Estabrook-Wahlquist Prolongations and Infinite-Dimensional Algebras

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I. Estabrook-Wahlquist Prolongations and Zero-Curvature Requirements

Motivated by a desire to find new solutions of physically-interesting partial differential equations, we think of a k-th order pde as a variety, Y, of a finite jet bundle, $J^{(k)}(M,N)$, with M the independent- and N the dependent-variables for the pde. From this geometric approach, we can look for point symmetries, contact symmetries, generalized symmetries, or even non-local symmetries, where the system is prolonged further, to a fiber space over J^∞, with fibers W, where vertical flows map solution spaces of one pde into another, satisfied by the additional dependent variables, w^A, that coordinatize the fibers. The compatibility conditions for such flows to exist are *"zero-curvature conditions."* Solutions of these conditions may be found using the tangent structure or the co-tangent structure, over $J^\infty \times W$. We describe both, but follow the approach via differential forms, following Cartan,[1] , Estabrook and Wahlquist,[2] and Pirani,[3] believing that it provides better guides for the intuition, for complicated (systems of) pde's.

For a vector-field presentation, we choose a commuting[4] basis, $\{e_a\}$, for tangent vectors over M, and lift them to the total derivative operators, D_a, over J^∞. Provided the system of pde's is involutive, they will still commute when restricted to the variety Y^∞, described by the pde's, this restriction being denoted by $\overline{D}_a$. The further prolongation into the fibers W requires the addition of some vector fields vertical with respect to the fibers, i.e., $\underline{X}_a = \sum X^A_a(\partial/\partial w^A)$, with the X^A_a functions of both the jet variables and the $\{w^A\}$. Requiring that the $\overline{D}_a + \underline{X}_a$ commute, when restricted to $Y^{(\ell)} \times W$, for some ℓ, ensures that the w^A can act as pseudopotentials for that pde:[5,6]

$$0 \;=\; [D_a + \underline{X}_a \,,\, D_b + \underline{X}_b\,]\Big|_{Y^\infty \times W} = \left\{\overline{D}_a(X^C_b) - \overline{D}_b(X^C_a)\right\} \frac{\partial}{\partial w^C} \;+\; [\underline{X}_a \,,\, \underline{X}_b\,] \;. \tag{1.1}$$

The general solution for the $\underline{X}_a$ describes all possible fiber spaces, or *coverings*,[1] for this pde, As the construction gives the $\underline{X}_a$ the "form" of a connection, it is reasonable to refer to these equations as "zero-curvature" requirements; it is, however, a generalization of the usual approach,[7,8] since the $\underline{X}_a$'s are still only elements of an **abstract** Lie algebra of vector fields, with neither coordinates, nor even their number yet determined.

As an identity in the jet coordinates, Eqs. (1.1) determine several independent equations. Their solution describes the $\underline{X}_a$ as linear combinations of vector fields $\underline{W}_\alpha$ with coefficients depending on coordinates for $Y^{(\ell)}$, with the w^A-dependence encoded within a set of commutation relations among the $\{\underline{W}_\alpha\}$, as vector fields within the algebra of vector fields over W. The smallest subalgebra that faithfully reproduces the linear independence, and the values, of those commutators is the general solution to the covering problem; we believe it is a universal object for the given pde and others related to it, so that it may be used to characterize related classes of pde's.[6,9,10]

The isolation and identification of such algebras is an important part of the process of determining and understanding all the solutions of nonlinear pde's. Vector-field realizations will generate Bäcklund transformations, inverse scattering problems, etc.[10,11] Faithful realizations will usually involve infinitely many pseudo-potentials, making their identification somewhat difficult, and the first researchers did not consider the entire infinite-dimensional algebras. However, beginning with the work by van Eck,[12] and Estabrook,[13] on identification of the universal algebra for the KdV equation, the search for the infinite-dimensional algebras involved has been extended considerably by Hoenselaers and co-workers[10,14,15] , by Omote[16] , and by the group at Twente, who seem to have made this a studied art-form.[17] Ťhe dual approach, via differential forms, created by Estabrook and Wahlquist and built on the ideas of Cartan, begins with the 'contact module,' $\Omega^k(M,N) \subseteq [J^k(M,N)]_*$, generated by the following set of 1-forms:

$$\Omega^k(M,N) : \left\{ \begin{array}{l} \theta^\mu = dz^\mu - z^\mu_a dx^a, \\ \qquad\qquad \cdots \\ \theta^\mu_{a_1 a_2 \ldots a_{k-1}} = dz^\mu_{a_1 a_2 \ldots a_{k-1}} - z^\mu_{a_1 a_2 \ldots a_{k-1} a_k} dx^{a_k} \end{array} \right\} \equiv \{\theta^\mu_\sigma \mid |\sigma| = 0, \ldots, k-1\} \tag{1.2}$$

