\}$ are explicit in terms of θ and ψ, we are unable to invert those equations explicitly because of the existence within them of the theta coefficients, and the (elliptic) theta functions, which are transcendental functions of μ, analytic for all positive values of μ. It is nonetheless useful to collect those equations together here, from the places where they have been derived in this text:

$$\begin{aligned} s &= s(\mu,\theta,\psi) = \tfrac{1}{2}F^2[w_1 + w_2 + (w_3 - w_1\sin^2\psi - w_2\cos^2\psi)\sin^2\theta]\ , \\ q &= q(\mu,\theta,\psi) = \tfrac{1}{4}P\cos\theta + \tfrac{1}{2}\mathbf{N}\ , \qquad \tilde{q} = \tilde{q}(\mu,\theta,\psi) = \tfrac{1}{4}\overline{P}\cos\theta + \tfrac{1}{2}\overline{\mathbf{N}}\ , \end{aligned} \tag{5.1}$$

along with definitions of the symbols involved:

$$P^2 = 2F^2\left[(w_1-w_2)u^2+(w_3-w_1)\right] \;, \qquad \overline{P}^2 = 2F^2\left[(w_1-w_2)\overline{u}^2+(w_3-w_1)\right] \;,$$
$$p = \log\tan(\theta/2) + i(\psi+\pi/2)\;, \qquad u = \cosh p = -2\frac{\sin\psi + i\cos\psi\cos\theta}{\sin\theta}\;, \qquad \overline{u} = \cosh\overline{p} = -2\frac{\sin\psi - i\cos\psi\cos\theta}{\sin\theta}\;. \tag{5.2}$$

The overbar is used here to indicate complex conjugation in the situation where we treat the variables $\{\mu,\theta,\psi,\varphi\}$ as real-valued. Since our goal is in fact to determine general complex-valued solutions, that approach is still valid in the more general case, where we note that the variable u may take on all values in the complex plane. The equation that determines the (potential) function Ω is then given by the following, in terms of functions of $\{\mu,\theta,\psi\}$:

$$e^{\Omega} = 2F^2\Big\{\left[(w_1-w_2)(\sin\psi+i\cos\psi\cos\theta)^2+(w_3-w_1)\sin^2\theta\right]$$
$$\left[(w_1-w_2)(\sin\psi-i\cos\psi\cos\theta)^2+(w_3-w_1)\sin^2\theta\right]\Big\}^{1/2} \tag{5.3}$$
$$= 2F^2\sin^2\theta\,\left|(w_1-w_2)u^2-(w_1-w_3)\right| = \sin^2\theta\,|P|^2\;.$$

We have shown explicitly that this parametrically-determined function Ω does indeed satisfy the sDiff(2) Toda equation, as described in Eqs. (1.2), where the function $\mathbf{N}(\mu,p)$ is given in Eqs. (3.17) while the general functions $w_i(\mu)$, that determine the dependence of the metric on the transverse coordinate μ, are given in terms of the conformal factor $F = c_0(\mu+d_0)$ and the solutions to the Halphén problem, $a_i(\mu)$, in Eqs. (2.5). The functions $a_i(\mu)$ themselves are given in their most general form in Eqs. (A3.16), which constitute a (conformal) Möbius transformation of the $i\mu$ upper complex half-plane of the simpler form given in Eqs. (2.4).

This is therefore the most general solution of the Plebański equation that allows SU(2) symmetry. It is hoped that this idea of parametric solutions to a very complicated nonlinear partial differential equation will be a useful one, and will engender ways to determine yet more general solutions, with smaller symmetry groups. The aim of determining such solutions has been the goal of M.B. Sheftel and co-workers for some years, with, for example, progress in that direction shown in fairly-recent work of theirs,[26] which uses partner symmetries.

Appendix I: Calculations for the curvature of a vacuum Bianchi IX metric

We believe it useful to show how the requirements of anti-self-duality of the conformal tensor, and the vanishing of the Ricci tensor, are turned into (ordinary) differential equations that must be satisfied by F and the three w_i's. Since the purpose of the conformal factor is to affect changes in the Ricci tensor, it is simpler to first determine these constraints by ignoring the function F, i.e., to set it just equal to $+1$, and then afterward use it to perform a conformal transformation of the metric, to achieve the desired results. We also note that since we are interested in the general case of complex-valued manifolds, for the sDiff(2)Toda equation, any particular form for the signature is not essential. Therefore, we will follow a fairly standard approach, and set up a tetrad with Riemannian signature, when variables are considered as real-valued, for our calculations, with $F = 1$:

$$\begin{gathered} \mathbf{g}\Big|_{F=1} \equiv (\underset{\sim}{\omega}^1)^2 + (\underset{\sim}{\omega}^2)^2 + (\underset{\sim}{\omega}^3)^2 + (\underset{\sim}{\omega}^4)^2 \;, \\ \underset{\sim}{\omega}^4 \equiv H\, d\mu \;, \quad \underset{\sim}{\omega}^i \equiv H \frac{\underset{\sim}{\sigma}_i}{w_I} \;, \quad i = 1,2,3 \;, \qquad H \equiv \sqrt{w_1 w_2 w_3} \;, \end{gathered} \tag{A1.1}$$

where we use an upper-case index along with a lower-case one when we need the same values, but to indicate that **no sum** on the values is intended.

