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Asymptotic properties of the *C*-metric

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Abstract
The aim of this paper is to analyse the asymptotic properties of the *C*-metric, using a general method specified in the work of Tafel and co-workers (Tafel and Pukas 2000 *Class. Quantum Grav.* **17** 1559–70, Tafel 2000 *Class. Quantum Grav.* **17** 4397–408, Tafel and Natorf 2004 *Class. Quantum Grav.* **21** 5397–407). By finding an appropriate conformal factor Ω, it allows the investigation of the asymptotic properties of a given asymptotically flat spacetime. The news function and Bondi mass aspect are computed, their general properties are analysed, as well as the small mass, small acceleration, small and large Bondi time limits.

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1. Introduction

The *C*-metric is commonly regarded as a spacetime describing two black holes accelerating in opposite directions, under the action of forces represented by conical singularities; see e.g. [4–7]. It admits boost and rotational symmetry and thus belongs to the large class of boost–rotation symmetric spacetimes [8]. While not strictly[3] asymptotically flat, the Bondi coordinates still exist and the news function and mass aspect can be computed. One can start from the news function and asymptotic properties of a general boost-rotation symmetric spacetime (see equation (26) in [10], (17) in [11] or (99) and (121) in [12]). However, here we shall adopt another method, and will give the final expression in the explicit form specific to the *C*-metric. The method created by Tafel and co-workers [1–3] is especially useful for such a computation since, unlike the traditional approach of finding an asymptotic form of a coordinate transformation to Bondi coordinates by comparing expansions of the metric

[3] Two generators of $\mathcal{J}$ are missing, but the spacetime is still locally asymptotically flat, see [9]; in fact for some parameter values $\mathcal{J}^+$ of the *C*-metric even allows some global spherical sections, see also [8].

tensor components, it provides a technique to find a specifically 'calibrated' form of Penrose's conformal factor, from which those quantities can be directly computed. These authors follow (a weakened version of) Penrose's definition of asymptotic flatness [13, 14] to obtain a particular embedding of a given manifold into an unphysical, conformal compactification. By making a set of constraints on the conformal factor, it can be made quite suitable for asymptotic analysis, so that it and its various derivatives can be used to determine the desired news function, Bondi mass and other asymptotic properties. Our goal here is to follow their format, determining a conformal factor that satisfies their 'calibration conditions', as applied to the C-metric, in appropriate coordinates, to obtain the desired asymptotic properties.

The definition used in [1–3] states that a spacetime, $\widetilde{\mathcal{M}}$, with a metric $\widetilde{g}$ is asymptotically flat at $\mathcal{J}^+$ (future null infinity) if and only if the following assumptions are satisfied.

(a) The physical spacetime $\widetilde{\mathcal{M}}$ is a submanifold of an unphysical spacetime $\mathcal{M}$. The metric, g, of $\mathcal{M}$ is conformally equivalent on $\widetilde{\mathcal{M}}$ to the metric $\widetilde{g}$ of $\widetilde{\mathcal{M}}$, i.e. we have a conformal transformation induced by the function Ω, which is always required to be positive on $\widetilde{\mathcal{M}}$:

$$g = \Omega^2 \widetilde{g}. \tag{1}$$

(b) A boundary of $\widetilde{\mathcal{M}}$ in $\mathcal{M}$ contains a three-dimensional null surface $\mathcal{J}^+$ such that Ω vanishes on that boundary although $\mathrm{d}\Omega$ must not, as well as a third condition on the derivative. Using the symbol $\hat{=}$ to denote that something is being evaluated on that boundary, we may write these conditions explicitly:

$$\Omega \hat{=} 0, \qquad \mathrm{d}\Omega \hat{\neq} 0 \quad \text{and} \quad g^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu} \hat{=} 0. \tag{2}$$

(c) The boundary $\mathcal{J}^+$ is diffeomorphic to the product $R \times S_2$. Thought of as a trivial bundle over S_2 the fibres are generated by the future-directed vector field

$$v = g^{\mu\nu}\Omega_{,\nu}\partial_\mu. \tag{3}$$

(d) The pullback of g under the natural embedding $\phi : \mathcal{J}^+ \to \mathcal{M}$ is the natural metric on the sphere:

$$\phi^* g = g_s. \tag{4}$$

(e) The Ricci tensor of $\tilde{g}$ satisfies (in the coordinates of $\mathcal{M}$) the boundary condition

$$\widetilde{R}_{\mu\nu} \hat{=} 2q\, \Omega_{,\mu}\Omega_{,\nu}, \tag{5}$$

where q is a function.

The method is then to follow a sequence of (allowed) transformations of the natural choice for the conformal factor Ω, which arranges in turn for the satisfaction of these requirements, as we describe in detail below for our metric of interest, a particular choice of the range of variables for the C-metric that can be described as two black holes accelerating oppositely. This will also involve various changes of coordinates, heading towards coordinates that have the Bondi–Sachs form [15, 16] (We also note that the news function for the C-metric has been computed in some earlier works, see e.g. [17] and general results in [11, 12, 18] using quite a different method, which can be used as an independent check of that portion of the results here.)

In the following text, we always use the metric signature $(- + ++)$, unlike in [1–3]; this will often lead to the associated sign changes in the equations used and referenced herein. Raising and lowering of the indices, and the covariant derivative, denoted by the symbols $_{|\mu}$, are all understood to be with respect to the unphysical metric $g = \Omega^2\tilde{g}$, unless specified otherwise. The symbol $\hat{=}$ is used in the same sense as above, i.e. to denote that the expression is being evaluated on the $\mathcal{J}^+$ boundary.

Once the final conformal factor $\Omega \equiv \Omega_F$ is obtained, it would allow us to compute the Bondi mass aspect M and the news function $c_{,u}$, important quantities characterizing the asymptotic properties of the spacetime, namely the total mass $m(u)$ and its change:

$$\begin{aligned} m(u) &= \frac{1}{4\pi}\int_{S_2} M(u,\Theta,\Phi)\,\mathrm{d}S = \frac{1}{4\pi}\int_{S_2}\hat{M}(u,\Theta,\Phi)\,\mathrm{d}S,\\ m_{,u}(u) &= -\frac{1}{4\pi}\int_{S_2} c_{,u}^2(u,\Theta,\Phi)\mathrm{d}S, \end{aligned} \tag{6}$$

where $\hat{M}$ is the so-called modified mass aspect and the integral is to be taken over a constant u slice of $\mathcal{J}^+$, i.e. a unit two-sphere. The modified mass aspect differs from M by a four-divergence constructed from the news tensor (see [2], equation (72)), thus allowing us to use it in the integral instead of M. The advantage of $\hat{M}$ lies in its simpler behaviour under the general BMS group, see [2], equation (36), and below. Also it is preferred in the construction of the asymptotic four-momentum, which has then correct transformation properties under the full BMS group, and not just under the Lorentz subgroup (see equations (7) and (8) in [2] for more details).

The BMS (Bondi–Metzner–Sachs) group mentioned above is the group of asymptotic symmetries for any asymptotically flat spacetime, also inducing a corresponding mapping of the $\mathcal{J}^+$ into itself. Notably, it is a much larger group than the Poincaré group, since it contains an infinite-dimensional Abelian normal subgroup, the supertranslations. The factor group obtained by quotienting it out is then isomorphic to the Lorentz group, see [19]. The action of any supertranslation on $\mathcal{J}^+$ manifests itself as an angle-dependent translation of the Bondi time, $u \to u + \alpha(\Theta,\Phi)$. This will become useful for the construction of the Schwarzschild limit in section 5.