where the summation convention is being used, and a choice for a local coordinate chart is $\left\{x^a, z^\mu, z^\mu_a, z^\mu_{a_1 a_2}, \ldots, z^\mu_{a_1 \ldots a_k}\right\}$, with z^μ_σ standing for any of these (jet) coordinates except the independent variables themselves, x^a. The contact module 'remembers' the relation the coordinates of the jet bundle would have when they are pulled back by the lift of a function over M: $u : U \subseteq M \to N \implies (j^k u)^*(\Omega^k) = 0$. The ideal, $\mathcal{I}$, is the differential closure of the pullback of the contact module to Y, and constitutes the Cartan description of the original pde. For 2 independent variables, $\{x, y\}$, the EW procedure first chooses a **proper**, closed subideal, $\mathcal{K} \subset \mathcal{I}$, generated by a set of **2-forms**, $\{\alpha^r\}$, that still is effective at describing the given pde.[18,19] The new variables $\{w^A\}$ are adjoined by appending to $\mathcal{K}$ contact forms, $\{\omega^A\}$, for each of these pseudopotentials, and maintaining the ideal closed:

$$\begin{array}{c} \omega^A = - dw^A + F^A dx + G^A dy, \\ dF^A \wedge dx + dG^A \wedge dy = f^A{}_r \, \alpha^r + \eta^A{}_B \wedge \omega^B \quad , \end{array} \qquad A = 1, \ldots, N \quad . \tag{1.3}$$

To show "equivalence" with the zero-curvature equations, we first consider all of $J^{(k)}$, i.e., without a pde, and then "restrict" to Y. For simplicity considering quasi-linear pde's, we first select $\mathcal{L}$, generated by the wedge-products of all 1-forms in Ω^k with the dx^a. For $0 \leq |\sigma| \leq k-1$, $\mathcal{L}$, contains exactly one copy of each of dz^μ_σ. Labelling its coefficient, in Eqs. (1.1), by $(f^A)^a_{z^\mu_\sigma}$, we have

$$F^A_{,z^\mu_\sigma} = (f^A)^1_{z^\mu_\sigma} \quad , \quad G^A_{,z^\mu_\sigma} = (f^A)^2_{z^\mu_\sigma} \quad , \tag{1.4}$$

for each jet coordinate z^μ_σ, with no repetitions. Writing $(\eta^A{}_B)_{z^\mu_\sigma}$ for the components of the 1-forms $\eta^A{}_B$, Eqs. (1.4) also gives us

$$\begin{aligned} 0 = (\eta^A{}_B)_{w^C}\, dw^C \wedge dw^B \ , &\quad 0 = (\eta^A{}_B)_{z^\mu_\sigma} dz^\mu_\sigma \wedge dw^B \ , \\ F^A_{,w^B} dw^B \wedge dx = -(\eta^A{}_B)_x dx \wedge dw^B \ , &\quad G^A_{,w^B} dw^B \wedge dy = -(\eta^A{}_B)_y dy \wedge dw^B \ , \\ \Longrightarrow \qquad \eta^A{}_B = F^A_{,w^B} dx + G^A_{,w^B} dy \quad . & \end{aligned} \tag{1.5}$$

The only remaining part of Eqs. (1.4) are the coefficients of the basis 2-form $dx \wedge dy$:

$$-F^A_{,y} + G^A_{,x} = -F^B\, G^A_{,w^B} + G^B\, F^A_{,w^B} + \sum_{|\sigma|=0}^{k-1} \left\{ -z^\mu_{\sigma y} F^A_{,z^\mu_\sigma} + z^\mu_{\sigma x} G^A_{,z^\mu_\sigma} \right\} \quad . \tag{1.6}$$

Introducing vertical vector fields, $\mathbf{F} \equiv (F^A)\partial/\partial w^A$ and $\mathbf{G} \equiv (G^A)\partial/\partial w^A$, so that the first two terms on the right hand side are the components of a commutator, this becomes

$$[D_x + \mathbf{F}, D_y + \mathbf{G}] = [\partial_x + \mathbf{F}, \partial_y + \mathbf{G}] + \sum_{|\sigma|=0}^{k-1} \left\{ -z^\mu_{\sigma y} \mathbf{F}_{,z^\mu_\sigma} + z^\mu_{\sigma x} \mathbf{G}_{,z^\mu_\sigma} \right\} = 0 \quad . \tag{1.6$'$}$$