To consider self-duality, of 2-forms, we also need a basis set for the vector spaces of 2-forms; we create the following two sets of triples

$$\underset{\sim}{E}^k_{\pm} \equiv \underset{\sim}{\omega}^i \wedge \underset{\sim}{\omega}^j \pm \underset{\sim}{\omega}^k \wedge \underset{\sim}{\omega}^4 \;, \quad i,j,k \;\text{ from } 1,2,3, \text{ and cyclic,} \tag{A1.2}$$

where the ones with the plus sign are a basis for self-dual 2-forms, and those with the minus sign are anti-self-dual. We use the word "cyclic" with a fairly standard meaning, i.e., to mean that the indices $\{i, j, k\}$ should always take distinct values and in cyclic order, so that they imply each of the three possible sets of values $1, 2, 3$, and $2, 3, 1$, and $3, 1, 2$.

The curvature, and the connection, of the manifold split into separate constituents for the self-dual and anti-self-dual parts: the connection for the two parts, which are determined by separate triplets of 1-forms, along with separate triplets of 2-forms to determine the curvature. The complete Cartan relations are of course written as follows:

$$d\underset{\sim}{\omega}^\alpha = \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\Gamma}^\alpha{}_\mu \;, \quad \underset{\sim}{\Omega}^\mu{}_\nu = d\underset{\sim}{\Gamma}^\mu{}_\nu + \underset{\sim}{\Gamma}^\mu{}_\sigma \wedge \underset{\sim}{\Gamma}^\sigma{}_\nu \equiv \tfrac{1}{2} R^\mu{}_{\nu\lambda\rho}\, \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\omega}^\rho \;. \tag{A1.3}$$

However, the separation into self-dual and anti-self-dual parts splits the connection and curvature into a pair of triplets of forms for each of them:

$$\underset{\sim}{G}^i_{\pm} \equiv \underset{\sim}{\Gamma}_{jk} \pm \underset{\sim}{\Gamma}_{i4} \;, \quad \underset{\sim}{\Omega}^i_{\pm} \equiv \underset{\sim}{\Omega}_{jk} \pm \underset{\sim}{\Omega}_{i4} = d\underset{\sim}{G}^i_{\pm} - \underset{\sim}{G}^j_{\pm} \wedge \underset{\sim}{G}^k_{\pm} \;, \tag{A1.4}$$

where again the upper subscripts are for the self-dual parts and the lower ones for the anti-self-dual parts. Inserting our tetrad, along with considerable calculation, gives us the two triplets of connections in the following explicit form:

$$\underset{\sim}{G}^i_{\pm} = \left(\frac{a_{\pm_I}}{H}\right)\underset{\sim}{\omega}^i \ , \qquad\qquad a_{\pm j} + a_{\pm k} \equiv \pm\frac{w_i'}{w_{_I}} + \frac{w_j w_k}{w_i} \ , \tag{A1.5}$$

where the new functions $a_{\pm k}(\mu)$ are defined by the above triplet of equations in terms of the original w_i's, and the prime denotes the derivative with respect to μ. [Note that within this approach the distinction between self-dual and anti-self-dual may be switched simply by switching the sign of the coordinate μ.] An immediate observation is that several of the connection coefficients are zero, since each of the two triplets of connections is spanned only by one basis vector, so that we have only 6 connection coefficients, instead of the maximal possible number of 24. These connections are then used to generate their respective curvatures, either self-dual or anti-self-dual, which gives us the following:

$$\begin{aligned}\underset{\sim}{\Omega}^i_{\pm} &= (R_{jkjk} \pm R_{jki4})\underset{\sim}{\omega}^{_J} \wedge \underset{\sim}{\omega}^{_K} + (R_{jki4} \pm R_{i4i4})\underset{\sim}{\omega}^{_I} \wedge \underset{\sim}{\omega}^{4} \equiv \mathcal{Z}_{\pm i}\underset{\sim}{\omega}^{_J} \wedge \underset{\sim}{\omega}^{_K} + \mathcal{K}_{\pm i}\underset{\sim}{\omega}^{_I} \wedge \underset{\sim}{\omega}^{4} \\ &= \frac{1}{H^2}\left\{\left(\frac{H^2}{w_{_I}}a_{\pm_i} - a_{\pm_J}a_{\pm_K}\right)\underset{\sim}{\omega}^j \wedge \underset{\sim}{\omega}^k - w_{_I}\left(\frac{a_{\pm_I}}{w_{_I}}\right)'\underset{\sim}{\omega}^i \wedge \underset{\sim}{\omega}^4\right\} \ .\end{aligned} \tag{A1.6}$$

Once again, since each of the $\underset{\sim}{\Omega}^i_{\pm}$ are spanned by only two of the basis 2-forms, namely $\underset{\sim}{\omega}^j \wedge \underset{\sim}{\omega}^k$ and $\underset{\sim}{\omega}^i \wedge \underset{\sim}{\omega}^4$, with all other possible coefficients zero, there appear to be only $3 \times 3 = 9$ non-zero components of the curvature tensor, namely R_{jkjk}, R_{jki4}, and R_{i4i4}. However, one of them is not linearly independent, since the first Bianchi identity causes those three of the form R_{jki4} to sum to zero, i.e., $R_{1234} + R_{2341} + R_{3412} = 0$, so only 8 of these are independent. There are also constraining relations between the convenient labels $\mathcal{Z}_{\pm i}$ and $\mathcal{K}_{\pm i}$. The most important of those are the following:

$$\mathcal{Z}_{+i} - \mathcal{Z}_{-i} = \mathcal{K}_{+i} + \mathcal{K}_{-i} \ , \qquad \sum_{i=1}^{3}(\mathcal{Z}_{+i} - \mathcal{Z}_{-i}) = 0 = \sum_{i=1}^{3}(\mathcal{K}_{+i} + \mathcal{K}_{-i}) \ . \tag{A1.7}$$