The news function $c_{,u}$ and the Bondi mass aspect M also appear in the asymptotic metric expansion in Bondi coordinates. For the simplest case of axial symmetry, without an electromagnetic field and without rotation, the Bondi metric reads (see e.g. [15])

$$\begin{aligned} \tilde{g}_B = &\left(-\frac{V}{r}\mathrm{e}^{2\beta} + U^2 r^2 \mathrm{e}^{2\gamma}\right)\mathrm{d}u^2 - 2\mathrm{e}^{2\beta}\mathrm{d}u\mathrm{d}r - 2Ur^2\mathrm{e}^{2\gamma}\mathrm{d}u\mathrm{d}\Theta\\ &+ r^2(\mathrm{e}^{2\gamma}\mathrm{d}\Theta^2 + \mathrm{e}^{-2\gamma}\sin^2\Theta\mathrm{d}\Phi^2), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \gamma &= \frac{c}{r} + O(r^{-3}), \qquad & U &= -(c_{,\Theta} + 2c\cot\Theta)\frac{1}{r^2} + O(r^{-3}),\\ \beta &= -\frac{c^2}{4r^2} + O(r^{-3}), \qquad & V &= r - 2M + O(r^{-1}), \end{aligned} \tag{8}$$

M is the Bondi mass aspect and the u-derivative of c is the news function. This metric can also be expressed as (see [2], equation (73))

$$\begin{aligned} \tilde{g} = &-\left(1 - \frac{2M}{r} + O(r^{-2})\right)\mathrm{d}u^2 - (2 + O(r^{-2}))\mathrm{d}u\mathrm{d}r - \left(n^B{}_{A|B} + O(r^{-1})\right)\mathrm{d}u\mathrm{d}x^A\\ &+ r^2\left(s_{AB} - \frac{1}{r}n_{AB} + O(r^{-2})\right)\mathrm{d}x^A\mathrm{d}x^B, \end{aligned} \tag{9}$$

s_{AB} being the metric of a two-sphere, M the Bondi mass aspect and n_{AB} the news tensor, related to the news function $c_{,u}$.

With the general formula for M and $c_{,u}$ determined, we then investigate their behaviour in various limits of the C-metric. In section 4 the small mass limit is computed, leading to the Minkowski spacetime with the black hole reduced to a uniformly accelerated particle, and in section 5 the small acceleration limit leading to the Schwarzschild limit is examined. Then, the large and small Bondi time limits are investigated in section 6 for the general case of the C-metric. Then in the last section, 7, the qualitative behaviour of the Bondi mass aspect M and the news function $c_{,u}$ that was observed in the small mass and small acceleration limits is investigated, and compared with the general case.

2. Conformal factor

The C-metric is usually given in the following standard form, which is also most useful for our calculations:

$$\tilde{g} = \frac{1}{A^2(x+y)^2}\left[-F(y)\mathrm{d}t^2 + \frac{\mathrm{d}y^2}{F(y)} + \frac{\mathrm{d}x^2}{G(x)} + G(x)\kappa^2\mathrm{d}\varphi^2\right], \tag{10}$$

where κ is the conicity parameter that determines the physical conicity (the ratio of circumference around a circle to 2π times the radius, see (B.1), (B.2)) on the axis of symmetry. The functions $-F(-z) = G(z)$ are, in the vacuum case, cubic polynomials, usually parametrized in one of the following two forms:

$$G(x) = -a(x-x_1)(x-x_2)(x-x_3) \tag{11a}$$

$$\equiv\ 1 - x^2 - 2mAx^3, \tag{11b}$$

where the first form is useful for explicit computations, while the latter is well adapted for taking limits, as given in sections 4 and 5. There is also no loss of generality associated with the particular gauge (11b); one can always perform a coordinate transformation which translates (11a) into (11b) [6]. The C-metric actually describes four distinct spacetimes [6, 7, 20] specified by the range of x and y coordinates. Our interest is in the most physically reasonable one, interpretable as black holes accelerated along a segment of the axis, with a conical singularity; it is defined by considering $x \in (x_2, x_3)$ and $y \in (-x, \infty)$, where x_2 and x_3 are the two largest roots of $G(x)$, see figure 1. The global conformal extension of this spacetime is schematically depicted in figures 2 and 3. From this and also from a combined

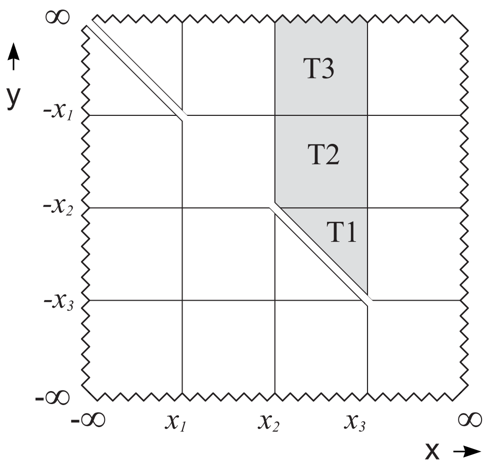


Figure 1. C-metric coordinate ranges.

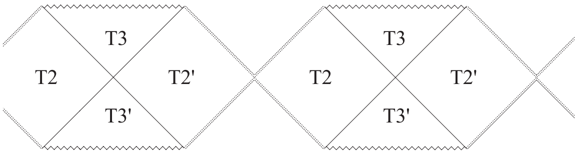


Figure 2. Conformal diagram of the $x = x_2$ slice.

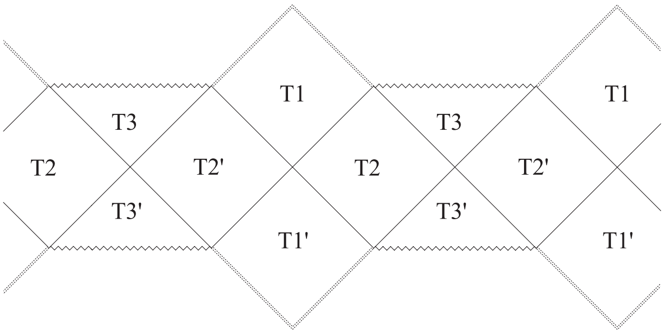


Figure 3. Conformal diagram of the $x = x_3$ slice.

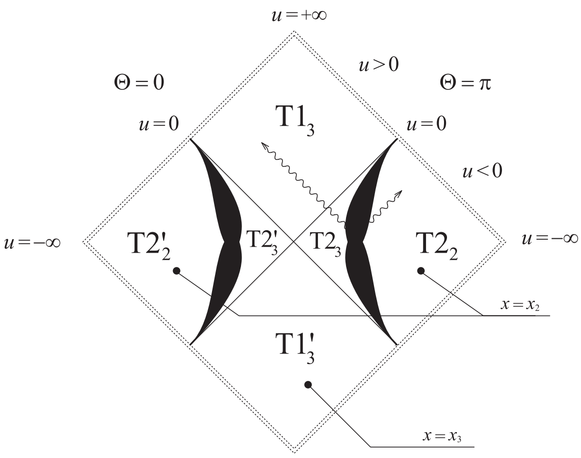


Figure 4. A schematic combination of the conformal slices $x = x_{2,3}$ into a single diagram, using a small mA flat space intuition. The interior structure of the black hole (blocks T3) and the other asymptotic regions are not shown, being located under the black hole horizon in the black area. The black hole horizon still hints at its bifurcation structure, as seen in figures 2 and 3. On $\mathcal{J}^+$, the angular coordinate Θ and the Bondi time $u \doteq \hat{u}$ are plotted.

figure 4, we can immediately see that the $x = x_3$ slice corresponds to the inner axial segment between the accelerating black holes, and the slice $x = x_2$ to the external part of the axis of the symmetry.

