Twisting Gravitational Waves and Eigenvector Fields for SL(2,$\mathbb{C}$) on an Infinite Jet

J. D. Finley, III
Dept. of Physics and Astronomy, University of New Mexico
Albuquerque, N.M., 87131 U.S.A. finley@tagore.phys.unm.edu

Abstract:

A system of coupled vector-field-valued PDE's is presented, the solutions to which would determine two coupled, infinite-dimensional vector-field realizations of the group SL(2,$\mathbb{C}$). While the general solution is (partially) presented, the complicated nature of that solution is deplored, and the hope expressed that someone can replace it by something much more natural.

The problem arises out of searches for Bäcklund transforms of a system of PDE's that describe twisting, Petrov type N solutions of Einstein's vacuum field equations.

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I. The Connection with Gravitational Waves

I have long considered it an honor to have been guided by Eyvind Wichmann during my studies in Berkeley. I am therefore very pleased to be here at this symposium to honor him and his work, and to present some questions about infinite-dimensional group representations. Certainly my strong interests in this area were nurtured by Professor Wichmann's many excellent class handouts on group representations and the importance of symmetries in physics.

While the subject of this paper revolves around questions concerning realizations of the (complex version of the) rotation group, it is appropriate to first give some indications of the context in which these questions first arose, which is a part of the classical theory of gravitation as described by Einstein's vacuum field equations. Working in Mexico City, Jerzy Plebański[1] and I have had a very long-term interest in realistic Petrov Type N solutions of these equations. These are the sort that would be appropriate for a description of the gravitational radiation emitted by a compact source, such as an exploding star, or simply a binary star system. Such a solution is characterized by a special direction in spacetime, a 4-vector field, that describes the world line of the radiation in question. Such a vector must be of zero length since the radiation moves at the same (local) speed as that of light. In order to support the proposal that the radiation has been emitted by a compact source, it is essential that the "wavefronts" associated with this direction should not be "plane," which generates the mathematical requirement on the vector field that it should have a non-zero value of the "twist." This requirement arranges for the wavefronts to retain some essential details of how they were created, thereby allowing observations to have some of the character of a telescope. This interesting physical problem has been seriously considered by many people. Nonetheless, only one solution is known,[2] and it is not asymptotically flat[3].

Our approach to this problem has its origin in the theory of complexified spacetimes often referred to by the name *hyperheavens*, or $\mathcal{HH}$ spaces.[4] Such a space is distinguished by the fact that it contains (at least) one congruence of null strings, i.e., completely null, totally geodesic, complex-valued, two-dimensional surfaces, which in the generic case has a non-zero expansion. This expansion picks out a special direction on any given leaf of the congruence, thereby determining an affine parameter, $p \equiv \phi^{-1}$, which can be used as one of the four coordinates needed for a local specification of the spacetime. Such a restriction on the space of solutions for Einstein's field equations causes those solutions to be determined by a single "Debye-type" potential function W required to satisfy a single non-linear partial differential equation, the hyperheavenly equation, thereby reducing greatly the effort required to solve the complete set of vacuum field equations which, otherwise, would constitute ten coupled PDE's in ten unknown functions.

In the case in question, the insistence that the solution be of Petrov Type N is what picks out the unique direction field for the propagation of the radiation, and also gives us completely the dependence of the potential function W on the affine parameter, p, reducing the problem to one in only three independent variables. A further simplification of the problem—in hopes of finding a new, interesting solution—may be obtained by asking that the wavefronts have a symmetry, i.e., to ask that the spacetime admit a Killing vector. This reduces the number of independent variables to only two, which allows the introduction of very powerful methods to find solutions via Bäcklund transforms, zero-curvature conditions, etc. One common approach to the determination of such transforms has been the creation of an Estabrook-Wahlquist prolongation structure.[5] My former student, Denis Khetselius worked on creating just such a structure for the twisting, Petrov type N, vacuum equations with one Killing vector.