This has the same form as Eqs. (1.1), except that we must still effect the restriction to some variety Y. The resulting EW ideal, $\mathcal{K}$, will be defined over $Y^{(k-1)} \equiv Y \cap J^{(k-1)}$, so that it reduces the problem in an important way; $\mathbf{F}$ and $\mathbf{G}$ will depend on several fewer variables—only the coordinates for $Y^{(k-1)}$, which we select by choosing "co-coordinates" for Y, i.e., those jet coordinates the pde's will be used to eliminate, as a method of (locally) defining $Y \subset J^{(k)}$.[20] If the restriction of $\mathcal{L}$ to Y removes all the k-th level coordinates, then it can be taken as $\mathcal{K}$. Otherwise, the remaining highest derivatives must still be removed from the system, which is always possible, although the methods depend on the particular pde. For a simple, quasi-linear evolution equation, we choose $z_y = H\, z_{(k)} + K$ as our co-coordinate, where H and K are functions over $Y^{(k-1)}$. Restriction to Y then causes $z_{(k)}$ to appear twice: in $(dz - z_y\, dy) \wedge dx$ and also $(dz_{(k-1)} - z_{(k)}\, dx) \wedge dy$. The following replacement process, followed by dropping the second 2-form above, reduces $\mathcal{L}$ to $\{Y^{(k-1)}\}^*$, as desired:

$$\begin{aligned} &(dz - z_y\, dy) \wedge dx \to \left\{ dz - (H\, z_{(k)} + K)\, dy \right\} \wedge dx \\ &\equiv dz \wedge dx + H\, dz_{(k-1)} \wedge dy - K\, dy \wedge dx \ \text{mod}\ (dz_{(k-1)} - z_{(k)}\, dx) \wedge dy\,. \end{aligned} \tag{1.7}$$

For an evolution equation, the above process is unique, and the EW process gives exactly the same results as that using vector fields; however, in general the situation is different. As a second example, we consider a pde that defines Y via the equation $z_{yy} = H\,z_{(k)} + J\,z_{(k-1),y} + K$, where the integers in parentheses indicate the number of x-derivatives, and H, J, and K are defined over $Y^{(k-1)}$. The earlier replacement process now **has more than one path allowed**:

$$
\begin{aligned}
(dz_y - z_{yy}\,dy)\wedge dx \quad &\longrightarrow \quad dz_y \wedge dx - K\,dy\wedge dx + H\,dz_{(k-1)}\wedge dy \\
&\qquad + J\begin{cases} dz^{\mu}_{(k-2),y}\wedge dy\,, & \text{Option 1} \\ -dz^{\mu}_{(k-1)}\wedge dx\,, & \text{Option 2 (equiv. to 1)} \\ \tfrac{1}{2}\big(z^{\mu}_{(k_a-1)}\wedge dy - z^{\mu}_{(k_b-1)}\wedge dx\big)\,, & \text{Option 3, symmetric,} \\ & \quad\text{but inequivalent.} \end{cases}
\end{aligned}
\tag{1.8}
$$

The two inequivalent paths each generate acceptable EW ideals, but lead to distinct prolongation structures. A third example pde with inequivalent prolongation structures is the sine-Gordon equation, as described below.

We describe the generators in $\mathcal{K}$ as a set of contact 2-forms for only those coordinates needed for $Y^{(k-1)}$, **and** a (set of) "dynamical 2-forms" for each pde in the system. The ideal $\mathcal{K}$ provides us a geometrically-motivated structure for knowing on which jet-variables we need no dependence; as well, the Lagrange multipliers, ${f^A}_r$, expressed in terms of derivatives of the F^A and G^B, as in Eqs. (1.4), tell us on which of the coordinates they **must** depend. (The curvature should vanish **only** when it is restricted to Y, so that the w^A are truly pseudopotentials for the pde; within the functional form of the curvature, the ${f^A}_r$ multiply the 2-form expression of the pde. The remaining information is then the commutator equation, Eq. (1.6′).

II. $\mathbf{u_{xy} = f(x, y; u)}$: The sine-Gordon and Robinson-Trautman equations

Using the generalized form of the equation, with any choice of f, we begin with the ideal, $\mathcal{L}$, as just described. Using the pde to replace u_{xy} within $\mathcal{L}$ leaves us with the largest possible EW ideal, with 4 generators. This ideal generates exactly the vector-field commutator equations,[6] and is too large to allow us to solve the resulting equations. However, Pirani,[3] has shown the existence of two distinct, useful sub-ideals:

$$
\mathcal{K}_1 : \left\{ \begin{matrix} (du - p dx)\wedge dy \\ (dp - f\,dx)\wedge dx \end{matrix} \right\}, \qquad \mathcal{K}_2 : \left\{ \begin{matrix} (du - p\,dx)\wedge dy \\ (du - q\,dy)\wedge dx \\ dp\wedge dx - dq\wedge dy + 2\,f\,dx\wedge dy \end{matrix} \right\}. \tag{2.1}
$$

For the case $f = f(u)$, only, such as the sine-Gordon equation, we have

$$
\begin{gathered}
[\mathbf{F}\,,\,\mathbf{G}] = -p\,\mathbf{G}_u + q\,\underline{\mathrm{F}}_u + f(u)(\underline{\mathrm{F}}_p - \mathbf{G}_q)\,, \\
\text{with}\;\; \underline{\mathrm{F}} = \underline{\mathrm{F}}(u,p;w^A)\,, \quad \mathbf{G} = \mathbf{G}(u,q;w^A)\,, \\
\textbf{and}\quad \mathcal{K}_1 : \quad \underline{\mathrm{F}}_u = 0 = \mathbf{G}_q\,, \qquad \mathcal{K}_2 : \quad \underline{\mathrm{F}}_p + \mathbf{G}_q = 0\,.
\end{gathered}
\tag{2.2}
$$