To divide these components further, we separate those 8 components into those that constitute the conformal tensor, $C_{\mu\nu\lambda\eta}$, and those that define the Ricci tensor, $\mathcal{R}_{\mu\nu}$, and its trace, $\mathcal{R}$:

$$\begin{aligned}C_{\mu\nu\lambda\eta} =& R_{\mu\nu\lambda\eta} - \tfrac{1}{2}\left(g_{\mu\lambda}\mathcal{R}_{\nu\eta} - g_{\mu\eta}\mathcal{R}_{\nu\lambda} + g_{\nu\eta}\mathcal{R}_{\mu\lambda} - g_{\nu\lambda}\mathcal{R}_{\mu\eta}\right) + \tfrac{1}{6}\left(g_{\mu\lambda}g_{\nu\eta} - g_{\mu\eta}g_{\nu\lambda}\right)\mathcal{R} \ , \\ &\mathcal{R}_{\nu\mu} = \mathcal{R}_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \ , \qquad \mathcal{R} \equiv R^{\mu\nu}{}_{\mu\nu} \ .\end{aligned} \tag{A1.8}$$

It is simpler, and equivalent, to divide the conformal tensor components into their self-dual and anti-self-dual parts on the basis of their definition as 2-forms, rather than their other pair of indices, using our two basis sets for 2-forms to accomplish this:

$$\mathcal{C}_{\pm\mu\nu} = \mathcal{C}_{i\pm\mu\nu} \mathcal{E}^i_{\pm} \; , \qquad \mathcal{C}_{\pm ij} = \mathcal{C}_{\pm k4} \; , \tag{A1.9}$$

where the second set of equalities is a generic statement about the symmetries of this tensor, which in the most general case only has 5 components for the self-dual part and another 5 for the anti-self-dual part. However, in our particular case, using the forms for the curvature tensor given in Eqs. (A1.6) we find that only the following two sets of triplets of components, $\mathcal{C}_{i\pm jk}$, are non-zero, i.e., those where all of i, j, and k are different. However, as well the sum of each triplet is also zero by the first Bianchi identity, so that we have only the two independent components for the self-dual part and another two for the anti-self-dual part:

$$\begin{aligned} 2H^2\, \mathcal{C}_{i\pm jk} &= 2\mathcal{W}_{\pm i} - \mathcal{W}_{\pm j} - \mathcal{W}_{\pm k} \; , \\ \mathcal{W}_{\pm i} &\equiv \mathcal{Z}_{i\pm} \pm \mathcal{K}_{i\pm} \\ &= H^2[R_{jkjk} \pm 2R_{jki4} + R_{i4i4}] = \mp a_{\pm i}{}' + a_{\pm i}(a_{\pm j} + a_{\pm k}) - a_{\pm j} a_{\pm k} \; . \end{aligned} \tag{A1.10a}$$

Once again only two of the three elements in either one of these two triplets is independent, since it is straightforward to see that

$$\mathcal{C}_{1\pm 23} + \mathcal{C}_{2\pm 31} + \mathcal{C}_{3\pm 12} = 0 \; . \tag{A1.10b}$$

The Ricci tensor of course involves components from both the self-dual and the anti-self-dual sides, which can most conveniently be described in terms of both the $\mathcal{W}_{\pm i}$ and also the additional parts $\mathcal{K}_{i\pm}$:

$$\begin{aligned} 2H^2\mathcal{R}_{44} &= \sum_{i=1}^{3} (\mathcal{K}_{+i} - \mathcal{K}_{-i}) = 2\sum_{i=1}^{3} \mathcal{K}_{+i} \; , \qquad 2H^2\mathcal{R} = 4\sum_{i=1}^{3} \mathcal{W}_{+i} \; , \\ 2H^2\mathcal{R}_{ii} &= -(a_{+i} - a_{-i})' + (a_{+j} + a_{-j})(a_{+k} + a_{-k}) \\ &= 2\left(\mathcal{W}_{+j} + \mathcal{W}_{+k} + 2\mathcal{K}_{+i} - H^2\mathcal{R}_{44}\right) \; . \end{aligned} \tag{A1.11}$$

We intend to require that the conformal curvature be anti-self-dual, which gives us **only** two constraints on the three $\mathcal{W}_{+i}$'s; in order to obtain a third one, we require as well that the scalar curvature, $\mathcal{R}$, vanish, which then requires all three of the $\mathcal{W}_{+i}$'s to vanish. This system of three equations, for the three unknown functions a_{+i}, is usually referred to as the Halphen system,[21] but also by various other names, including often Darboux and/or Brioschi, since there was considerable interest in their solution in the late part of the 19th century. Once we have distinct forms for the a_{+i}'s then Eqs. (A1.5) constitute a triplet of first-order differential equations to determine the form of the three w_i's which must also be solved. A general solution of these 6 equations has been studied by several authors; we prefer the particular form of the solution used by Babich and Korotkin[18]. In the case where the Ricci scalar vanishes and all three of the a_{+i}'s are distinct they show that the difference $w_i - a_{+i}$ is independent of i. Referring to that difference as $g = g(\mu) = 1/(\mu + q_0)$, with q_0 an arbitrary constant, we will now use that information to greatly simplify the previously-given forms for the traceless part of the Ricci tensor. We can use the ode's that we have assumed satisfied to re-write the auxiliary quantities $\mathcal{K}_{+i}$, in a very simple form:

$$\begin{aligned} \mathcal{K}_{+i} \equiv & -a'_{+i} + a_{+i}\frac{w'_I}{w_I} = a_{+j}a_{+k} - a_{+i}\frac{w_j w_k}{w_I} = (w_j - g)(w_k - g) - (w_i - g)\frac{w_j w_k}{w_i} \\ &= g^2 - g\left(w_j + w_k - \frac{w_j w_k}{w_i}\right) = -g(g + w'_i/w_I) \ , \end{aligned} \tag{A1.12}$$