The reduction of the hyperheavenly equation to this case[6,7] leaves one with two unknown functions of 2 (complex) variables, $F(v,s)$, and $x = x(v,s)$, which must satisfy a triplet of nonlinear, second-order partial differential equations. These equations may be presented in a way that is linear in each of the variables separately, thereby either illuminating or obscuring some of the difficulty of the problem; however, in order to do this, one must use a nonholonomic basis for the derivatives. We therefore agree to begin on some larger manifold, where we treat all three of x, v, and s as coordinates and F a function of them all, **but** then project downward to the physical variables in two different ways. We use ∂_2 as the derivative with respect to s, holding v constant, i.e., with $\{v,s\}$ as the choice for the two independent variables and $x = x(v,s)$ as a function of them, but also make an alternate choice where we choose $\{x,s\}$ as the two independent variables and take $v = v(x,s)$ as the dependent function, indicating this choice of derivative with respect to s, holding x constant, by the symbol ∂_3. We may "explain" these derivative choices by the following differential, and also show their (non-zero) commutator:

$$dF = F_v dv + F_2 ds = F_{x'} dx' + F_3 ds = \tfrac{F_2}{x_2} dx + \tfrac{F_3}{v_3} dv \,, \tag{1.1}$$

$$[\partial_2, \partial_3] = \tfrac{x_{23}}{x_2}(\partial_3 - \partial_2) = -\tfrac{v_{32}}{v_3}(\partial_3 - \partial_2) \,, \tag{1.2}$$

where the function x_{23} is the physical twist of the problem, which we need to be non-zero. Using subscripts to denote partial derivatives in this (nonholonomic) basis the **type N** equations—to be solved—are

$$
\begin{aligned}
F_{33} - \gamma F = 0 \ , \\
({\partial_2}^2 - \Delta) x_v F = 0 \ , \\
x_{23}(F_{23} + F_{32}) + x_{223} F_3 \ + x_{233} F_2 + \tfrac{1}{2} x_{2233} F = 0 \ , \\
.
\end{aligned}
\tag{1.3}
$$

Of course the symbol x_v above denotes the derivative of x with respect to v; however, in this basis it may be replaced by its equivalent, the ratio $-x_2/v_3$. As well there are two *gauge functions*, Δ and γ, of only one variable, which may be allowed into the problem. In the simplest case they could be chosen to be simply x and v, respectively. However, there may be some use in the freedom they represent, which may be described by the following equations:

$$
\Delta_2 \neq 0 = \Delta_3 \ , \qquad \gamma_2 = 0 \neq \gamma_3 \ . \tag{1.4}
$$

Lastly, one must admit that the system as presented is not yet involutive,[1] but has yet one **integrability condition** other than just the equations themselves (and of course their derivatives):

$$
F_{223} + 2\tfrac{x_{23}}{x_2}(F_{23} - \gamma F) + \gamma F_2 - \left\{ \tfrac{x_{233}}{x_2} + 2(\tfrac{x_{23}}{x_2})^2 \right\} (F_2 - F_3) = 0 \ . \tag{1.5}
$$

II. Zero-Curvature Prolongations for Nonlinear PDE's

Our current desire is to obtain non-trivial solutions of this system of equations. The preferred method would be to determine a Bäcklund transform via a zero-curvature relation and Estabrook-Wahlquist prolongation structures. We therefore give a very brief description of how this process is implemented.[8,9] To begin with, we think of a k-th order system of PDE's as a variety, Y, of a finite jet bundle, $J^{(k)}(M, N)$, with M the (space of) independent variables and N the dependent-variables. From this geometric approach, we can look for point symmetries or contact symmetries directly on Y; by prolonging to the infinite jet space, we may determine generalized symmetries. However, to determine the non-local symmetries that generate Bäcklund transformations, we must prolong the system yet further, to a fiber space over J^∞. We label the fibers W, supposing that there will exist vertical flows that map solution spaces of one PDE into another, this one being satisfied by the dependence of the fiber coordinates, w^A, on the independent variables. The compatibility conditions for such flows to exist are referred to as *"zero-curvature conditions."*

Solutions of these conditions may be found using the tangent structure or the co-tangent structure, over $J^\infty \times W$. For a vector-field presentation, we choose a commuting basis, $\{e_a\}$, for tangent vectors over M, and lift them to the total derivative operators, D_a, over J^∞. When they are restricted to the variety Y^∞, which is the lift of the original PDE's, we denote that restriction by $\overline{D}_a$. The further prolongation into the fibers W requires the addition of some vector fields vertical with respect to the fibers, which we may denote by $\mathbf{X}_a = \sum X_a^A(\partial/\partial w^A)$, with the X_a^A functions of both the jet variables and the $\{w^A\}$. It is the insistence that these prolonged total derivatives, $\overline{D}_a + \mathbf{X}_a$, still commute, that ensures that the w^A can act as pseudopotentials for that PDE:[10]

$$0 \;=\; [D_a + \mathbf{X}_a\,,\, D_b + \mathbf{X}_b\,]\Big|_{Y^\infty \times W} = \left\{\overline{D}_a(X^C_b) - \overline{D}_b(X^C_a)\right\}\frac{\partial}{\partial w^C} + [\mathbf{X}_a\,,\,\mathbf{X}_b\,]\,. \quad (2.1)$$