These equations are of quite a different character than those for evolution equations. For an evolution equation, the constraints resolve the dependence over Y by the solution of algebraic equations; however, for higher order equations, vector-field-valued pde's must be solved, substantially increasing the difficulty of the problem.[21] Nonetheless, with relatively minor assumptions, the reductions generated by these smaller algebras allow us to resolve these equations as the "flow" of one vector field along the direction described by another, and to express the solutions explicitly in terms of the adjoint operation of one field upon another.[22] We describe the general solutions of each, labelling the resultant algebras by $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and the prolongations $\{\mathbf{F},\mathbf{G}\}$ within them as $\{\mathcal{F}_i,\mathcal{G}_i \mid i=1,2\}$. We also show that $\mathfrak{A}_2$ is gauge equivalent to a subalgebra of $\mathfrak{A}_1$, and that a subalgebra of that is homomorphic to $A_1^{(1)}$. Important earlier work on *infinite* versions of these algebras was done by Hoenselaers,[10,15] by Leznov and Saveliev,[8] by Dodd and Gibbon[23] for $\mathcal{A}_1$, and by Shadwick,[24] for $\mathcal{A}_2$.

With details in Ref. 22, the solution for $\mathfrak{A}_1$ may be found by first defining $\mathbf{Z} \equiv \mathcal{G}_1 + (\mathcal{G}_1)_{uu}$, which requires $[\mathcal{F}\,,\,\mathbf{Z}] = -p\,\mathbf{Z}_u$. Expanding $\mathcal{F}$ about the origin and writing $\mathbf{F}_n$ as the coefficient of $p^n/n!$, the flow equations tell us that

$$\mathbf{Z} = \mathbf{e}^{-u\,(\mathrm{ad}\,\underline{\mathbf{F}}_{\mathbf{1}})}\mathbf{Q}_0 \quad , \qquad [\underline{\mathrm{F}}_n\,,\,(\mathrm{ad}\,\mathbf{F}_{\mathbf{1}})^m\mathbf{Q}_0] = 0, \quad , \quad n \neq 1\,,\, m = 0,1,2,\dots \quad , \tag{2.3}$$

where $\mathbf{Q}_0 \in W_*$ is a "constant" of the integration. Continuing in this mode, the general solution is

$$\mathcal{F}_1 - \underline{\mathrm{F}}_0 + p\underline{\mathrm{F}}_1 = \int_0^p ds\,(p-s)\,e^{s\,(\mathrm{ad}\,\mathbf{G}_{\mathbf{1}})}\mathbf{K}_0 = \sum_{n=0}^{\infty}\frac{p^{n+2}}{(n+2)!}(\mathrm{ad}\,\mathbf{G}_{\mathbf{1}})^n\mathbf{K}_0\,,$$

$$\mathcal{G}_1 - \mathbf{G}_0\cos u - \mathbf{G}_1\sin u = \int_0^u dw\sin(u-w)e^{-w\,(\mathrm{ad}\,\underline{\mathbf{F}}_{\mathbf{1}})}\mathbf{Q}_0 = \sum_{n=0}^{\infty}(-\cos u)^{(-n-2)}(\mathrm{ad}\,\mathbf{F}_{\mathbf{1}})^n\mathbf{Q}_0$$

and requirements on **(only)** some of the commutators:

$$[\underline{\mathrm{f}}_0,\underline{\mathrm{g}}_0] = 0\,,\ [\underline{\mathrm{f}}_0,\underline{\mathrm{g}}_1] = -\underline{\mathrm{f}}_1\,,\ [\underline{\mathrm{f}}_0,\underline{\mathrm{q}}_n] = 0\,,\ [\underline{\mathrm{f}}_1,\underline{\mathrm{g}}_0] = \underline{\mathrm{g}}_1\,,\ [\underline{\mathrm{f}}_1,\underline{\mathrm{g}}_1] = -\underline{\mathrm{g}}_0 + \underline{\mathrm{q}}_0 - \underline{\mathrm{k}}_0\,,$$
$$[\underline{\mathrm{f}}_1,\underline{\mathrm{q}}_n] = -\underline{\mathrm{q}}_{n+1}\,,\ [\underline{\mathrm{k}}_m,\underline{\mathrm{g}}_0] = 0\,,\ [\underline{\mathrm{k}}_m,\underline{\mathrm{g}}_1] = -\underline{\mathrm{k}}_{m+1}\,,\ [\underline{\mathrm{k}}_m,\underline{\mathrm{q}}_n] = 0$$
$$\mathbf{K}_n \equiv (\mathrm{ad}\,\mathbf{G}_{\mathbf{1}})^n\ \mathbf{K}_0 \quad , \qquad \mathbf{Q}_m \equiv (-1)^m(\mathrm{ad}\,\mathbf{F}_{\mathbf{1}})^m\ \mathbf{Q}_0\ .$$