Since infinity is located at $x = -y$, and since we would also like to use a null (or retarded) time coordinate, it is convenient to change the metric into the following form:

$$\tilde{g} = \frac{1}{\rho^2}\left[-F\mathrm{d}w^2 + \frac{2}{A}\,\mathrm{d}w\mathrm{d}\rho - 2\mathrm{d}w\mathrm{d}x + \frac{\mathrm{d}x^2}{G} + G\kappa^2\mathrm{d}\varphi^2\right], \tag{12}$$

by the transformation $\rho = A(x + y)$, $w = t + \int \mathrm{d}y/F(y)$.

We now begin our sequence of choices for the switch to an unphysical, conformally equivalent metric $g = \Omega^2\tilde{g}$, such that all the properties (a)–(e), above, are satisfied. The first, quite natural choice is $\Omega \equiv \Omega_0 = \rho = A(x + y)$. Therefore the unphysical metric is now given by the following:

$$g = -F\mathrm{d}w^2 + \frac{2}{A}\,\mathrm{d}w\mathrm{d}\rho - 2\mathrm{d}w\mathrm{d}x + \frac{\mathrm{d}x^2}{G} + G\kappa^2\mathrm{d}\varphi^2, \tag{13}$$

with its contravariant form as follows:

$$g^{-1} = 2A\,\partial_w\partial_\rho + A^2(F + G)\,\partial_\rho^2 + 2AG\,\partial_\rho\partial_x + G\partial_x^2 + \frac{1}{G\kappa^2}\,\partial_\varphi^2. \tag{14}$$

However, condition (d) above is not satisfied, i.e. the pullback[4] of this metric on $\mathcal{J}^+$, namely

$$\phi^* g \equiv g_2 = G\mathrm{d}w^2 - 2\mathrm{d}w\mathrm{d}x + \frac{\mathrm{d}x^2}{G} + G\kappa^2\mathrm{d}\varphi^2 = \left(\sqrt{G}\,\mathrm{d}w - \frac{\mathrm{d}x}{\sqrt{G}}\right)^2 + G\kappa^2\mathrm{d}\varphi^2, \tag{15}$$

is not the metric g_S of a unit 2-sphere. To correct this, we improve our Ω_0 by multiplying it with another factor, so that the resulting new choice, namely $\Omega_0\Omega_S$, will satisfy this condition, which requires

$$\Omega_S^2\left[G\mathrm{d}w^2 - 2\mathrm{d}w\mathrm{d}x + \frac{\mathrm{d}x^2}{G} + G\kappa^2\mathrm{d}\varphi^2\right] = \frac{4\,\mathrm{d}\xi\mathrm{d}\bar{\xi}}{(1 + \xi\bar{\xi})^2} = \mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2 \tag{16}$$

presented for both stereographic and standard angular coordinates, which are related as

$$\xi = \mathrm{e}^{\mathrm{i}\Phi}\tan\frac{\Theta}{2}, \qquad \bar{\xi} = \mathrm{e}^{-\mathrm{i}\Phi}\tan\frac{\Theta}{2}\,. \tag{17}$$

In order to relate the C-metric coordinates to the asymptotic angular coordinates as simply as possible, we choose $\varphi = \Phi$. (If we need some other particular coordinates on $\mathcal{J}^+$ later, we can always use some BMS transformation later to transform to them.) Comparing first the coefficients at $\mathrm{d}\varphi^2$ and $\mathrm{d}\Phi^2$, and then the rest, we obtain

$$\begin{aligned}
&\Omega_S^2 = \frac{\sin^2\Theta}{G\kappa^2}, \qquad \frac{\kappa^2\mathrm{d}\Theta^2}{\sin^2\Theta} = \mathrm{d}w^2 - \frac{2\mathrm{d}w\mathrm{d}x}{G} + \frac{\mathrm{d}x^2}{G^2} = \left(\mathrm{d}w - \frac{\mathrm{d}x}{G}\right)^2\\
&\quad\Rightarrow \text{we can choose}\quad \frac{\kappa\mathrm{d}\Theta}{\sin\Theta} = \mathrm{d}w - \frac{\mathrm{d}x}{G} \equiv \kappa\mathrm{d}S\\
&\quad\Rightarrow \kappa S = w - \int\frac{\mathrm{d}x}{G} \equiv w - \mathrm{Gi}(x).
\end{aligned} \tag{18}$$

Integrating this, we obtain

$$\begin{aligned}
&S = \ln\left(\frac{1 - \cos\Theta}{\sin\Theta}\right) \;\Rightarrow\; \mathrm{e}^S = \tan\frac{\Theta}{2} \;\Rightarrow\; \mathrm{ch}\,S = \frac{1}{\sin\Theta}, \quad \mathrm{sh}\,S = -\cot\Theta,\\
&\quad\Omega_S = \frac{1}{\mathrm{ch}\,S\,\kappa G^{1/2}}.
\end{aligned} \tag{19}$$

[4] Using expansion $F(y) = F(\frac{\rho}{A} - x) = -G(x - \frac{\rho}{A}) = -G(x) + G'(x)\frac{\rho}{A} + O\left(\frac{\rho^2}{A^2}\right)$. Also note that in our coordinates, pullback of a form simply means to disregard all components containing $\mathrm{d}\rho$.

Using the relation[5] between S and Θ, we can express the stereographic coordinates as follows:

$$\xi = \mathrm{e}^{\mathrm{i}\Phi}\mathrm{e}^{S}, \qquad \bar{\xi} = \mathrm{e}^{-\mathrm{i}\Phi}\mathrm{e}^{S}. \tag{20}$$

As already noted there is a sequential approach to modifying the conformal factor and the coordinates so as to obtain a form that presents clearly that the metric has the desired asymptotic properties. Following Tafel's approach [1], we next need to satisfy his equations (34) and (38), which, in our $(-+++)$ signature, take the form

$$\Omega^{-2}\Omega^{|\mu}\Omega_{|\mu} \mathrel{\hat{=}} 1, \quad \Omega^{-1}\,\Omega^{|\mu}{}_{\mu} \mathrel{\hat{=}} 2. \tag{21}$$

The first may be understood as a calibration condition, which transforms Ω into a form suitable for direct insertion into general formulae for the news function and mass aspect. The latter equation is related to the condition on the determinant of the resulting Bondi–Sachs coordinates (see equations (7) and (38) of [1]). If conditions (a)–(e) are satisfied, then the latter follows from the former.