As an identity in the jet coordinates, Eqs. (2.1) determine several independent equations. Their solution describes the $\mathbf{X}_a$ as linear combinations of vector fields $\mathbf{W}_\alpha$ with coefficients depending on coordinates for $Y \subset J^{(k)}(M,N)$. The w^A-dependence is encoded within a set of commutation relations among the $\{\mathbf{W}_\alpha\}$, considered as vector fields within the entire algebra of vector fields over W. The smallest subalgebra generated by the $\mathbf{W}_\alpha$ that faithfully reproduces the linear independence, and the values, of those commutators is the general solution to the covering problem, and will allow Bäcklund transforms for those equations. As the construction gives the $\mathbf{X}_a$ the "form" of a connection, it is reasonable to refer to these equations as "zero-curvature" requirements; it is, however, a generalization of the more usual approach,[11,12] since the $\mathbf{X}_a$'s are still only elements of an **abstract** Lie algebra of vector fields, with neither coordinates, nor even their number yet determined.

III. Simple Vector-Field Flows

Since the zero-curvature equations involve the solutions of vector-field-valued PDE's, it is worth commenting on some simpler cases first. As well I note that this is again an area of research where I had considerable guidance and training from Professor Wichmann. The simplest sort of a flow equation for a vector field may be written simply as

$$\mathbf{Z}_{,u} = [\mathbf{F},\mathbf{Z}]\;, \quad \text{with } \mathbf{F}_{,u} = 0\;. \quad (3.1)$$

Locally, on the tangent bundle of a manifold the geometric picture that goes with this differential equation is the following. The vector fields $\mathbf{Z}$ and $\mathbf{F}$ are two directions, in the neighborhood of a point, with $\mathbf{F}$ the tangent vector for a curve Γ_F with parameter u. The equation describes the "Lie-dragging" of $\mathbf{Z}$, along this curve. Taking the initial value as $\mathbf{Q} \equiv \mathbf{Z}(0)$, we may write down the well-known solution to this equation:

$$\begin{aligned}\mathbf{Z}(u) \;&=\; e^{u\,(\mathrm{ad\,F})}\mathbf{Q} \equiv \sum_{n=0}^{\infty}\frac{(u)^n}{n!}(\mathrm{ad}\;F)^n\mathbf{Q}\\ &=\; \mathbf{Q} + u[\mathbf{F},\mathbf{Q}] + \tfrac{1}{2}u^2[\mathbf{F},[\mathbf{F},\mathbf{Q}]] + \ldots\;.\end{aligned} \quad (3.2)$$

A more general case is given by the following situation, where both the (unknown) vector fields are being dragged, but in different directions. More precisely, we may take $\mathbf{A},\mathbf{R}$ as vertical vector fields over a fiber bundle, but with dependence on disjoint base manifold variables:

$$\begin{gathered}\mathbf{A} = A^D(w,x)\partial_{w^D}\;, \mathbf{R} = R^D(w,u)\partial_{w^D}\;,\\ \big[\mathbf{A},\,\mathbf{R}\big] = \mathbf{A}_{,x} + \mathbf{R}_{,u}\;.\end{gathered} \quad (3.3)$$

The general solution of this problem is given[13,9] by the following somewhat complicated set of equations, along with a set of constraints on the initial values:

$$\mathbf{A}(x) - \mathbf{A}_0 = \int_0^x dz\, e^{-z\,(\mathrm{ad}\,\mathbf{R}_0)}\mathbf{A}_1 = \sum_{m=0}^{\infty} \frac{(-x)^{m+1}}{(m+1)!}(\mathrm{ad}\,\mathbf{R_0})^m \mathbf{A}_1\,,$$