Alternatively, the equations that define $\mathfrak{A}_2$ first tell us that

$$\mathfrak{A}_2 : \begin{cases} \mathcal{F}_2 = \frac{1}{2}p\mathbf{R} + \mathbf{B} \quad , \qquad \mathcal{G}_2 = -\frac{1}{2}q\mathbf{R} + \mathbf{C}, \qquad \mathbf{R}_u = 0 \quad , \\ \left[\frac{1}{2}\mathbf{R}\,,\,\mathbf{B}\right] = \mathbf{B}_u \quad , \qquad \left[\frac{1}{2}\mathbf{R}\,,\,\mathbf{C}\right] = -\mathbf{C}_u \quad , \qquad [\mathbf{B}\,,\,\mathbf{C}] = \mathbf{R}\,f(u). \end{cases} \tag{2.4}$$

Integration of the differential equations gives two new, vertical vector fields such that

$$\mathbf{B} = e^{+\frac{1}{2}u((\mathrm{ad}\,\mathbf{R}))}\mathbf{E}\,, \quad \mathbf{C} = e^{-\frac{1}{2}u((\mathrm{ad}\,\mathbf{R}))}\mathbf{J} \quad . \tag{2.5}$$

Defining iterated commutators, $\mathbf{E}_m \equiv (+\mathrm{ad}\mathbf{R})^m\mathbf{E}\,,\ \mathbf{J}_n \equiv (-\mathrm{ad}\mathbf{R})^n\mathbf{J}$, and setting c_n as the coefficients multiplying $u^n/n!$ in the power series for $f(u)$, the content of the commutator equation becomes simply

$$\big[\mathbf{E}_{k-m},\, \mathbf{J}_m\big] = c_k\,\mathbf{R}\,, \quad \forall\, m \ni 0 \leq m \leq k \quad . \tag{2.6}$$

Even when divided by the countably infinite set of relations in Eqs. (2.6), the free algebra generated by $\{\mathbf{J},\mathbf{R},\mathbf{E}\}$, is so far unidentified as an already-studied algebra. In fact, those relations do not appear to be consistent with the usual sorts of gradings, so that some distinct approach to infinite-dimensional algebras may be required. We mention two quite different avenues that can be followed at this point. If one no longer requires all the $\underset{\sim}{\mathrm{E}}_m$ to be linearly independent, then the homomorphic mapping to $A_1^{(1)}$ can be demonstrated. Alternatively, maintenance of linear independence seems to lead in the direction of Toeplitz algebras of operators over Banach spaces.

Viewing $\mathbf{F}$ and $\mathbf{G}$ as the components of a Lie-algebra-valued connection 1-form, Γ, over the covering spaces, a gauge transformation generated by a vertical vector field, say $\mathbf{S}$, would transform Γ by $\Gamma_t \equiv e^{t((\mathrm{ad}\,\mathbf{S}))}\Gamma - d(t\,\mathbf{S})$. Choosing $\underset{\sim}{\mathrm{S}} = -\frac{1}{2}\mathbf{R}$, transforms $\mathfrak{A}_2$ into that part of $\mathfrak{A}_1$ generated by setting $\underset{\sim}{\mathrm{K}}_0 = 0$.

$$\Xi:\ (\Phi_{-\frac{1}{2}u})_*(\mathfrak{A}_2) \to (\mathfrak{A}_1)\Big|_{\mathbf{K}=0}:\ \begin{cases} \mathbf{R}\to\mathbf{F}_1\,,\ \underset{\sim}{\mathrm{E}}_0\to\mathbf{F}_0\,,\ \underset{\sim}{\mathrm{J}}_0\to\mathbf{G}_0\,, \\ \qquad \underset{\sim}{\mathrm{J}}_1\to -\,\mathbf{G}_1\,,\ \ldots\,,\ \underset{\sim}{\mathrm{J}}_j\to\ \mathbf{Q}_{j-2}-\underset{\sim}{\mathrm{J}}_{j-2}\,. \end{cases} \tag{2.7}$$

Hoenselaers' homomorphism[10] of part of $\mathfrak{A}_1$ into $A_1^{(1)}$ can now be extended to $\mathfrak{A}_2$:

$$\mathbf{E}_m \equiv (\mathrm{ad}\,\underset{\sim}{\mathbf{R}})^m\mathbf{E}\ \underset{\Xi}{\longrightarrow}\ (\mathrm{ad}\,\mathbf{F_1})^m\underset{\sim}{\mathrm{F}}_0\ \underset{\Psi}{\longrightarrow}\ \begin{cases} (-1)^{\frac{m-2}{2}}\,J_1^{(1)}, & \text{for m even,} \\ (-1)^{\frac{m-1}{2}}\,J_2^{(1)}, & \text{for m odd,} \end{cases}\ ,$$

$$[\underset{\sim}{\mathrm{E}}_m,\underset{\sim}{\mathrm{E}}_n]\ \underset{\Xi}{\longrightarrow}\ [(\mathrm{ad}\,\mathbf{F_1})^m\underset{\sim}{\mathrm{F}}_0\,,\,(\mathrm{ad}\,\mathbf{F_1})^n\underset{\sim}{\mathrm{F}}_0]\ \underset{\Psi'}{\longrightarrow}\ \begin{cases} 0\,, & m+n \text{ even,} \\ (-1)^{\frac{m+n-1}{2}}\,J_3^{(2)}\,, & m+n \text{ odd.} \end{cases}\ .$$