This then allows simple expressions for the components of the Ricci tensor:

$$\begin{aligned} H^2\mathcal{R}_{44} = & \sum_{i=1}^{3}\mathcal{K}_{+i} = -g\sum_{i=1}^{3}(g + w'_i/w_I) = -g\left(3g + \sum_{i=1}^{3}\frac{w'_i}{w_I}\right) = -g(3g + 2H'/H) \ , \\ H^2\mathcal{R}_{ii} = & \left(\mathcal{W}_{+j} + \mathcal{W}_{+k} + 2\mathcal{K}_{+i} - H^2\mathcal{R}_{44}\right) \\ &= [\mathcal{K}_{+i} + g(3g + 2H'/H)] = g\{g + 2[\log(H/w_i)]'\} \ . \end{aligned} \tag{A1.13}$$

Since the sDiff2(Toda) equation generates metrics which have zero Ricci tensor, and the standard form of the Bianchi IX metric described above still has non-zero diagonal components of that tensor, we must now implement the conformal transformation of the metric generated by our function $F = F(\mu)$, to arrange for the Ricci tensor to vanish as well. We consider the re-scaled metric, and correspondingly re-scaled tetrad basis 1-forms, denoting the re-scaled tensors with a "hat" over the relevant symbols:

$$\delta_{\mu\nu}\,\hat{\underset{\sim}{\omega}}^\mu \underset{s}{\otimes} \hat{\underset{\sim}{\omega}}^\nu = \hat{\mathbf{g}} \equiv F^2\mathbf{g} = F^2\delta_{\mu\nu}\,\underset{\sim}{\omega}^\mu \underset{s}{\otimes} \underset{\sim}{\omega}^\nu \ , \quad \Longrightarrow \quad \hat{\underset{\sim}{\omega}}^\mu = F\,\underset{\sim}{\omega}^\mu \ . \tag{A1.14}$$

Such a transformation will leave invariant the tetrad components of the conformal tensor, $\hat{C}^\alpha{}_{\beta\gamma\delta}$. However, the general transformation of the tetrad components of the Ricci tensor and, separately and usefully, its trace, $\mathcal{R}$, is given as follows:

$$\begin{aligned} \hat{\mathcal{R}}_{\beta\delta} &= \mathcal{R}_{\beta\delta} + X_{\beta\delta} \ , \qquad \hat{\mathcal{R}} = F^{-2}\mathcal{R} + \mathcal{X} \ , \\ X_{\beta\delta} &= F\left(2\delta^\alpha_\beta\delta^\zeta_\delta + g_{\beta\delta}g^{\alpha\zeta}\right)\nabla_\alpha\nabla_\zeta F^{-1} - 3F^2\,g_{\beta\delta}\left[g^{\alpha\zeta}(\nabla_\alpha F^{-1})(\nabla_\zeta F^{-1})\right] \ , \\ \mathcal{X} &= -6F^{-3}g^{\beta\delta}\nabla_\beta\nabla_\delta F \ . \end{aligned} \tag{A1.15}$$

Since we want to maintain unchanged the current zero value for the Ricci scalar, we see that this is straightforward provided the function F is "harmonic." Now we show that a conformal factor which depends only on μ, therefore not disturbing our symmetry, is sufficient to annul the Ricci tensor as is desired. Such a dependence of course simplifies the equations greatly, giving us the following:

$$\begin{aligned} \mathcal{X}_{44} &= -\frac{1}{H^2}\left[3\frac{F''}{F} - 3\left(\frac{F'}{F}\right)^2 - 2\frac{F'H'}{FH}\right] , \qquad \mathcal{X}_{4i} = 0 , \\ \mathcal{X}_{ij} &= -\frac{1}{H^2}\delta_{ij}\left\{\frac{F''}{F} + \left(\frac{F'}{F}\right)^2 + 2\frac{F'}{F}[\log(H/w_I)]'\right\} , \qquad \mathcal{X} = -6\frac{F''}{H^2F} . \end{aligned} \tag{A1.16}$$

In this case the requirement that F be harmonic simply reduces to the requirement that it be a linear function of μ:

$$0 = \mathcal{X} \propto F'' \quad \Longrightarrow \quad F = c\mu + d , \tag{A1.17}$$

where c and d are constants. We next go to the earlier-determined, non-zero forms for the Ricci tensor components themselves, from Eqs. (A1.13), to see if this form for F will allow the transformed (trace-free) Ricci tensor to vanish. Beginning with $\mathcal{R}_{44}$, where we are now including the requirement that $F'' = 0$, we have

$$H^2F^2\hat{\mathcal{R}}_{44} = H^2\mathcal{R}_{44} + H^2X_{44} = -g(3g + 2H'/H) + (F'/F)[3F'/F + 2H'/H] . \tag{A1.18}$$

For this to vanish we need only to require that $F'/F = g$:

$$F = c_0(\mu + d_0) \quad \Longrightarrow \quad F'/F = \frac{1}{\mu + d_0} , \qquad \text{but} \;\; g = 1/(\mu + q_0) . \tag{A1.19}$$