$$\mathbf{R}(u,v) - \mathbf{R}_0(v) = \int_0^u dw\, e^{w\,(\mathrm{ad}\,\mathrm{A}_0)}\mathbf{R}_1 = \sum_{k=0}^{\infty} \frac{(+u)^{k+1}}{(k+1)!}(\mathrm{ad}\,\mathbf{A_0})^k \mathbf{R}_1\,, \tag{3.4a}$$

where $\mathbf{A}_0$, $\mathbf{R}_0$ and either of $\mathbf{A}_1$ or $\mathbf{R}_1$ may be freely chosen, with the other being determined by the relation that connects them:

$$\mathbf{A}_1 - \mathbf{R}_1 = [\mathbf{R}_0\,, \mathbf{A}_0]\ . \tag{3.4b}$$

The constraints are the following doubly countable collection:

$$\big[\mathbf{A}_{m+1}\,, \mathbf{R}_{k+1}\big] = 0\ , \quad \forall\ k, m = 0, 1, 2, \ldots \quad , \tag{3.4c}$$

where $\mathbf{A}_m$ is the coefficient of $x^m/(m)!$ in for $\mathbf{A}(x)$, with the same idea for $\mathbf{R}_k$.

IV. Systems of PDE's for Vector-fields, for Type N

Having given this background, I may now introduce the advertised system of vector-field-valued PDE's associated with $\boldsymbol{sl}(2,\,\mathbb{C})$, which was originally discovered by Denis Khetselius, who received his Ph.D. in 1996,[8] for his work on the twisting type N prolongation problem associated with the equations given earlier.

From the point of view of Estabrook and Wahlquist, following Cartan, he rewrote the equations as a first-order system. The underlying manifold then had 2 independent variables, 4 dependent variables, and an additional 13 jet variables, needed to describe a differential system with fourteen 2-forms in the co-tangent bundle. At an early step in the calculations, he showed that the entire structure would lose its relationship to the original system of PDE's unless the associated fibers of pseudopotentials were infinite dimensional.[13] The structure was then expanded in terms of an infinite series in powers of the "twist" variable, x_{23}.

However, this is not today's talk. Rather I want to discuss some of the structure of his results already at the zero-th level in the twist variable, which relate to the underlying rotational symmetry of the problem. At this point he found himself searching for two pair of vertical vector fields that depended, disjointly, on 4 and 2 jet variables:

$$\mathcal{E}_i(w^A, a, b, e, f)\,,\ \mathcal{M}_i(w^A, u, h)\,; \quad i = 1, 2\ . \tag{4.1}$$

They were required to be solutions of a system of PDE's that seriously generalizes the earlier, "two-direction" flow problem:

$$\begin{aligned}
(a\partial_b + e\partial_f)\mathcal{E}_1 - (u\partial_h)\mathcal{M}_1\ &= [\mathcal{E}_1, \mathcal{M}_1]\ ,\\
(b\partial_a + f\partial_e)\mathcal{E}_1 - (u\partial_h)\mathcal{M}_2\ &= [\mathcal{E}_1, \mathcal{M}_2]\ ,\\
(a\partial_b + e\partial_f)\mathcal{E}_2 - (h\partial_u)\mathcal{M}_1\ &= [\mathcal{E}_2, \mathcal{M}_1]\ ,\\
(b\partial_a + f\partial_e)\mathcal{E}_2 - (h\partial_u)\mathcal{M}_2\ &= [\mathcal{E}_2, \mathcal{M}_2]\ .
\end{aligned} \tag{4.2}$$

Each of these 4 equations is of the form we have already discussed, so that the earlier method may be applied. However, they are seriously coupled together, which causes many compatibility equations, which we want now to uncover, and try to understand.

Each of the two different pairs of (first-order) differential operators constitutes a pair of generators, analogous to $J_{\pm}$, for a realization of the Lie algebra $\boldsymbol{sl}(2,\mathbb{C})$, in their respective variable spaces:

$$\{\mathcal{L}_1, \mathcal{L}_2\} = \{(a\partial_b + e\partial_f), (b\partial_a + f\partial_e)\} \quad \text{and} \quad \{\mathcal{A}_1, \mathcal{A}_2\} = \{u\partial_h, h\partial_u\} \ . \tag{4.3}$$