Triple commutators of the $\mathbf{E}_m$'s among themselves will generate elements of $A_1^{(1)}$ at the third level, etc. While interesting, this homomorphism loses much information contained within the larger algebra; for example, $\Psi'(\underset{\sim}{\mathrm{E}}_{m+2}) = -\Psi'(\underset{\sim}{\mathrm{E}}_m)\,,\,\Psi'(\underset{\sim}{\mathrm{J}}_{k+2}) = -\Psi'(\underset{\sim}{\mathrm{J}}_k)$, and it eliminates any information carried by the generators $\underset{\sim}{\mathrm{Q}}_i$ and $\underset{\sim}{\mathrm{K}}_j$.

A quite distinct approach would maintain linearly independent at least those of the generators that appear in $\mathcal{F}_2$ and $\mathcal{G}_2$, which leads us to consider that subalgebra spanned, as a vector space, on the countable list of generators $\{\mathbf{R},\mathbf{E}_m,\mathbf{J}_n \mid m,n = 0,1,\ldots\}$, therefore requiring the double, triple, etc. commutators to be linear combinations of these, such as $[\underset{\sim}{\mathrm{E}}_m,\underset{\sim}{\mathrm{E}}_n] = \left(A^i{}_m\right)_n\,J_i$. This is a quite distinct approach from that which led to $A_1^{(1)}$, where only $\{\underset{\sim}{\mathrm{R}},\underset{\sim}{\mathrm{E}}_0,\underset{\sim}{\mathrm{E}}_1,\underset{\sim}{\mathrm{J}}_0,\underset{\sim}{\mathrm{J}}_1\}$, of this original set, were linearly independent, and it was their commutators that generated the higher levels of the Kac-Moody algebra. Solving the constraints on the coefficients determining the linear combinations leads to the creation[22] of a Banach algebra of Toeplitz operators made from countable sums of those coefficients that define the commutators. This mode of thinking causes us to re-describe the countable set of repeated commutators, $\underset{\sim}{\mathrm{E}}_m$, in terms of a function $\underset{\sim}{\mathrm{E}}(t)$, defined within a Banach spaces of functions on the circle,[25] along with a set of integral equations, reminiscent of Weiner-Hopf equations,[25] for functions defined on $S^1 \times S^1$. The resulting prolongation forms, $\mathcal{F}_2$ and $\mathcal{G}_2$, would then be expressed as integrals over that circle:

$$\mathcal{F}_2 = \tfrac{1}{2} p\mathbf{R} - \frac{1}{2\pi i}\oint \frac{dt}{t}\, \underset{\sim}{\mathrm{E}}(t) e^{\frac{1}{2}ut} \quad , \quad \mathcal{G}_2 = -\tfrac{1}{2} q\mathbf{R} - \frac{1}{2\pi i}\oint \frac{dt}{t}\, \underset{\sim}{\mathrm{J}}(t) e^{-\frac{1}{2}ut} \tag{2.8}$$

We may now return to the case where f does depend on the independent variables; an important example for this case is the Robinson-Trautman equation for diverging, non-twisting, Petrov type III solutions of the vacuum Einstein field equations:

$$\text{RT equation of type III:} \;\; u_{xy} = \tfrac{1}{2}(x+y)e^{-2u} \quad . \tag{2.9}$$

Using the symmetric subideal, $\mathcal{K}_2$, in Eqs. (2.1), the RT version of Eqs. (2.2) is

$$\begin{gathered} \mathbf{F} = p\,\mathbf{Z} + \mathbf{B} \quad , \quad \mathbf{G} = -q\,\mathbf{Z} + \mathbf{C} \quad , \quad \mathbf{Z}_u = 0 \quad , \\ [\mathbf{Z}\,,\,\mathbf{C}] = -\mathbf{C}_u + \mathbf{Z}_y \quad , \quad [\mathbf{Z}\,,\,\mathbf{B}] = +\mathbf{B}_u + \mathbf{Z}_x \quad , \\ [\mathbf{B}\,,\,\mathbf{C}] = \mathbf{B}_y - \mathbf{C}_x + (x+y)e^{-2u}\mathbf{Z} \quad . \end{gathered} \tag{2.10}$$