Our $\Omega' \equiv \Omega_0\Omega_S$ unfortunately does not yet satisfy (21), and therefore needs to be modified further. To get such an Ω we correct Ω' using the gauge freedoms given in equation (35) of [1] and equation (52) of [2], which we repeat below:

$$\Omega = \Omega' + \eta\Omega'^2, \qquad \eta \mathrel{\hat{=}} -\tfrac{1}{2}\hat{u}^{|\mu}{}_{\mu} - \Omega'^{-1}\,(1 - \Omega'^{|\nu}\hat{u}_{|\nu}). \tag{22}$$

The coordinate $\hat{u}$ is an asymptotic Bondi time, which coincides with the Bondi time coordinate on $\mathcal{J}^+$. The function $\hat{u}$ can be obtained from equation (16) of [1] and equation (48) of [2], which for our Ω' reads

$$\Omega'^{|\mu}\partial_\mu = \partial_{\hat{u}} \mathrel{\hat{=}} \partial_u, \quad \Omega'^{|\mu}\hat{u}_\mu \mathrel{\hat{=}} 1. \tag{23}$$

Substituting $\Omega' = \Omega_0\Omega_S$ leads directly to a differential relation:

$$A\,\Omega_S^{-1}(G\partial_x + \partial_w) = \partial_{\hat{u}}, \tag{24}$$

which has as a solution the following:

$$\hat{u} = \frac{1}{AK\,\mathrm{ch}\,S}\int \frac{\mathrm{d}x}{G^{3/2}} + \alpha(\xi,\bar{\xi}) \equiv \frac{1}{AK\,\mathrm{ch}\,S}\,\mathrm{Gj}(x) + \alpha(\xi,\bar{\xi}), \tag{25}$$

where we have denoted the integral of $G^{-3/2}$ as Gj. The explicit form and some other properties of this function can be found in the appendix, see (D.1), (D.2) and also figure D1. The quantity $\alpha(x^A)$ includes the integration constant of the function Gj. It is an arbitrary function of the asymptotic angular coordinates only, i.e. it does not depend on the Bondi time $\hat{u}$, and corresponds to the supertranslations contained in the BMS group of the $\mathcal{J}^+$ coordinate transformations.

This result for $\hat{u}$ together with (22) allows us to obtain[6] η and therefore the final conformal factor:

$$\Omega_F = \Omega_0\Omega_S(1 + \eta\Omega_0\Omega_S)\,, \quad \Omega_0 = \rho, \quad \Omega_S = \frac{1}{K\,\mathrm{ch}\,S\sqrt{G}}\,,$$
$$\eta = \frac{\mathrm{Gj}}{2AK\,\mathrm{ch}\,S}(1 - \mathrm{sh}^2 S) + \frac{KG'}{4A\sqrt{G}}\,\mathrm{ch}\,S - \frac{\mathrm{sh}\,S}{A\sqrt{G}} - \frac{1}{2}\triangle\alpha, \tag{26}$$

[5] It is also interesting to note that on $\mathcal{J}^+$ the s coordinate behaves as $K\mathrm{d}S = \mathrm{d}t + \frac{\mathrm{d}y}{F} - \frac{\mathrm{d}x}{G} = \mathrm{d}t + O(\rho) \mathrel{\hat{=}} \mathrm{d}t$.

[6] Interestingly, had we used the coordinates (w,ρ,Θ,φ), instead of the more direct choice (w,ρ,x,φ), the formula (22) for $\hat{u}$ would have simplified significantly, giving $\eta = -\frac{1}{2}\triangle\hat{u}$, where $\triangle$ is the Laplace operator on the 2-sphere and the derivatives are to be taken while holding w constant. This is probably because in the (w,ρ,Θ,φ) coordinates the angular part of the metric is exactly that of the 2-sphere, and so are the corresponding $\Gamma^A{}_{BC}$. Effectively, what happens is that the $\Omega^{-1}\,(1 - \Omega^{|\nu}\hat{u}_{|\nu})$ term precisely cancels with $-\frac{1}{2}(\hat{u}^{|w}{}_w + \hat{u}^{|\rho}{}_\rho)$.

where $\triangle$ is the Laplace operator on the 2-sphere: $\triangle = (1+\xi\bar{\xi})^2\partial_\xi\partial_{\bar{\xi}} = \partial_\Theta\partial_\Theta + \cot\Theta\,\partial_\Theta + \sin^{-2}\Theta\,\partial_\Phi\partial_\Phi$.

Using $\Omega \equiv \Omega_F$ in the transition $\tilde{g} \to g = \Omega^2\tilde{g}$ now finally ensures that both conditions (21) are satisfied. The dependence on α is also in agreement with the transformation properties of η under the supertranslations, see [2], equation (32). From now on, we will simply denote Ω_F as Ω.

3. Mass aspect and the news function

The final conformal factor $\Omega \equiv \Omega_F$ will now be used, according to the scheme outlined at the end of section 1, to obtain the Bondi mass aspect M and the news function $c_{,u}$. According to the procedure described in section 3 of [2], we start with the modified mass aspect $\hat{M}$. In our case the explicit formula [2], equation (44), for the modified mass aspect and its change under the supertranslations subgroup of the BMS transformations yields, for the final conformal factor Ω,

$$\begin{aligned}\hat{M} &\hat{=} \Omega^{-1}\left(1 - 2\Omega^{-2}\Omega^{|\mu}\Omega_{|\mu} + \frac{1}{2}\Omega^{-1}\Omega^{|\mu}{}_{\mu}\right)\\ &= \frac{\mathrm{ch}^3 S}{4A\kappa\sqrt{G}}\left[\frac{\kappa^4 G'^3}{8} - \kappa^2 G' - \frac{1}{4}\kappa^4 GG'G'' + \frac{1}{6}\kappa^4 G^2 G''' - \sqrt{G}\mathrm{Gj}\right] - \frac{1}{4}(\triangle+2)\triangle\alpha\end{aligned} \tag{27}$$

and its u-derivative

$$\hat{M}_{,u} \hat{=} -\frac{1}{4}\,\mathrm{ch}\,S^4\left[1 + \frac{1}{2}\kappa^2 GG'' - \frac{1}{4}\kappa^2 G'^2\right]^2. \tag{28}$$

The behaviour under the supertranslation α simply follows from the α-dependence of expression (26) used in Ω_F, in accord with the general formula [2], equation (41).

To find the news function $c_{,u}$, we first compute the news tensor as described in [2], equation (67) or (61):

$$\begin{aligned} n &= -\varphi^*(L_v g) = -2\varphi^*(u_{|\mu\nu}\mathrm{d}x^\nu\mathrm{d}x^\mu), \qquad v \equiv u^{|\nu}\partial_\nu\\ \dot{n} &= -\varphi^*(\Omega^{-1}L_v g) = -2\varphi^*(\Omega^{-1}\Omega_{|\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu)\end{aligned} \tag{29}$$

using also [2], equation (69), which in our case reads

$$n'_{AB} = n_{AB} - 2\alpha_{|AB} + \triangle\alpha g_{SAB}.$$

This leads[7] to the news tensor n_{AB} being

$$n_{\xi\xi} \hat{=} -\frac{1}{\xi\xi}\frac{1}{2A\kappa\,\mathrm{ch}\,S}\left(\mathrm{Gj} + \frac{\kappa^2 G'}{2\sqrt{G}}\right) - 2\,\alpha_{|\xi\xi}, \qquad n_{\bar{\xi}\bar{\xi}} = \bar{n}_{\xi\xi}\ , \quad n_{\xi\bar{\xi}} = 0 \tag{30}$$

which is consistent with (28), seen by using [2], equation (66).