Therefore, by setting the two previously arbitrary integration constants, q_0 and d_0 equal to each other, we have accomplished what was desired for $\mathcal{R}_{44}$. Since everything is now determined, we must now hope that this will also allow the remainder of the transformed components of the Ricci tensor to vanish:

$$\begin{aligned} H^2F^2\hat{\mathcal{R}}_{ii} &= H^2\mathcal{R}_{ii} + H^2X_{ii} \\ &= g(g + 2H'/H - 2w_i'/w_I) - (F'/F)[(F'/F) + 2H'/H - 2w_i'/w_I] . \end{aligned} \tag{A1.20}$$

We see that, yes, once again the choice that $F'/F = g$, is sufficient to cause these components to vanish as well, which tells us that the metric $\hat{\mathbf{g}} = (c_0/g)^2\mathbf{g}$ should also be capable of being generated by a solution to the Plebański equation, where $\mathbf{g}$ is the Bianchi IX metric described in Eqs. (2.1).

Appendix II: The (3-dimensional) Hodge dual in the spherical coordinates

We first recall that the Levi-Civita tensor, in 3 dimensions and with an arbitrary metric g, with Riemannian signature, is given by

$$\eta_{\alpha_1\alpha_2\alpha_3} = \sqrt{g}\,\epsilon[\alpha_1,\alpha_2,\alpha_3]\ , \qquad g \equiv \det(g_{ab}) \qquad \eta^{\alpha_1\alpha_2\alpha_3} = \frac{1}{\sqrt{g}}\,\epsilon[\alpha_1,\alpha_2,\alpha_3]\ , \tag{A2.1}$$

where the ϵ-symbol is the usual Levi-Civita alternating symbol that takes on only the values ± 1 and 0. Then the general form of the dual of a 1-form or a 2-form, under our metric $\boldsymbol{\gamma}$, is given by

$$\begin{aligned} \underset{\sim}{\alpha} = \tfrac{1}{2}\alpha_{ab}\,\underset{\sim}{\omega}^a \wedge \underset{\sim}{\omega}^b \quad &\Longleftrightarrow \quad *\,(\underset{\sim}{\alpha}) = \tfrac{1}{2}\boldsymbol{\gamma}^{ac}\boldsymbol{\gamma}^{bd}\alpha_{ab}\,\eta_{cdf}\,\underset{\sim}{\omega}^f\ , \\ \underset{\sim}{\beta} = \beta_a\,\underset{\sim}{\omega}^a \quad &\Longleftrightarrow \quad *\,(\underset{\sim}{\beta}) = \tfrac{1}{2}\boldsymbol{\gamma}^{ac}\beta_a\,\eta_{cdf}\,\underset{\sim}{\omega}^d \wedge \underset{\sim}{\omega}^f\ , \end{aligned} \tag{A2.2}$$

Our metric is given explicitly in Eqs. (3.4), but is perhaps best here presented in the following symbolic form:

$$\boldsymbol{\gamma} = ((\boldsymbol{\gamma}_{ij})) \ \Longrightarrow \ \begin{array}{c} \\ \mu \\ \theta \\ \psi \end{array} \begin{array}{c} \begin{array}{ccc} \mu & \theta & \psi \end{array} \\ \begin{pmatrix} \gamma_{\mu\mu} & 0 & 0 \\ 0 & \gamma_{\theta\theta} & \gamma_{\theta\psi} \\ 0 & \gamma_{\theta\psi} & \gamma_{\psi\psi} \end{pmatrix} \end{array}\ , \tag{A2.3a}$$

along with its determinant:

$$\det\boldsymbol{\gamma} = \gamma_{\mu\mu}\left(\gamma_{\theta\theta}\gamma_{\psi\psi} - \gamma_{\theta\psi}^2\right) = \gamma_{\mu\mu}(F^4\sin^2\theta\,\gamma_{\mu\mu}) = (\gamma_{\mu\mu}\,F^2\sin\theta)^2 = (M\,F^6\sin\theta)^2\ , \tag{A2.3b}$$

where the value of M is noted in Eq. (3.2) and in Eq. (3.4) it is noted that $\gamma_{\mu\mu} = MF^4$.

The simple form of this makes it very easy to present the inverse metric:

$$\boldsymbol{\gamma}^{-1} = \left((\boldsymbol{\gamma}^{ij})\right) \ \Longrightarrow \ \frac{\ell^2}{\gamma_{\mu\mu}}\begin{pmatrix} \ell^{-2} & 0 & 0 \\ 0 & \gamma_{\psi\psi} & -\gamma_{\theta\psi} \\ 0 & -\gamma_{\theta\psi} & \gamma_{\theta\theta} \end{pmatrix}\ , \qquad \ell \equiv \frac{1}{F^2\sin\theta}\ ,\ . \tag{A2.3c}$$

We then begin by calculating the duals of the three basis 2-forms and also, reciprocally, the three basis 1-forms:

$$\left.\begin{aligned} *(d\theta\wedge d\psi) &= \ell\,d\mu\ , \\ *(d\mu\wedge d\theta) &= \ell\left[\gamma_{\theta\psi}d\theta + \gamma_{\psi\psi}d\psi\right]/\gamma_{\mu\mu}\ , \\ *(d\mu\wedge d\psi) &= -\,\ell\left[\gamma_{\theta\theta}d\theta + \gamma_{\theta\psi}d\psi\right]/\gamma_{\mu\mu}\ , \end{aligned}\right\} \quad \left\{\begin{aligned} *d\mu &= \ell^{-1}d\theta\wedge d\psi\ , \\ *d\theta &= \ell\left[\gamma_{\psi\psi}\,d\psi\wedge d\mu - \gamma_{\theta\psi}\,d\mu\wedge d\theta\right]\ , \\ *d\psi &= \ell\left[\gamma_{\theta\theta}\,d\mu\wedge d\theta - \gamma_{\theta\psi}d\psi\wedge d\mu\right]\ , \end{aligned}\right. \tag{A2.4}$$

where the various coefficients of the 3-metric, $\boldsymbol{\gamma}$, may be found in Eqs. (3.4).

