Equations for Complex-Valued, Twisting, Type N, Vacuum Solutions, with one or two Killing/homothetic vectors

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Abstract: $\mathfrak{h}\mathfrak{h}$-spaces, i.e., complex spacetimes, of Petrov type N×N are determined by a trio of pde's for two functions, λ and a, of three independent variables (and also two gauge functions, chosen to be two of the independent variables if one prefers). As in common integrable systems, these form a second order, linear system for λ; howver, here the integrability conditions, involving a, are more complicated than is common. Therefore, with the hope of finding new solutions, these equations are now constrained to also admit both one and two homothetic or Killing vectors. The case with one Killing and one homothetic vector reduces these equations to two ode's for two unknown functions of the one remaining variable.

In addition, we also describe in detail the explicit forms of the metric, tetrad, connections, and curvature for twisting $\mathfrak{h}\mathfrak{h}$-spaces of Petrov type N×N, modulo the determining equations. This simplifies considerably the process of obtaining these details cleanly from earlier articles on the subject, thus simplifying access to the research area.

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1. Simpler Description of the Local Metric

The goal of understanding general classes of solutions of Petrov Type N, with non-zero twist, is one that is still not realized. The use of $\mathfrak{h}\mathfrak{h}$-spaces to forge a different path toward this goal was developed to a reasonable form in 1992.[1] This work pushes that path a step further, in a direction that has interested the more standard analysis for some time: to look for solutions that admit one Killing and also one homothetic vector, to simplify the task.[2]

A general $\mathfrak{h}\mathfrak{h}$-space is a complex-valued solution of the Einstein vacuum field equations that admits (at least) one congruence of null strings, i.e., a foliation by completely null, totally geodesic two-dimensional surfaces.[3] Those solutions with algebraically-degenerate, real Petrov type have two distinct such congruences. We describe them with coordinates $\{p, v, y, u\}$, where p is an affine, null coordinate along one null string, v specifies local wave surfaces, and y and u are transverse coordinates, in those surfaces. The metric is determined by x and λ, functions of $\{v, y, u\}$, which must satisfy three quasilinear pde's. (Alternatively, one may reverse the roles of v and x, choosing $\{x, y, u\}$ as independent variables, and treating $v = v(x, y, u)$ and $F \equiv v_x\,\lambda[v(x, y, u), y, u]$ as dependent variables.) These three equations are most easily presented using a mixed (non-holonomic) basis for the derivatives in these three variables:

$$\partial_1 \equiv \partial_v \ , \quad \partial_2 \equiv \partial_y \ , \quad \partial_3 \equiv \partial_u + a\partial_v, \text{ with } a \equiv -x_u/x_v \ , \tag{1}$$

where ∂_2 is the derivative with respect to y in the $\{v, y, u\}$ coordinate system while ∂_3 is the derivative with respect to u in the $\{x, y, u\}$ coordinate system, i.e., $v_2 = 0$ and $x_3 = 0$. This then implies the commutators

$$[\partial_1, \partial_2] = 0 \ , \quad [\partial_1, \partial_3] = a_1\,\partial_1 \ , \quad [\partial_2, \partial_3] = a_2\,\partial_1 \ , \tag{2}$$

where the function $a \equiv v_3$ determines the twist of the metric, it being proportional to $a_2 = v_{32} \propto x_{23}$, which we insist remain nonzero.

The constraining pde's then have the following form in terms of λ and a:

$$\begin{gathered} \lambda_{22} = \Delta\lambda \ , \qquad \lambda_{33} + 2a_1\lambda_3 + a_{31}\lambda = \gamma\lambda \ , \\ a_2(\lambda_{23} + \lambda_{32}) + a_{22}\lambda_3 + a_{32}\lambda_2 + \tfrac{1}{2}a_{322}\lambda = 0 \ , \end{gathered} \tag{3}$$

where two gauge functions, Δ and γ, have also been introduced that determine the left- and right-curvatures:

$$\begin{gathered} \Delta = \Delta(x, y) \ \text{ so that } \Delta_3 = 0\,, \text{ while } C^{(1)} \propto \Delta_1 \neq 0 \ , \\ \gamma = \gamma(v, u) \ \text{ so that } \gamma_2 = 0\,, \text{ while } \overline{C}^{(1)} \propto \gamma_1 \neq 0 \ . \end{gathered} \tag{4}$$

These functions do simply describe some gauge freedom in the defining equations, since they may be chosen arbitrarily, modulo the constraints above, as explained in more detail below.