Now we can directly compare the angular parts of the Bondi metric forms (7) and (9):

$$\xi^2(\sin\Theta)^{-2}n_{\xi\xi} + \mathrm{c.c.} = n_{\Theta\Theta} = -(\sin\Theta)^{-2}n_{\Phi\Phi} = -2c, \tag{31}$$

leading to

$$\begin{aligned} c &= -\frac{1}{2A\kappa\sin\Theta}\left[\mathrm{Gj} + \frac{\kappa^2 G'}{2\sqrt{G}}\right] + \frac{1}{2}\alpha_{|\Theta\Theta} - \frac{1}{2}\sin^{-2}\Theta\alpha_{|\Phi\Phi},\\ c_{,u} &= -\frac{1}{2\sin^2\Theta}\left[1 + \frac{1}{2}\kappa^2 GG'' - \frac{1}{4}\kappa^2 G'^2\right] = -\frac{\kappa^2}{2\sin^2\Theta}\left[1 + \frac{1}{2}\,GG'' - \frac{1}{4}G'^2\right] + \frac{\kappa^2-1}{2\sin^2\Theta}.\end{aligned} \tag{32}$$

[7] The same simplification can occur as in the case of η, see remark (6). Also, the α correction vanishes identically for $n_{\xi\bar{\xi}}$, ensuring the tracelessness of n_{AB}.

Next we use our result for c to obtain the Bondi mass aspect M from the reduced mass aspect $\hat{M}$, using the formula [2], equation (72),

$$M = \hat{M} - \tfrac{1}{4}n^{AB}{}_{|AB} = \hat{M} + \tfrac{1}{2}(c_{,\Theta\Theta} + 3c_{,\Theta}\cot\Theta - 2c - c_{,\Phi\Phi}\sin^{-2}\Theta), \qquad (33)$$

which we have expressed in terms of the news function $c(u, \Theta, \Phi)$, using relation (31).

Together with one of the two Bondi's supplementary conditions[8], assuming[9] $c = c(u, \Theta)$ only, i.e. it is axially symmetric,

$$M_{,u} = -c_{,u}^2 + \tfrac{1}{2}(c_{,\Theta\Theta} + 3c_{,\Theta}\cot\Theta - 2c)_{,u},$$

we obtain an interesting relation:

$$M_{,u} = -c_{,u}^2 - \tfrac{1}{4}\left(n^{AB}{}_{|AB}\right)_{,u} \Rightarrow \hat{M}_{,u} = -c_{,u}^2, \qquad (34)$$

in agreement with (28) and [2], equation (66). An explicit calculation of $M_{,u}$ then leads to

$$M_{,u} = -\frac{1}{4\sin^4\Theta}\left[1 + \frac{1}{2}\kappa^2 GG'' - \frac{1}{4}\kappa^2 G'^2\right]^2 - \frac{5A^2\kappa^4 G^3 G' G'''}{16\sin^4\Theta}\left(\alpha_{,\Theta} + \cos\Theta\frac{\mathrm{Gj}}{A\kappa}\right)^2 + \frac{A\kappa^3 G^{\frac{5}{2}} G'''}{8\sin^4\Theta}\left(\alpha_{,\Theta\Theta}\sin\Theta - 3\alpha_{,\Theta}\cos\Theta - (2\cos^2\Theta + 1)\frac{\mathrm{Gj}}{A\kappa}\right). \qquad (35)$$

Substituting (32) into (33), and using the partial derivatives (A.1), we finally obtain the value for M itself:

$$M = \frac{1}{4\sin^3\Theta}\left[1 + \frac{1}{2}\kappa^2 GG'' - \frac{1}{4}\kappa^2 G'^2\right]\left(\alpha_{,\Theta\Theta}\sin\Theta - \alpha_{,\Theta}\cos\Theta - \frac{\mathrm{Gj}}{A\kappa}\right) - \frac{A\kappa^3 G^{\frac{5}{2}} G'''}{8\sin^3\Theta}\left(\alpha_{,\Theta} + \cos\Theta\frac{\mathrm{Gj}}{A\kappa}\right)^2 + \frac{1}{8A\sqrt{G}\sin^3\Theta}\left[\frac{1}{4}\kappa^3 G'^3 - \kappa G' - \frac{1}{2}\kappa^3 GG'G'' + \frac{1}{3}\kappa^3 G^2 G'''\right]. \qquad (36)$$

It is interesting to note the absence of a pure α correction term: after putting the α terms from (30), (27) together into definition (33) one would expect to obtain

$$-\tfrac{1}{2}\Delta\Delta\alpha - \tfrac{1}{2}\Delta\alpha + \tfrac{1}{2}\alpha_{|AB}{}^{AB},$$

in (36), but surprisingly, due to the simplicity of the Riemann tensor in two dimensions, this correction term is identically zero for any $\alpha(\Theta, \Phi)$. Therefore the mass aspect $M(\hat{u}, \Theta, \Phi)$ transforms itself under the supertranslations $\hat{u} \to \hat{u}' = \hat{u} + \alpha(\Theta, \Phi)$ only due to the explicit change of $\hat{u}$, and due to the change of the partial derivatives[10] in $\frac{1}{4}n_{AB}{}^{|AB}$.

Last but not least, there is an interesting 'alternative' result for M. If we had used $\frac{\partial x}{\partial\Theta}|_{w=\text{const.}}$ instead of $\frac{\partial x}{\partial\Theta}|_{\hat{u}=\text{const.}}$ in (33), we would have ended with a much more simple expression:

$$M^{(w)} = -\frac{\kappa^3 G^{\frac{3}{2}} G'''}{12A\sin^3\Theta} = m\kappa^3 G^{\frac{3}{2}}\,\mathrm{ch}^3 S = m\Omega_S^{-3}.$$

However, its interpretation is not clear, since w is not a Bondi time[11].

[8] A consequence of the vacuum Einstein equation, namely $R_{00} = 0$, in coordinates (7), see [15] for further discussion.

[9] This is just a technical simplification, since the version of Bondi's analysis covered in textbooks typically assumes axial symmetry. Of course, the C-metric case covered here is also axially symmetric.

[10] In our case this manifested itself via (A.1). Had we been able to express n_{AB} in terms of $\hat{u}$, Θ and Φ only, α would have emerged as a result of $\frac{\partial\hat{u}}{\partial\Theta'} = -\alpha_{,\Theta}$ and $\frac{\partial\hat{u}}{\partial\Phi'} = -\alpha_{,\Phi}$ in the transformation $\hat{u}' = \hat{u} + \alpha$, $\Theta' = \Theta$ and $\Phi' = \Phi$.

[11] On the other hand w becomes the Bondi time asymptotically in the Schwarzschild limit, see section 5. It might be possible to regard this mass aspect as a mass aspect for a stationary observer adapted to (at rest with respect to) the accelerated black hole, which becomes an asymptotic Schwarzschild observer in the $A \to 0$ limit.