It is then straightforward to write down the duals of an arbitrary 1-form $\underset{\sim}{\alpha}$ and 2-form $\underset{\sim}{\beta}$:

$$\underset{\sim}{\alpha} \equiv H_1\, d\mu + H_2\, d\theta + H_3\, d\psi \ , \quad \underset{\sim}{\beta} = J_1\, d\theta \wedge d\psi + J_2\, d\mu \wedge d\theta + J_3\, d\mu \wedge d\psi$$

$$\Longleftrightarrow$$

$$\begin{aligned} *(\underset{\sim}{\alpha}) =& \ell^{-1} H_1 d\theta \wedge d\psi + \ell(\gamma_{\theta\theta} H_3 - \gamma_{\theta\psi} H_2) d\mu \wedge d\theta - \ell(\gamma_{\psi\psi} H_2 - \gamma_{\theta\psi} H_3) d\mu \wedge d\psi \ , \\ *(\underset{\sim}{\beta}) =& \ell J_1 d\mu + \frac{\ell}{\gamma_{\mu\mu}} \left[\gamma_{\theta\psi} J_2 - \gamma_{\theta\theta} J_3\right] d\theta + \frac{\ell}{\gamma_{\mu\mu}} \left[\gamma_{\psi\psi} J_2 - \gamma_{\theta\psi} J_3\right] d\psi \ . \end{aligned} \tag{A2.5}$$

Using Eqs. (3.2-3) for $1/V$ and $\underset{\sim}{\omega}$, as well as the expressions for the coefficients of the metric $\boldsymbol{\gamma}$, in the expression Eq. (3.7), we obtain the expression for ds as given in Eq. (3.8) in the main text.

Appendix III: Integration for $q = q(\mu, \theta, p)$, and general Theta Functions

We intend here to show that the solution of the three (compatible) differential equations for q that are described in Eqs. (3.15) is in fact that given in Eq. (3.16), i.e.,

$$\begin{gathered} 2q = \tfrac{1}{2} P \cos\theta + \mathbf{N}(\mu, p) \ , \qquad P \sin\theta\, (2\, dq) = \underset{\sim}{e}_+ = \mathcal{A}\, d\mu + \mathcal{B}\, d\theta + \mathcal{C}\, dp \ , \\ P^2 \equiv 2F^2[(w_1 - w_2)\cosh^2 p - (w_1 - w_3)] \ . \end{gathered} \tag{A3.1}$$

We begin by inserting just the first term of the form for $2q$ into the equation for $\underset{\sim}{e}_+$. All the dependence on the variables is given explicitly in this case, and we obtain:

$$\begin{aligned} P \sin\theta\, d(\tfrac{1}{2} P \cos\theta) &= -\tfrac{1}{2} P^2 \sin^2\theta\, d\theta + \tfrac{1}{4} \sin\theta \cos\theta\, d(P^2) \\ &= -\tfrac{1}{2} e^{2f}\, d\theta + F^2 \sin\theta \cos\theta (w_1 - w_2) \cosh p \sinh p\, dp + \tfrac{1}{4} \sin\theta \cos\theta\, \frac{\partial P^2}{\partial \mu}\, d\mu \\ &= -\tfrac{1}{2} e^{2f}\, d\theta + F^2 \sin\theta \cos\theta (w_1 - w_2) u \sqrt{u^2 - 1}\, dp \\ &\qquad + F^2 \sin\theta \cos\theta [w_3 (w_1 - w_2) u^2 + w_2 (w_3 - w_1)]\, d\mu \ . \end{aligned} \tag{A3.2}$$

These do in fact satisfy exactly all the $d\theta$ terms in $\underset{\sim}{e}_+$ and all those other terms in $\underset{\sim}{e}_+$ that have an explicit $\cos\theta$. Therefore we are left with the two remaining equations to determine the yet-unknown function $\mathbf{N} = \mathbf{N}(\mu, p)$:

$$\begin{aligned} P \frac{\partial \mathbf{N}}{\partial \mu} &= F^2 w_3 (w_1 - w_2) \cosh p \sinh p \ , \\ P \frac{\partial \mathbf{N}}{\partial p} &= F^2 [(w_1 - w_2) \cosh^2 p - w_1] \ . \end{aligned} \tag{A3.3}$$