Since the sine-Gordon equation is well-studied, we were interested in "all" of its subtleties. On the other hand, no interesting solutions of the RT equation are known; therefore we will look, first, for as simple a solution as possible. Obviously the prolongation structure must depend on $\{x, y\}$. We have considered two complementary cases in Ref. 26, integrating there these three-term pde's that generalize our earlier flow equations. We also show there that (at least) subalgebras for each case are gauge equivalent. Here, we only follow the simplest case, which assumes that $\mathbf{F}_y = 0 = \mathbf{G}_x$, thereby requiring that $\mathbf{Z}$ be independent of both x and y, and reducing our pde's to simple flows, which integrate as before. The additional assumption that all commutators of $\mathbf{Z}$ with their (respective) constants of integration are parallel gives us a system quite similar to the generators for $\mathbf{sl}(2)$, but with $\{x, y\}$-dependence:

$$\begin{gathered} \mathbf{B}(x,u) = e^{+u(\mathrm{ad}\,\mathbf{Z})}\,\mathbf{R}(x) \quad , \quad \mathbf{C}(y,u) = e^{-u(\mathrm{ad}\,\mathbf{Z})}\,\mathbf{S}(y) \quad , \\ [\mathbf{Z}, \mathbf{S}] = \mathbf{S} \quad , \quad [\mathbf{Z}, \mathbf{R}] = -\mathbf{R} \quad , \quad \Longrightarrow \quad [\underset{\sim}{\mathrm{R}}(x), \underset{\sim}{\mathrm{S}}(y)] = (x+y)\,\mathbf{Z} \quad . \end{gathered} \tag{2.11}$$

As the independent variables occur linearly, the simplest, non-trivial solution is for $\underset{\sim}{\mathrm{R}}$ and $\mathbf{S}$ to be linear polynomials, which causes the entire EW prolongation algebra to have a contragredient form:

$$
\begin{gathered}
\mathbf{R}(x) \equiv -\mathbf{f}_1 - x\mathbf{f}_2 \quad , \quad \mathbf{S}(y) \equiv +\mathbf{e}_2 + y\mathbf{e}_1 \quad , \\
[\mathbf{Z}, \mathbf{e}_i] = \mathbf{e}_i \quad , \quad [\mathbf{Z}, \mathbf{f}_i] = -\mathbf{f}_i \quad , \; i = 1, 2 \quad , \\
[\mathbf{e}_2, \mathbf{f}_1] = 0 = [\mathbf{e}_1, \mathbf{f}_2] \quad , \quad [\mathbf{e}_1, \mathbf{f}_1] = \mathbf{Z} = [\mathbf{e}_2, \mathbf{f}_2] \quad .
\end{gathered} \tag{2.12}
$$

The three Lagrange multipliers are now proportional to $\{\mathbf{R}(x), \mathbf{S}(y), \mathbf{Z}\}$. We must therefore determine a realization of the algebra defined by these 5 generators, which maintains these three linearly independent. At some level, this is a simple task, since this is in fact just the simplest contragredient algebra of infinite growth, usually referred to as K_2,[27] when one identifies our $\mathbf{Z}$ with its generator h. The generic contragredient algebra[27] has the (standard) form

$$
[\mathbf{e}_i, \mathbf{f}_j] = \delta_{ij}\mathbf{h}_i \quad , \quad [\mathbf{h}_i, \mathbf{h}_j] = 0 \quad , \quad [\mathbf{h}_i, \mathbf{e}_j] = A_{ij}\mathbf{e}_j \quad , \quad [\mathbf{h}_i, \mathbf{f}_j] = -A_{ij}\mathbf{f}_j \quad , \tag{2.13}
$$

where the A_{ij} are elements of a (generalized Cartan) matrix A. Our algebra K_2 is just the algebra described by the matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, divided by its center, $h_1 - h_2$. However, since no realizations of K_2 have yet been discovered, this is not the end of the task.

The infinite growth of K_2 is of course the difficulty To see how it affects the problem directly, we note that the Jacobi identity requires $[\mathbf{Z}, \{(\mathrm{ad}\,\mathbf{e_1})^n\mathbf{e}_2\}] = (n+1)\{(\mathrm{ad}\,\mathbf{e_1})^n\mathbf{e}_2\}$, and also $[\mathbf{f}_2, \{(\mathrm{ad}\,\mathbf{e_1})^{m+1}\mathbf{e}_2\}] = -\frac{1}{2}m(m+1)\{(\mathrm{ad}\,\mathbf{e_1})^m\mathbf{e}_2\}$. The first equality shows us the growth of the dimension of the space of i-th level commutators, **unless** the objects $\{(\mathrm{ad}\,\mathbf{e_1})^n\mathbf{e}_2\}$ were to vanish from some value of n onward. The second equality tells us that this would cause a downward cascade, leaving us with zero values for our Lagrange multipliers. To give precise definitions, we define an (integer)-graded Lie algebra as one that can be presented as a direct sum of subspaces, which the Lie bracket operation "preserves"; i.e., for $\mathcal{G} = \oplus\mathcal{G}_i$, we have $[\mathcal{G}_i\,,\,\mathcal{G}_j] \subseteq \mathcal{G}_{i+j}$. If d_n is the dimension of $\sum_{j=-n}^{n}\mathcal{G}_j$, then the (Gel'fand-Kirillov) growth,[28] r, of $\mathcal{G}$ is defined as $\overline{\lim}_{n\to\infty}\{\log d_n/\log n\}$. For K_2 one finds that d_n grows like 2^n, so that the resulting growth is infinite. More recently, Kirillov[29] has introduced the notion of algebras of intermediate growth, where $\log d_n$ grows like n^δ, for some $0 < \delta < 1$, and has shown that $Vect(R^m)$ is an algebra of intermediate growth, with $\delta = m/(m+1)$. We conclude from this that K_2, which has $\delta = 1$, will not have a realization within $Vect(R^m)$ for any finite value of m.