4. Small mass limit

In this section we find the limit of the news function $c_{,u}$ and the mass aspect M for small mass m. Substituting $G(x)$ in the form of (11b) into (32), the news function $c_{,u}$ reduces as follows:

$$c_{,u} = \frac{\kappa^2 - 1}{2\sin^2\Theta} - \frac{A\kappa^2 m x}{\sin^2\Theta}[x^2 - 3] + O(m^2). \tag{37}$$

To obtain this in terms of the asymptotic Bondi coordinates only, we express Gj(x) from (25). Then, using expansion (A.4) together with (A.5), we arrive to

$$c_{,u} = \frac{\kappa^2 - 1}{2\sin^2\Theta} \pm \frac{2\kappa^2 A}{\sin^2\Theta} m + \frac{\hat{u} A^2\kappa^3}{\sin^2\Theta} \cdot \frac{2\hat{u}^2 A^2\kappa^2 + 3\sin^2\Theta}{(\hat{u}^2 A^2\kappa^2 + \sin^2\Theta)^{3/2}} m + O(m^2) \tag{38}$$

where κ denotes the physical conicity between particles (i.e. $\kappa \equiv \kappa_{\text{ext}}$) for the '$-$' sign, or outside the particles ($\kappa \equiv \kappa_{\text{in}}$) for the '+' sign respectively. This is in agreement with the special case of $\kappa_{\text{ext}} = 1$ presented in [20]. It is closely related to news functions given in [10, 11, 18]. Note that the first term corresponds precisely to the news function of an infinite cosmic string; see equation (7) in [21] and (25) in [22].

It is of interest to see how exactly the possible singularities of the function $c_{,u}$ are distributed on $\mathcal{J}^+$. First, with the help of our equation (25) , we realize that sign $\hat{u}$ = sign Gj, and from the properties of the Gj(x) function (see figure D1) correspondence between the roots $x = x_{2,3}$ and poles $\Theta = 0, \pi$ emerges as

$$\begin{aligned} \hat{u} > 0&:\ \Theta = 0, \Theta = \pi \ \leftrightarrow\ x = x_3, \\ \hat{u} < 0&:\ \Theta = 0, \Theta = \pi \ \leftrightarrow\ x = x_2. \end{aligned} \tag{39}$$

In general, we have a 2 : 1 relation between Θ and the x coordinate; for any $x_2 \leqslant x \leqslant x_3$ we have two corresponding values, Θ and $\pi - \Theta$. In the schematic conformal pictures (figures 4–7), this naturally emerges as $\Theta = 0$ on one side and $\Theta = \pi$ on the other.

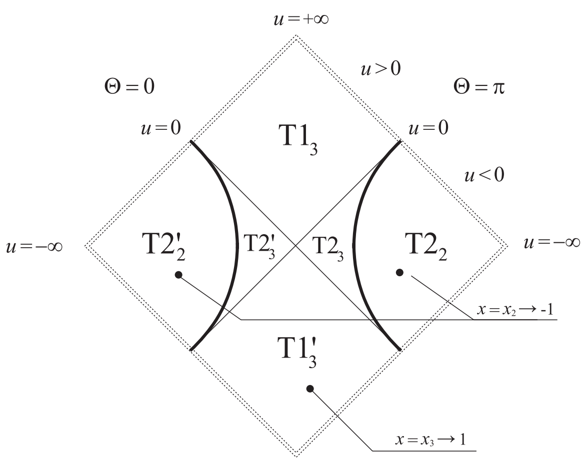


Figure 5. Conformal diagram depicting the Minkowski limit $m \to 0$. On $\mathcal{J}^+$, the Bondi angular coordinate Θ is shown, together with the Bondi null time u. The black hole is reduced to a uniformly accelerated pointlike particle.

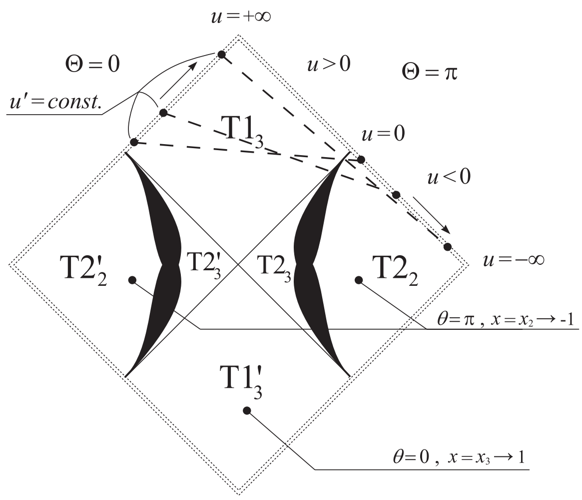


Figure 6. Conformal diagram representing the Schwarzschild limit $A \to 0$ and the supertranslation $\hat{u}' \to \hat{u} + \alpha$, from the perspective of the original Bondi time coordinate $\hat{u}$.

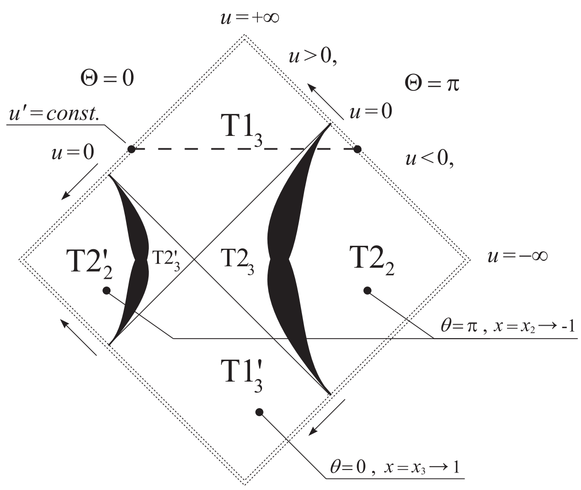


Figure 7. Same as in figure 6, but from the perspective of the translated Bondi time coordinate $\hat{u}'$. The emerging conformal diagram of a single black hole (blocks $T2_3$, $T2_2$ and the black hole interior) is now clearly recognized. See also related comment after (54).

It is obvious that singular behaviour can only occur on the axis, i.e. for $\Theta = 0, \pi$. Expanding around the poles yields the following:

$$\kappa_{\text{ext}} = 1: \quad \begin{cases} c_{,u} = 2mA(\operatorname{sign}\hat{u} + 1)\dfrac{1}{\Theta^2} + \dfrac{2}{3}mA(\operatorname{sign}\hat{u} + 1) + O(\Theta^2) \\ c_{,u} = 2mA(\operatorname{sign}\hat{u} + 1)\dfrac{1}{(\pi - \Theta)^2} + \dfrac{2}{3}mA(\operatorname{sign}\hat{u} + 1) + O((\pi - \Theta)^2), \end{cases}$$

$$\kappa_{\text{in}} = 1: \quad \begin{cases} c_{,u} = 2mA(\operatorname{sign}\hat{u} - 1)\dfrac{1}{\Theta^2} + \dfrac{2}{3}mA(\operatorname{sign}\hat{u} - 1) + O(\Theta^2) \\ c_{,u} = 2mA(\operatorname{sign}\hat{u} - 1)\dfrac{1}{(\pi - \Theta)^2} + \dfrac{2}{3}mA(\operatorname{sign}\hat{u} - 1) + O((\pi - \Theta)^2). \end{cases} \tag{40}$$

Therefore by choosing the conical singularity to exist only between the particles, it appears on $\mathscr{J}^+$ only for $u > 0$, that is above the acceleration Cauchy horizon $y = -x_1$, and vice versa (see figures 1 and 4). It is interesting that, as shown in section 7, this situation occurs not only in the small m limit, but also persists in the full C-metric.