These constraints are sufficient to completely satisfy Einstein's equations. However, for them to be **in involution**, we must also differentiate them, and solve, to determine either[4]

a) λ_{22}, λ_{33}, a_{322}, and also $\partial_3\lambda_{22}$, $\partial_2\lambda_{33}$, and $\partial_1 a_{322}$, **or**

b) all **six** λ_{ij}, along with **eight** equations between the three λ_i which involve derivatives of a up through the 6th-order.

As (the only known) explicit example, Hauser's solution[5] is given by

$$
\begin{gathered}
a = y + u\,,\ \Delta = \frac{3}{8x}\,, \gamma = \frac{3}{8v}\,,\ x + v = \tfrac{1}{2}(y+u)^2\,,\\
\lambda = (y+u)^{3/2} f(t)\,, \quad it + 1 \equiv 4v/(y+u)^2\,,\\
\text{with} \quad 16(1+t^2)f'' + 3f = 0\ ,\ f \text{ a hypergeometric function.}
\end{gathered}
\tag{5}
$$

In these coordinates its Killing vector is $\widetilde{K} = \partial_u - \partial_y$, and its homothetic vector is given by $\widetilde{H} = 3p\,\partial_p + y\,\partial_y + u\,\partial_u + 2v\,\partial_v$.

Returning **to our general form**, we also choose a null tetrad, and specify the associated non-zero components of the curvature:

$$
\begin{gathered}
\mathbf{g} = \underset{\approx}{\omega}^1 \otimes \underset{\approx}{\omega}^2 + \underset{\approx}{\omega}^2 \otimes \underset{\approx}{\omega}^1 + \underset{\approx}{\omega}^3 \otimes \underset{\approx}{\omega}^4 + \underset{\approx}{\omega}^4 \otimes \underset{\approx}{\omega}^3\,, \qquad \text{with } \underset{\approx}{\omega}^1 \ \equiv\ p\,du\,,\\
\underset{\approx}{\omega}^2 \ \equiv\ Z\,dy + a_1\,\underset{\approx}{\omega}^3\,,\ \underset{\approx}{\omega}^3 \ \equiv dv - a\,du\,,\ \underset{\approx}{\omega}^4 \ \equiv\ dp + E\,du - Q\,\underset{\approx}{\omega}^3\ ,\\
\text{where } E \equiv \lambda(\lambda\,a_{32} + 2\lambda_3\,a_2)\,,\ Z \equiv p/\lambda^2 + a_2\,,\ Q \equiv p/\lambda^2 + \lambda^2(\lambda_2/\lambda)_3\ ,\\
\text{and} \ \ 2R_{1313} = 2\gamma_1/p = C^{(1)}\ , \quad 2R_{2323} = 2(\lambda^2/Z)\Delta_1 = \overline{C}^{(1)}\ ,
\end{gathered}
\tag{6}
$$

so that the curvature is indeed of type $N \otimes N$.

The forms of these constraining pde's are unchanged under any one of the following coordinate transformations.[1]

I. Replace $\{v,y,u\}$ by $\{\overline{v},y,u\}$, with $\overline{v} = \overline{V}(v,u)$ arbitrary but invertible, along with F, x, γ, Δ scalars, while $\overline{\lambda} = \lambda/\overline{V}_v$ and $\overline{a} = \overline{V}_v a + \overline{V}_u$;

II. Replace $\{x,y,u\}$ by $\{\overline{x},y,u\}$, with $\overline{x} = \overline{X}(x,y)$ arbitrary but invertible, and λ, a, v, γ, Δ scalars, while $\overline{F} = F/\overline{X}_x$;

III. Replace $\{v,y,u\}$ by $\{v,\overline{y},u\}$, with $\overline{y} = \overline{Y}(y)$ arbitrary but invertible, and x, a, γ scalars, while λ [or F] scales as $\overline{\lambda} = \sqrt{\overline{Y}_y}\,\lambda$, and Δ has an additional term: $\Delta = \overline{\Delta} + \{\sqrt{\overline{Y}_y}\}_{yy}$.