Therefore each pair generates a third such operator, completing the (usual) generators for $\boldsymbol{sl}(2,\mathbb{C})$. In different language, treated as a system of first-order operators, the elements of each pair "conspire" to include their various (non-zero) commutators—as integrability conditions. As well the commutators of the original unknown functions enter the picture. In this way we end up with a system of **nine equations**:

$$\mathcal{L}_j\mathcal{E}_i - \mathcal{A}_i\mathcal{M}_j = [\mathcal{E}_i, \mathcal{M}_j] \ , \quad \forall i, j = +, 0, - \quad , \tag{4.4}$$

where we have given names as follows:

$$\left\{\begin{aligned} \mathcal{L}_+ &\equiv a\partial_b + e\partial_f \ , \\ \mathcal{L}_- &\equiv b\partial_a + f\partial_e \ , \\ \mathcal{L}_0 &\equiv [\mathcal{L}_+, \mathcal{L}_-] \\ &= a\partial_a + e\partial_e - b\partial_b - f\partial_f \ , \end{aligned}\right\} \qquad \left\{\begin{aligned} \mathcal{A}_+ &\equiv u\partial_h \ , \\ \mathcal{A}_- &\equiv h\partial_u \ , \\ \mathcal{A}_0 &\equiv [\mathcal{A}_+, \mathcal{A}_-] \\ &= h\partial_h - u\partial_u \ , \end{aligned}\right\} \tag{4.5}$$

$$\left\{\begin{aligned} \mathcal{E}_+ \equiv \mathcal{E}_1 \ , \ \mathcal{E}_- \equiv \mathcal{E}_2 \ , \\ \mathcal{E}_0 \equiv [\mathcal{E}_+, \mathcal{E}_-] \ , \end{aligned}\right\} \qquad \left\{\begin{aligned} \mathcal{M}_+ \equiv \mathcal{M}_1 \ , \ \mathcal{M}_- \equiv \mathcal{M}_2 \ , \\ \mathcal{M}_0 \equiv [\mathcal{M}_+, \mathcal{M}_-] \ , \end{aligned}\right\}$$

To better relate these tangent vectors to the more usual matrix representations of this algebra, put the coordinates h and u into a vector, $h^A \equiv (h, u)^T$, and take $(\mathbf{S}_i)^A{}_B$ as the usual three 2×2 matrices representing the generators. Then the differential operators $\mathcal{A}_i$ are simply

$$\mathcal{A}_i = h^B (\mathbf{S}_i)^A{}_B \frac{\partial}{\partial h^A} \ ,$$

so that we can see that they are the lift of the defining 2×2 matrix representation to the tangent bundle. In a similar way, we may write the tangent-vector fields $\mathcal{L}_j$ as coming from a reducible, 4-dimensional representation, with the variables $\{a, b, e, f\}$ as homogeneous coordinates for the group manifold, $S^3 \subset C^4$. In that case the quantity $s \equiv af + eb$ is the radius squared for that S^3, and is a characteristic variable for all the $\mathcal{L}_i$.

However, there are **still more integrability conditions.** They arise because of the remaining commutators of the still-to-be-determined vector fields, $\mathcal{E}_i$ and $\mathcal{M}_j$. The commutators of each pair of these vector fields have, *a priori,* no requirements on them, so that their closure is an infinite-dimensional **free** algebra. In principle we could continue writing down all the integrability conditions imposed by that general free algebra. However, it seems useful to require it to follow the behavior of the differential operators; i.e., we are led to consider eliminating any additional commutators by reducing this infinite-dimensional algebra down to its smallest interesting constituent, $\boldsymbol{sl}(2,\mathbb{C})$.

The standard approach to this problem is to divide out the the free algebra by the Serre relations, i.e., by the ideal generated by the vanishing of the $\boldsymbol{sl}(2,\mathbb{C})$ commutation relations. However, to surely ascertain what we are discarding, we first write these divisors in the following form

$$\begin{aligned} \mathcal{H}_\pm \equiv\ & [[\mathcal{M}_+,\mathcal{M}_-],\mathcal{M}_\pm] \mp 2\mathcal{M}_\pm \ , \\ \mathcal{J}_\pm \equiv\ & [[\mathcal{E}_+,\mathcal{E}_-],\mathcal{E}_\pm] \mp 2\mathcal{E}_\pm \ . \end{aligned} \tag{4.6}$$

With that notation the next set of integrability conditions are the following first-order differential equations:

$$\mathcal{A}_i\mathcal{H}_\pm = [\mathcal{H}_\pm\, , \mathcal{E}_i]\ , \quad \mathcal{L}_j\mathcal{J}_\pm = [\mathcal{J}_\pm, \mathcal{M}_j] \tag{4.7}$$

The obvious solution given by the vanishing of the divisors does not seem particularly egregious, so that we now append to our problem the additional **assumption** that they do in fact vanish. In that case the $\mathcal{M}_j$ and, separately, the $\mathcal{E}_i$ are **also** realizations of $\boldsymbol{sl}(2,\mathbb{C})$, each in terms of their respective variables, and the w^A, but with a form determined by solving the PDE's.