Also, perhaps not very surprisingly, this behaviour translates into the properties of the Bondi mass aspect M. Using the same approach as in the case of the news function, application of (25), (A.4) and (A.5) for (36), while assuming $\alpha = 0$, yields

$$\begin{aligned} M(\hat{u}, \Theta) = {} & \frac{m}{2U^5 \sin^3\Theta}(u'^2(3\cos^2\Theta - 1) - 1) \\ & - \frac{m^2 A u'}{U^6 \sin^3\Theta}(8u'^6 + 24u'^4 + u'^2(15\cos^2\Theta + 20) + 4) \pm \frac{m^2 A}{U^7 \sin^3\Theta} \\ & \times (8u'^8 + 28u'^6 + u'^4(6\cos^2\Theta + 34) - u'^2(9\cos^2\Theta - 19) + 5) + O(m^3), \\ \text{where} \quad & U = \sqrt{u'^2 + 1}, \quad u' = \frac{\hat{u}A}{\sin\Theta}, \end{aligned} \tag{41}$$

and the '$-$' sign denotes the case with the axis regular outside the particles, i.e. $\kappa_{\text{ext}} = 1$, while the '+' sign assumes the axis being regular between the particles, i.e. $\kappa_{\text{in}} = 1$.

To investigate the integrability of our mass aspect M, we expand (41) near the poles $\Theta = 0$ $(x = x_3)$ and $\Theta = \pi$ $(x = x_2)$, obtaining

$$\begin{aligned} \kappa_{\text{ext}} = 1: & \begin{cases} M = -8m^2A^2\hat{u}(1 + \operatorname{sign}\hat{u})\dfrac{1}{\Theta^4} - \dfrac{16}{3}m^2A^2\hat{u}(1 + \operatorname{sign}\hat{u})\dfrac{1}{\Theta^2} + O(1) \\ M = -8m^2A^2\hat{u}(1 + \operatorname{sign}\hat{u})\dfrac{1}{(\pi - \Theta)^4} - \dfrac{16}{3}m^2A^2\hat{u}(1 + \operatorname{sign}\hat{u})\dfrac{1}{(\pi - \Theta)^2} + O(1) \end{cases} \\ \kappa_{\text{in}} = 1: & \begin{cases} M = -8m^2A^2\hat{u}(1 - \operatorname{sign}\hat{u})\dfrac{1}{\Theta^4} - \dfrac{16}{3}m^2A^2\hat{u}(1 - \operatorname{sign}\hat{u})\dfrac{1}{\Theta^2} + O(1) \\ M = -8m^2A^2\hat{u}(1 - \operatorname{sign}\hat{u})\dfrac{1}{(\pi - \Theta)^4} - \dfrac{16}{3}m^2A^2\hat{u}(1 - \operatorname{sign}\hat{u})\dfrac{1}{(\pi - \Theta)^2} + O(1)\,. \end{cases} \end{aligned} \tag{42}$$

As expected, the behaviour is qualitatively the same as in the case of $c_{,u}$.

Furthermore, we find the mass change[12] $M_{,u}$ to be

$$\begin{aligned} M_{,u}(\hat{u}, \Theta) = {} & -\frac{3mu'A}{2U^7 \sin^4\Theta}(u'^2(3\cos^2\Theta - 1) - 1 - 2\cos^2\Theta) \\ & - \frac{m^2A^2}{U^8 \sin^4\Theta}(8u'^8 + 32u'^6 + 15u'^4(4 - 3\cos^2\Theta) + 5u'^2(9\cos^2\Theta + 8) + 4) \\ & \pm \frac{m^2A^2u'}{U^9 \sin^4\Theta}(8u'^8 + 36u'^6 + 6u'^4(11 - 3\cos^2\Theta) \\ & + u'^2(41 + 69\cos^2\Theta) - 18\cos^2\Theta + 3) + O(m^3), \\ \text{where} \quad & U = \sqrt{u'^2 + 1}, \quad u' = \frac{\hat{u}A}{\sin\Theta} \end{aligned} \tag{43}$$

again; the '$-$' sign denotes $\kappa_{\text{ext}} = 1$, while the '+' occurs for $\kappa_{\text{in}} = 1$.

[12] The function M is fortunately sufficiently continuous that the $\hat{u}$-derivative commutes with the $m \to 0$ limit.

On first sight, one may expect, according to (6), an inconsistency with (38), because of the non-zero term $O(m)$ here in $M_{,u}$. It is, however, still true that $M^{(\mathrm{tot.})}_{,u} = -\int c^2_{,u}\mathrm{d}S = \int M_{,u}\mathrm{d}S$ (if it exists) because the aforementioned term $O(m)$ integrates to zero. This confirms that the lowest order of mass change is indeed $O(m^2)$, as suggested by $c_{,u}$, and in analogy with the electromagnetic radiation case, where the Poynting vector, and therefore the energy radiated per unit time, is proportional to the charge of the accelerated particle squared.

The expansion near the poles is completely analogous to the case of M and can be simply obtained by applying the derivative with respect to $\hat{u}$ to (40):

$$\kappa_{\mathrm{ext}} = 1: \quad \begin{cases} M_{,u} = -8m^2A^2(1+\operatorname{sign}\hat{u})\dfrac{1}{\Theta^4} - \dfrac{16}{3}m^2A^2(1+\operatorname{sign}\hat{u})\dfrac{1}{\Theta^2} + O(1) \\ M_{,u} = -8m^2A^2(1+\operatorname{sign}\hat{u})\dfrac{1}{(\pi-\Theta)^4} - \dfrac{16}{3}m^2A^2(1+\operatorname{sign}\hat{u})\dfrac{1}{(\pi-\Theta)^2} + O(1), \end{cases}$$

$$\kappa_{\mathrm{in}} = 1: \quad \begin{cases} M_{,u} = -8m^2A^2(1-\operatorname{sign}\hat{u})\dfrac{1}{\Theta^4} - \dfrac{16}{3}m^2A^2(1-\operatorname{sign}\hat{u})\dfrac{1}{\Theta^2} + O(1) \\ M_{,u} = -8m^2A^2(1-\operatorname{sign}\hat{u})\dfrac{1}{(\pi-\Theta)^4} - \dfrac{16}{3}m^2A^2(1-\operatorname{sign}\hat{u})\dfrac{1}{(\pi-\Theta)^2} + O(1). \end{cases} \tag{44}$$