IV. Replace $\{v,y,u\}$ by $\{v,y,\overline{u}\}$, with $u = U(\overline{u})$ arbitrary but invertible, and x, Δ scalars, while λ [or F] scales as $\overline{\lambda} = \sqrt{\overline{U}_u}\,\lambda$, and also $\overline{a} = a/\overline{U}_u$, and $\gamma = \overline{\gamma} + \{\sqrt{\overline{U}_u}\}_{uu}$.

We refer to $\gamma = \gamma(u,v)$ and $\Delta = \Delta(x,y)$ as gauge functions since transformations I and II would allow them to be replaced by v and x, respectively. However, we save that freedom for now.

2. Killing's Equations

We reduce the generality of the pde's by insisting that the metric allow some symmetries. An arbitrary homothetic vector, $\widetilde{V}$, constrains the metric and curvature as follows:

$$\mathcal{L}_{\tilde{V}}\, g_{\alpha\beta} \equiv V_{(\alpha;\beta)} = 2\chi_0\, g_{\alpha\beta}\,, \qquad \mathcal{L}_{\tilde{V}}\, \underset{\sim}{\Gamma}^{\alpha}{}_{\beta} \;=\; 0 \;=\; \mathcal{L}_{\tilde{V}}\, \underset{\sim}{\Omega}^{\alpha}{}_{\beta}\,. \tag{7}$$

When put together with the pde's for the metric functions, Eqs.(3), via GRTensor and Maple, these constraints require any prospective homothetic vector to be determined by only two functions, $K = K(u)$ and $B = B(v,u)$:

$$\widetilde{V} = +(2\chi_0 - B_{,v})p\partial_p + \frac{(\partial_u + a\partial_v - a_{,v})(B - aK)}{a_{,y}}\,\partial_y + K\partial_u + B\partial_v\,, \tag{8}$$

along with various constraints on λ, a, Δ, and γ, relative to K and B. We may however use our coordinate freedom(s) to simplify those equations further:

$$\begin{aligned} &\text{Under Transformation I,}\quad \overline{v} = \overline{V}(v,u) \;\Longrightarrow \overline{K} = K\,,\quad \overline{B} = K\overline{V}_{,u} + B\overline{V}_{,v}\,;\\ &\text{under Transformation IV,}\quad \overline{u} = \overline{U}(u) \;\Longrightarrow \overline{K} = \overline{U}_{,u}\,K\,,\quad \overline{B} = B\,. \end{aligned} \tag{9}$$

Therefore, when $K \neq 0$, we may always choose coordinates so that $B = 0$ and K is a constant, say $+1$, and then ask for the constraints on $\{\lambda, a, \Delta, \gamma\}$ that are implied by this.

3. When one Homothetic Vector is Permitted

With $K = +1$, $B = 0$, Killing's equations require that

$$\begin{aligned} &\gamma_{,v} = \gamma_{,v}(v)\,,\ \Delta_{,v} = \Delta_{,v}[x(v,s)]\,,\ a = a(v,s)\,,\\ &\lambda = e^{\chi_0 u}L(v,s)\,,\quad \text{where}\ \ s \equiv y + u \qquad \text{and}\ \ \widetilde{V} = 2\chi_0 p\,\partial_p + \partial_u - \partial_y\,. \end{aligned} \tag{10}$$

When $\chi_0 = 0$ this is the usual Killing vector, namely a translation in the 2-plane which is the local wavefront. This clearly reduces the independent variables in the pde's to only two, so that the first 3 equations determine all the second derivatives of λ, and requiring only one more equation, solved for a 4th derivative of a, to complete the involutive set. The new version of the constraining equations may be written in terms of only ∂_1 and ∂_2, or in terms of only ∂_2 and ∂_3, with $\partial_1 \;\to\; \frac{1}{a}(\partial_3 - \partial_2)$.[6]

4. One Homothetic Vector plus the Killing Vector

We now indeed insist that our metric allows one true Killing vector, in the form described above for the metric quantities, with $\chi_0 = 0$.[7] In addition we also ask for a second (homothetic) symmetry, $\widetilde{H}$, which will have the form shown in Eqs.(8) with its own K, B, and χ_0 not necessarily zero. Its existence is additionally constrained by the fact that the commutator of two homothetic vectors must be a Killing vector:[8]

$$[\widetilde{K}, \widetilde{H}] \propto \widetilde{K} \qquad \Longrightarrow \qquad \partial_u B = 0 = \partial_u^2 K \tag{11}$$