Nonetheless, the next step in the process of finding new solutions is to determine explicit realizations of this algebra, use the variables in the carrier space as pseudopotentials, pick out a Bäcklund transformation, take the one existing solution, and begin to generate new ones, as has been done many times before with many other pde's. Since this is indeed the minimal prolongation algebra, it seems reasonable to suppose that finding new solutions is equivalent to evolving realizations of this algebra. We hope to encourage listeners to achieve a faithful realization of K_2.

III. Even More Complicated Vector-Field PDE's

Quite an interesting system of vector-field-valued pde's has recently arisen in investigations of a student, Denis Khetselius, who originally came to UNM from Dubna. His work on twisting, Petrov type N vacuum solutions lead him to the following (involutive) set of **coupled** equations:

$$\begin{gathered}\mathcal{L}_j\mathcal{E}_i - \mathcal{A}_i\mathcal{M}_j = [\mathcal{E}_i, \mathcal{M}_j]\,, \quad \forall i,j = +,0,- \quad , \\ \mathcal{L}_+ \equiv a\partial_b - e\partial_f\,,\ \mathcal{L}_- \equiv b\partial_a - f\partial_e\,, \mathcal{A}_+ \equiv u\partial_w\,,\ \mathcal{A}_- \equiv w\partial_u\,, \\ \mathcal{L}_0 \equiv [\mathcal{L}_+, \mathcal{L}_-]\,, \quad \mathcal{A}_0 \equiv [\mathcal{A}_+, \mathcal{A}_-]\,,\end{gathered} \tag{3.1}$$

where the vector fields $\mathcal{M}_j$ and, separately, the $\mathcal{E}_i$ generate a realization of $\mathbf{sl}(2,\mathbb{C})$ in their (pseudopotential-type) variable spaces. Notice, of course, that the differential operators $\mathcal{L}_j$ and $\mathcal{A}_i$ also constitute realizations of the generators for $\mathbf{sl}(2,\mathbb{C})$. (The quantity $s \equiv af + eb$ is a characteristic for all the $\mathcal{L}_i$. If one treats the 4 variables as complex, projective coordinates, in $\mathbb{C}^4$, for the group manifold, s is the radius variable.)

We can write the most general solution to this system of pde's, which is quite "messy." It can be done in a number of different, but equivalent, ways, depending upon the order of the integrations performed. However, one could hope for a much less coordinate-dependent approach to such a problem. To emphasize the meaning of this quest, we first study a slightly reduced version, obtained by assuming the $\mathcal{M}_j$ to be constant:

$$\mathcal{L}_j\mathcal{E}_i = [\mathcal{M}_j, \mathcal{E}_i] = \{\mathrm{ad}\,\mathcal{M}_j\}\mathcal{E}_i\ . \tag{3.2}$$

These equations may be treated as saying that the $\mathcal{E}_i$ are eigenvectors of the "total angular momentum" operators, with eigenvalue zero, taking the ad-action of the $\mathcal{M}_j$ as a realization of (the negative of) the usual 'spin'-operators for $\mathbf{sl}(2,\mathbb{C})$. Such a point of view ought, it seems, to generate "nice" expressions for the solutions; however, we have not yet seen them.

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4. Some problems are more convenient if one uses a non-holonomic basis on the tangent space, such that $[\mathbf{e}_a\,,\,\mathbf{e}_b] = \mathbf{C}^c{}_{[ab]}\,\mathbf{e}_c$. Fiber coordinates are then defined by the scheme

$u_a \equiv \mathbf{e}_a(u),\ u_{ab} \equiv \mathbf{e}_b(\mathbf{e}_a(u))$ etc., with a unique choice of order for the indices. This behavior naturally lifts to the corresponding total derivatives, $[D_a\, ,\, D_b] = \text{Ç}^c_{[ab]} D_c$, so that the zero-curvature equations take the form $D_{[a}(\mathbf{X}_{b]}) + [\mathbf{X}_a, \mathbf{X}_b] = \text{Ç}^c{}_{[ab]}\mathbf{X}_c$.

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