That assumption puts the system into involution, i.e., **all** compatibility conditions are now listed, and we can begin to consider the integration of the system. However, it turns out that these reasonably "pretty" and "simple-appearing" equations have solutions that look terrible, and which have a presentation that is very coordinate-dependent! Therefore, although I will in fact describe the general solution to the problem, I propose to first consider a rather simpler question, by looking at the subcase. where we forget the dependence of the $\mathcal{E}_j$ on the jet variables, which reduces the system to merely the following triplet of equations, since the subscript on $\mathcal{M}$ is no longer relevant, the equations being the same for different values of it:

$$\mathcal{A}_i\,\mathcal{M} + [\mathcal{E}_i, \mathcal{M}] = 0\ , \quad \forall i = +,0,- \quad . \tag{4.8}$$

These equations may be interpreted as asking for "eigenvector fields" of the "angular-momentum" operators, $\mathcal{A}_i$, in the infinite-dimensional fibers where the $\mathcal{M}$ reside. In this question the $\mathcal{E}_i$ are independent of the jet variables, so that it might be thought that their ad-action, on the $\mathcal{M}$, is like the action of the usual 'spin'-operators. Then the entire equation says that $\mathcal{M}$ is an eigenvector of the "total angular momentum" operators, **with eigenvalue zero**:

$$\{\mathcal{A}_i + \mathrm{ad}\,\mathcal{E}_i\}\,\mathcal{M}\ = 0\ , \tag{4.9a}$$

or, in a "cleaner" viewpoint on the problem, to ask for invariant vector-field-valued functions, under the action of $\boldsymbol{sl}(2,\mathbb{C})$, i.e., to require

$$e^{-i\theta\, a^j(\mathcal{A}_j + \mathrm{ad}\mathcal{E}_j)}\mathcal{M} = \mathcal{M}\ . \tag{4.9b}$$

There **should be** "nice" expressions for quantities of this type, I believe. However, I have not been able to find them. Nonetheless, not having the "nice" expressions, the alternative is to simplify proceed directly, using the techniques discussed earlier. Perhaps the forms so obtained are in fact acceptably "nice." However, their current presentation has more coordinate-dependence than I think is reasonable. The result may be written in several distinct, equivalent forms:

$$\begin{aligned} \mathcal{M} =& e^{-(h/u)(\mathrm{ad}\,\mathcal{E}_+)}\, e^{-(\ln u)(\mathrm{ad}\,\mathcal{E}_0)}\,\mathbf{Z}_+ \\ =& e^{-(\ln u)(\mathrm{ad}\,\mathcal{E}_0)}\, e^{-(uh)(\mathrm{ad}\,\mathcal{E}_+)}\,\mathbf{Z}_+\;, \\ & \text{along with the constraint}\;\; [\mathbf{Z}_+\,,\mathcal{E}_-]=0\,. \end{aligned} \tag{4.10a}$$

or the equally valid forms

$$\begin{aligned} \mathcal{M} =& e^{-(u/h)(\mathrm{ad}\,\mathcal{E}_-)}\, e^{+(\ln h)(\mathrm{ad}\,\mathcal{E}_0)}\,\mathbf{Z}_- \\ =& e^{+(\ln h)(\mathrm{ad}\,\mathcal{E}_0)}\, e^{-(uh)(\mathrm{ad}\,\mathcal{E}_-)}\,\mathbf{Z}_-\;, \\ & \text{along with the constraint}\;\; [\mathbf{Z}_-\,,\mathcal{E}_+]=0\,. \end{aligned} \tag{4.10b}$$

Was that result acceptable? If so, then let us now consider the very next level of simplicity for the equations. Consider the case when the $\mathcal{M}_i$ are independent of $\{u,h\}$, instead of the other way around, just discussed. Then our system reduces to the triplet,

$$\mathcal{L}_i\mathcal{E} = -[\mathcal{M}_i,\mathcal{E}] = -\{\mathrm{ad}\,\mathcal{M}_i\}\mathcal{E}\;. \tag{4.11}$$