We begin this part of the problem by first determining an integral for the equation involving $\partial \mathbf{N} / \partial p$:

$$
\begin{aligned}
\sqrt{2}\,\mathbf{N} = F \int dp \sqrt{(w_1 - w_2)\cosh^2 p - (w_1 - w_3)} \\
\qquad - F \int dp \frac{w_3}{\sqrt{(w_1 - w_2)\cosh^2 p - (w_1 - w_3)}} \\
= \sqrt{w_1 - w_3} F \int dp \sqrt{k^2 \cosh^2 p - 1} - \frac{w_3}{\sqrt{w_1 - w_3}} F \int \frac{dp}{\sqrt{k^2\cosh^2 p - 1}} \\
= F \left[\sqrt{(w_1 - w_3)} E(u,k) + \frac{w_3}{\sqrt{w_1 - w_3}} F(u,k) \right] ,
\end{aligned}
\tag{A3.4}
$$

where $F(u,k)$ and $E(u,k)$ are the standard first and second incomplete elliptic integrals, and $(w_1 - w_2)/(w_1 - w_3) = k^2$ is the Jacobi modulus for Jacobi elliptic functions, as will be shown below:

$$
\begin{aligned}
F(z,k) \equiv \int_0^z \frac{da}{\sqrt{(1-a^2)(1-k^2a^2)}} = \int_0^{\sin^{-1} z} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \int_0^{\mathrm{sn}^{-1}(z,k)} dw , \\
E(z,k) \equiv \int_0^z da \frac{\sqrt{1-k^2a^2}}{\sqrt{1-a^2}} = \int_0^{\sin^{-1} z} d\theta \sqrt{1 - k^2\sin^2\theta} = \int_0^{F(z,k)} dw \, \mathrm{dn}^2(w,k) .
\end{aligned}
\tag{A3.5}
$$

Although this is indeed the desired solution, it may be put into much more reasonable forms provided we now make explicit use of our solutions for the functions w_i, and also the a_i, as noted in Eqs. (2.4) and (2.5), in terms of $F = c_0(\mu + d_0)$ and the theta coefficients. However, those coefficients may also be expressed in terms of complete elliptic integrals:[23]

$$
\begin{aligned}
\pi a_3(\mu) = 2\pi i \frac{d}{d\mu} \log \vartheta_2(0 \mid i\mu) = -2\,K(k)E(k) , \\
\pi a_2(\mu) = 2\pi i \frac{d}{d\mu} \log \vartheta_3(0 \mid i\mu) = -2\,K(k)[E(k) - k'^2 K(k)] , \\
\pi a_1(\mu) = 2\pi i \frac{d}{d\mu} \log \vartheta_4(0 \mid i\mu) = -2\,K(k)[E(k) - K(k)] , \\
w_i(\mu) = a_i(\mu) + \frac{d}{d\mu} \log F(\mu) = a_i(\mu) + \frac{1}{\mu + d_0} , \\
k'^2 \equiv 1 - k^2 , \qquad \mu = -i\tau = K'(k)/K(k) ,
\end{aligned}
\tag{A3.6}
$$

where the extra factor of π appears because of our normalization, following Hancock,[23] for the arguments of the theta functions. Normalizations vary considerably from author to author, concerning the arguments of these functions. We will present ours at the end of this section.

This allows us to present the two differences of functions w_i that appear in our integral in a different way, more easily showing the values for the integration being performed for $\mathbf{N}$:

$$
w_1 - w_3 = \frac{2}{\pi} K^2(k) , \qquad w_1 - w_2 = \frac{2}{\pi} k^2 K^2(k) , \tag{A3.7}
$$

which allows for the following re-presentation of the result for $\mathbf{N}$:

$$\sqrt{\pi}\,\mathbf{N} = c_0(\mu + d_0)\,[K(k)E(u,k) - E(k)F(u,k)] + \tfrac{\pi}{2}c_0\frac{F(u,k)}{K(k)} \ , \tag{A3.8}$$

It is of course true that there might also be some "constant of integration," which would depend on μ. We show that no such constant is needed by inserting this value for $\mathbf{N}$ back into the differential equation involving its derivative with respect to μ, and finding that it gives exactly the desired right-hand side; i.e., the pde is satisfied exactly with the value given above.

We would like, however, to present this result also in some other formats, where the dependence on μ is made more explicit. The simplest next step is to turn it into a form involving the Jacobi Zeta function, $Z(w,k)$,[23,24] and then use its relationship to the theta functions with both arguments non-zero.

$$\begin{aligned} Z(w,k) &= E(w,k) - \frac{E(k)}{K(k)}w \ , \\ K(k)Z[F(u,k),k] &= K(k)E[F(u,k),k] - E(k)F(u,k) \ , \\ K(k)Z[2K(k)z;k] &= \tfrac{1}{2}\frac{d}{dz}\log\,\vartheta_4(z \mid i\mu) \ , \\ K(k)Z(a,k) &= \Pi_1[1,a,k] \equiv \Pi_1(a,k) \ , \end{aligned} \tag{A3.9}$$

where the last line shows the relationship between the Jacobi Zeta function and the complete elliptic integral of the third kind, in the form originally given by Jacobi, and used by Whittaker and Watson.[24] When the first argument is 1 the integral is referred to as *complete*, and the value of that argument is often not shown; however, especially with Π_1, the first argument is often[24] given in terms of $F(z,k)$ instead of z as has been done here, so that then that argument would be $K(k)$. The more usual form of the integral of the third kind is the form due to Legendre, $\Pi(z,\alpha^2,k)$, which was used by Olivier,[16] in his integration for the coordinates q, and $\tilde{q}$, in the special case when our conformal factor F is simply a constant. The Legendre form is related to the Jacobi form as follows, where the coefficients are various Jacobi elliptic functions,[24] and we also give their basic definitions, as integrals:

$$\begin{aligned} \Pi(z,\alpha^2,k) &\equiv \int_0^z \frac{dt}{(1-\alpha^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{F(z,k)} \frac{dv}{1-\alpha^2\,\mathrm{sn}^2(v,k)} \ , \\ \Pi_1(z,a,k) &\equiv k^2\,\mathrm{sn}(a,k)\,\mathrm{cn}(a,k)\,\mathrm{dn}(a,k)\int_0^{F(z,k)} dv\,\frac{\mathrm{sn}^2(v,k)}{1-k^2\,\mathrm{sn}^2(a,k)\,\mathrm{sn}^2(v,k)} \ , \ , \\ \Pi_1[z,a,k] &= \frac{\mathrm{cn}(a,k)\,\mathrm{dn}(a,k)}{\mathrm{sn}(a,k)}\left\{\Pi[z,k^2\,\mathrm{sn}^2(a,k),k] - z\right\} \ . \end{aligned} \tag{A3.10}$$