Since we have used some gauge freedom to simplify our Killing vector, much less freedom remains. Nonetheless, while maintaining the simple form of our (first) Killing vector, that freedom is still sufficient to allow us to set

$$\begin{gathered} K \longrightarrow u\,, \quad \text{and} \quad B \longrightarrow 2v \\ \Longrightarrow \quad \widetilde{H} = 2(\chi_0 - 1)p\,\partial_p + y\,\partial_y + u\,\partial_u + 2\,v\,\partial_v\,, \end{gathered} \tag{12}$$

where the constant 2 is simply a convenient choice.

Killing's equations are then completely satisfied by the following "scaling" equations for each dependent variable and the concommitant ones for their derivatives:

$$\begin{aligned} \widetilde{H}(a) &= a\,, \qquad & \widetilde{H}(\lambda) &= (\chi_0 - 1)\,\lambda\,, \\ \widetilde{H}(\gamma) &= -2\,\gamma\,, & \widetilde{H}(\Delta) &= -2\,\Delta\,. \end{aligned} \tag{13}$$

Since a and λ already depend only on v and $s \equiv y + u$, these constraints reduce them in terms of functions of only one variable:

$$a = s\,A(q)\,, \qquad \lambda = s^{\chi_0 - 1} L(q)\,, \qquad q \equiv \frac{v}{s^2}\,, \tag{14}$$

while the gauge functions are almost completely determined:

$$\gamma = \gamma_0/v = s^{-2}G(q), \quad \text{i.e., } G = \gamma_0/q\,, \quad \Delta = s^{-2}D(q), \text{ where } (A - 2q)(\ln D)' = 2\,. \tag{15}$$

Of course the original constraint equations must still be resolved. They are now 3 **ode's** for the two functions, L and A. To display them, we take a new form for the similarity variable,

$$r \equiv \tfrac{1}{2}\ln(q) \;\Leftrightarrow\; q \equiv e^{2r} \;\text{ and set }\; W = W(r) \equiv (A - 2q)/(2q)\,. \tag{16}$$

The first two equations present very nicely, in a simple, factorized form:

$$
\begin{aligned}
(\frac{d}{dr} + \chi_0 - 1)(\frac{d}{dr} + \chi_0 - 2)L \; &= D\,L = e^{+2\int dr(1/W)}\,L\;, \\
(\frac{d}{dr}W + \chi_0 + 1)(\frac{d}{dr}W + \chi_0)L \; &= GL = e^{-2p}L\;.
\end{aligned}
\tag{17}
$$

The third one, while still linear in L, is somewhat more complicated, and not immediately factorizable:

$$
\begin{aligned}
4WZ\,\frac{d^2L}{dr^2} + 2\left[(\mathbf{A}+\mathbf{B})(WZ) + (2\chi_0 - 1)Z\right]\frac{dL}{dr} & \\
+ \left\{\mathbf{A}(\mathbf{B}(WZ)) + 2(\chi_0 Z_r - \eta_0\, Z)\right\}L = 0\;, & \\
\text{with}\;\; \mathbf{A} \equiv \frac{d}{dr} + 4\,, \quad \mathbf{B} \equiv \frac{d}{dr} - (2\chi_0 - 5)\;, & \\
Z \equiv \frac{dW}{dr} + W + 1,\;\text{and}\;\; \eta_0 \equiv 2{\chi_0}^2 - 6\chi_0 + 1\;. &
\end{aligned}
\tag{18}
$$

Not all three equations, for two functions, are necessary. Involutivity now needs only two of these equations. A possible choice is the following pair:

a.) a Riccati equation for $H \equiv L_{,r}/L$, which is derived by using $\Delta_3 = 0$ to eliminate it from the presentation:

$$
\begin{aligned}
W[dW/dr + (2\chi - 1)(W+1)]\,\frac{dH}{dr} & \\
- 2W[dW/dr + W + 1]H^2 + \mu H + \nu = 0\;, &
\end{aligned}
\tag{19}
$$

where μ and ν are fairly complicated polynomials in $(WW_r)_r$, W_r, W, and constants; and

b.) the equation above, Eq. (17b), with W and G, which is second order and linear for L, but which could also be seen as a Riccati equation for H.

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