The more geometrical forms for these equations have, basically, the same structure as above, except that now there are more jet variables involved in the differential operators, i.e., the equations are built over a large matrix representation:

$$\begin{aligned} \{\mathcal{L}_i + \mathrm{ad}\,\mathcal{M}_i\}\,\mathcal{E} &= 0\;, \\ e^{-i\theta\, a^j(\mathcal{L}_j+\mathrm{ad}\mathcal{M}_j)}\mathcal{E} &= \mathcal{E}\;. \end{aligned} \tag{4.11}$$

Direct integration of these equations gives more complicated forms, and **also** several alternative, equivalent choices. However, now let me present just one of the various choice for the form of this solution:

$$\mathcal{E}_j = e^{-(b/a)(\mathrm{ad}\,\mathcal{M}_-)}\, e^{(ac/s)(\mathrm{ad}\,\mathcal{M}_+)}\, e^{(\ln a)(\mathrm{ad}\,\mathcal{M}_0)}\,\mathbf{H}_j(s)\;. \tag{4.12}$$

This is more complicated than the previous one, as perhaps it should be: it has more variables and larger matrices, but it does also have a larger number of distinct forms, related to ordering, but which I defer.

We should now return to the original problem, where we maintain both sets of indices. We can also completely integrate this system; **however**, the results involve a number of integrations, and the appearance of the solutions depends on their order. This leaves open to doubt their optimal presentation. Nonetheless, as an example, here are two of them:

$$\begin{aligned} \mathcal{M}_- = \mathbf{K}_- &+ \sum_{n=0}^{\infty}\frac{(-\ln h)^{n+1}}{(n+1)!}(\mathrm{ad}\,\mathbf{W_0})^n[\mathbf{W}_-,\mathbf{Y}_-] \\ &- h^2\sum_{n=0}^{\infty}\frac{(-e/h)^{n+1}}{(n+1)!}(\mathrm{ad}\,\mathbf{W}_+)^n\left\{e^{-(\ln h)(\mathrm{ad}\,\mathbf{W}_0)}\mathbf{Y}_-\right\}\;, \end{aligned} \tag{4.13a}$$

$$
\begin{aligned}
\mathcal{E}_+ = {} & e^{-(b/a)(\mathrm{ad}\,\mathbf{K}_+)}\, e^{(ea/s)(\mathrm{ad}\,\mathbf{K}_-)}\, e^{-(\ln a)(\mathrm{ad}\,\mathbf{K}_0)}\, \mathbf{W}_+ \\
& + e^{-(b/a)(\mathrm{ad}\,\mathbf{K}_+)} e^{(ea/s)(\mathrm{ad}\,\mathbf{K}_-)} \sum_{n=0}^{\infty} \frac{-\ln a)^{n+1}}{(n+1)!} (\mathrm{ad}\,\mathbf{K}_0)^n \mathbf{Y}_0 \\
& + e^{-(b/a)(\mathrm{ad}\,\mathbf{K}_+)} \sum_{n=0}^{\infty} \frac{(ea/s)^{n+1}}{(n+1)!} (\mathrm{ad}\,\mathbf{K}_-)^n \mathbf{Y}_- \\
& + \sum_{n=0}^{\infty} \frac{(-b/a)^{n+1}}{(n+1)!} (\mathrm{ad}\,\mathbf{K}_+)^n \mathbf{Y}_+ \ .
\end{aligned}
\tag{4.13b}
$$

The other elements in the solution have the same general structure as the ones presented here. However, as before, it is still true that one may equivalently write out the $\mathcal{M}_i$ in terms of the variables $\{h/e, \ln e\}$, instead of $\{e/h, \ln h\}$. One may also use other variables for the $\mathcal{E}_j$.

I truly wonder how can such very "pretty" and "simple-appearing" equations have solutions that look so "nasty." Surely there should be presentations which are less coordinate-dependent! Perhaps the simpler versions, involving only invariant vector-field valued quantities, truly are in the literature somewhere? Nonetheless, I have yet to find them, and would ask that someone help guide me in the right direction. However, I doubt that this is the case for these more complicated questions, involving two sets of indices. Perhaps they are questions involving the "direct product" of two different "spin" representations, but I do not know.

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