This allows us to re-write our desired function $\mathbf{N}$ in several different, equivalent ways, where we choose the following one as most useful for our purposes:

$$\sqrt{\pi}\,\mathbf{N}(\mu,p) = c_0\left[\pi z + \tfrac{1}{2}(\mu+d_0)\frac{d}{dz}\log\vartheta_4(z\mid i\mu)\right]\Big|_{z=\frac{1}{2}F(\cosh p,k)/K(k)}\ , \qquad (A3.11)$$

where we recall here the relation of p and u to the original spherical coordinates, θ and ψ,

$$p \equiv \log\tan(\theta/2) + i\psi\ , \qquad u \equiv \cosh p = -\frac{\sin\psi + i\cos\psi\cos\theta}{\sin\theta}\ , \qquad (A3.12)$$

and, in terms of these variables, our important quantity $P^2(\mu,p)$ is given by

$$P^2(\mu,p) = \frac{e^{2f}}{\sin^2\theta} = -\tfrac{1}{\pi}\left[2c_0(\mu+d_0)K(k)\,\mathrm{dn}(a,k)\right]^2\ . \qquad (A3.13)$$

Completing our picture we put here the complete definitions for the theta functions, which are everywhere analytic functions of their first argument, z, periodic with period 1, while they are analytic in the upper half plane for their second argument, τ. As well, they have power series expressions which converge very fast and are generators of the usual Jacobi elliptic functions in that those functions are ratios of the theta functions:

$$\begin{aligned}
\vartheta_4(z|\tau) &= 1 + 2\sum_{n=1}^{+\infty}(-1)^n\, q^{n^2}\, \cos(2\pi n z)\ ,\\
\vartheta_3(z|\tau) &= 1 + 2\sum_{n=1}^{+\infty} q^{n^2}\cos(2\pi n z)\,,\\
\vartheta_2(z|\tau) &= \quad 2\sum_{m=0}^{+\infty} q^{(m+\frac{1}{2})^2}\, \cos[(2m+1)\pi z]\ ,\\
\vartheta_1(z|\tau) &= \quad 2\sum_{m=0}^{+\infty}(-1)^m q^{(m+\frac{1}{2})^2}\, \sin[(2m+1)\pi z]\\
&\qquad\qquad q \equiv e^{i\pi\tau} = e^{-\pi\mu}\ ,
\end{aligned} \qquad (A3.14)$$

and we note again that there are various other normalizations for the arguments of these functions, often including the factor π into the argument,[24] so that, then, they have period π.

A last thing to do here is to provide more detail as to how one acquires the other 3 parameters for the general solution[18] to the Halphen problem, via a Möbius transformation of the τ (upper) half-plane, accompanied by appropriate transformations of the dependent functions, where in this brief section we use the "overbar," as in $\overline{\tau}$, to indicate the result after the transformation, rather than it having any relation to the complex conjugation operation:

$$\left.\begin{aligned} \tau &\longrightarrow \overline{\tau} \equiv \frac{a\tau+b}{c\tau+d}\ , \\ a_{+i} &\longrightarrow \overline{a}_{+i}(\overline{\tau}) \equiv (c\tau+d)^2 a_{+i}[\tau(\overline{\tau})] + c(c\tau+d)\ , \\ w_i &\longrightarrow \overline{w}_i(\overline{\tau}) \equiv (c\tau+d)^2 w_i[\tau(\overline{\tau})]\ , \end{aligned}\right\} ; \qquad ad-bc=+1\ . \tag{A3.15}$$

Therefore, when we include these 3 parameters, the general solution has the following form, where we retain the overbars:

$$\begin{aligned} \overline{a}_{+i}(\overline{\tau}) &= 2\frac{d}{d\overline{\tau}}\log\vartheta_{5-i}\left(\frac{d\overline{\tau}-b}{a-c\overline{\tau}}\right) + \frac{c}{a-c\overline{\tau}} \\ \overline{w}_i(\overline{\tau}) &= \frac{(c\tau+d)^2}{\tau+q_0} + \overline{a}_{+i}(\overline{\tau}) - c(c\tau+d) = \overline{a}_{+i}(\overline{\tau}) + \frac{1}{\overline{\tau}+\overline{q}_0}\ ; \quad \overline{q}_0 = \frac{aq_0-b}{d-cq_0}\ . \end{aligned} \tag{A3.16}$$

If one makes the particular, allowed choice of the 3 parameters in the transformation, of $d = a = 1$ and $c = 0 = b$, then this more general form reduces to our earlier, particular form, as expected.

References:

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2. The choice of anti-self-dual rather than self-dual is simply following earlier history. The difference is simply a choice of orientation, or, from the point of view of the (nonlinear) pde's to be discussed in this paper, it is simply an overall sign change of the independent variable on which the important functions depend.
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4. K.P. Tod and R.S. Ward, "Self-dual metrics with self-dual Killing vectors," Proc. R. Soc. London **A368**, 411-427 (1979). See also, for instance, G.W. Gibbons and Malcolm J. Perry, "New gravitational instantons and their interactions," Phys. Rev. D **22**, 313-321 (1980).
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