5. Schwarzschild limit

It is also possible to investigate the other situation, which is the case where m is kept finite and non-zero while $A \to 0$. Intuitively, this should lead to a single static black hole, i.e. to the Schwarzschild solution. To show that this is indeed true, we start with the physical metric (10). In order to get the correct limit of the metric tensor, we have to parametrize the coordinates and the parameters A, m, κ, characterizing the C-metric solution. Perhaps the most physically plausible way to do this is by using A as a parameter[13], while holding the horizon area $\mathcal{A}$ and the conicity $\kappa_{\mathrm{ext,in}}$ on one of the axial segments constant. This leads to (see (B.2), (C.3))

$$\begin{aligned} \kappa_{\mathrm{ext}} = 1: &\quad \kappa = 1 + 2m'A + \tfrac{11}{2}m'^2A^2 - 4m'^3A^3 + O(A^4) \\ m &= m' - m'^2A - \tfrac{21}{2}m'^3A^2 + \tfrac{71}{2}m'^4A^3 + O(A^4) \\ \kappa_{\mathrm{in}} = 1: &\quad \kappa = 1 - 2m'A + \tfrac{11}{2}m'^2A^2 + 4m'^3A^3 + O(A^4) \\ m &= m' + m'^2A - \tfrac{21}{2}m'^3A^2 - \tfrac{71}{2}m'^4A^3 + O(A^4), \end{aligned} \tag{45}$$

where the horizon area $\mathcal{A} = 16\pi m'^2$. Together with a simple coordinate rescaling:

$$\begin{aligned} y &= y'A^{-1} \\ t &= t'A \\ x &= x' \end{aligned} \quad \Rightarrow \quad \begin{aligned} &A^2F = -A^2 + y'^2 - 2my'^3 = -A^2 + F' + O(A), \quad F' = y'^2(1-2m'y'), \\ &G = 1 - x'^2 - 2mAx'^3 = G' - 2mAx'^3, \quad G' = 1 - x'^2, \end{aligned} \tag{46}$$

[13] In the $m \to 0$ limit, A is precisely the acceleration of the test particle which the black hole becomes.

the metric $\tilde{g}$ takes the following reduced form:

$$\begin{aligned}\tilde{g} &= \frac{1}{A^2(x+y)^2}\left[-F\mathrm{d}t^2 + \frac{\mathrm{d}y^2}{F} + \frac{\mathrm{d}x^2}{G} + G\kappa^2\mathrm{d}\varphi^2\right] \\ &= \frac{1}{(Ax'+y')^2}\left[(F'+O(A))\mathrm{d}t'^2 + \frac{\mathrm{d}y'^2}{F'+O(A)} + \frac{\mathrm{d}x'^2}{G'-2mAx'^3} + (G'-2mAx'^3)\kappa^2\mathrm{d}\varphi^2\right] \\ &\underset{A\to 0}{\to} \frac{1}{y'^2}\left[-F'\mathrm{d}t'^2 + \frac{\mathrm{d}y'^2}{F'} + \frac{\mathrm{d}x'^2}{G'} + G'\mathrm{d}\varphi^2\right],\end{aligned} \tag{47}$$

which is indeed the Schwarzschild metric[14] as can be seen using the transformation

$$\begin{aligned} y' &= 1/R \\ x' &= \cos\theta \end{aligned} \quad \Rightarrow \quad \tilde{g} = -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{\left(1-\frac{2M}{R}\right)} + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{48}$$

where the Schwarzschild mass M equals the mass m', confirming the condition $\mathcal{A} =$ constant.

Now we can proceed to compute the limits of various asymptotic quantities, i.e. obtain the asymptotic expansion of the metric in Bondi coordinates. The strategy is to obtain a given quantity $f(\hat{u}, \Theta)$ as a series in A with coefficients being a function of the limiting coordinates θ and u' only. This is however not as straightforward as the $m \to 0$ case, as we will see when computing the limit of $\hat{u}$, which we would like to coincide with the Bondi time for the Schwarzschild metric. To check whether this can be satisfied, it is useful to realize the relation between the Schwarzschild Bondi time and the limit of the w coordinate (see (12)):

$$\begin{aligned} w &= t + \int \frac{\mathrm{d}x}{F} = At' + \int \frac{A\mathrm{d}y'}{F'+O(A)} = \left(t' + \int \frac{\mathrm{d}y'}{F'}\right) A + O(A^2) \equiv w'A + O(A^2) \\ &\text{while also:}\quad t' + \int \frac{\mathrm{d}y'}{F'} = t' - \int \frac{\mathrm{d}R}{1-\frac{2M}{R}} \equiv t' - r^* \equiv u', \end{aligned} \tag{49}$$

where u' is precisely the Bondi null time[15] on $\mathcal{J}^+$ of the Schwarzschild metric, with u' constant, being the future directed null cones, as required. In addition, by a suitable redefinition[16] of $t' \to t' + O(A)$, we can make the $w' \leftrightarrow u'$ correspondence exact, i.e. $w' = u'$.

The idea is now to use (18) to express x as a function of Θ and w', and insert this into (25) to obtain a relation between $\hat{u}$ and u' as a function of Θ only. The terms lower than $O(A)$ of this function will then be the needed supertranslation.

We start with the expansion of (18), using (A.2) and (C.3):

$$\begin{aligned} \kappa_{\mathrm{ext}} = 1: &\qquad S = -\operatorname{arctanh} x + \left(u' - 2m'\ln(1-x) - \frac{m'}{1-x^2}\right) A + O(A^2) \\ \kappa_{\mathrm{in}} = 1: &\qquad S = -\operatorname{arctanh} x + \left(u' - 2m'\ln(1+x) - \frac{m'}{1-x^2}\right) A + O(A^2). \end{aligned} \tag{50}$$

Inverting it, we obtain, with the help of (19),

$$\begin{aligned} \kappa_{\mathrm{ext}} = 1: &\qquad x(u', \Theta) = \cos\Theta + [\sin^2\Theta(u' - 2m'\ln(1-\cos\Theta)) - m']A + O(A^2) \\ \kappa_{\mathrm{in}} = 1: &\qquad x(u', \Theta) = \cos\Theta + [\sin^2\Theta(u' - 2m'\ln(1+\cos\Theta)) - m']A + O(A^2). \end{aligned} \tag{51}$$

[14] To be more precise, it is (for $K \neq 1$) the Schwarzschild metric with a conical singularity.
[15] This can also be verified by comparing the Schwarzschild metric in the (u, r, Θ, Φ) coordinates with (9), or by a direct application of (23).
[16] This, of course, does not spoil the limit (47).

Since we also have $x = x' = \cos\theta$, this expansion confirms the expected coincidence of Θ and θ in the limit $A \to 0$. Using (51), we can now express $\hat{u}$ (25) as

$$\begin{aligned} \kappa_{\rm ext} = 1: \qquad \hat{u} &= \frac{\cos\Theta}{A} + u' - 2m'\cos\Theta - 2m'\ln(1-\cos\Theta) - 3m' + O(A) \\ \kappa_{\rm in} = 1: \qquad \hat{u} &= \frac{\cos\Theta}{A} + u' + 2m'\cos\Theta - 2m'\ln(1+\cos\Theta) - 3m' + O(A). \end{aligned} \tag{52}$$

Apparently the limit of $\hat{u}$ for $A \to 0$ is diverging and is obviously not the Bondi time for the Schwarzschild metric. However using a specific supertranslation, it can be corrected:

$$\hat{u} \;\to\; \hat{u}' = \hat{u} + \alpha(\Theta),$$

$$\alpha(\Theta) = \begin{cases} \kappa_{\rm ext} = 1: -\dfrac{\cos\Theta}{A} + 2m'\cos\Theta + 2m'\ln(1-\cos\Theta) + 3m' \\ \kappa_{\rm in} = 1: -\dfrac{\cos\Theta}{A} - 2m'\cos\Theta + 2m'\ln(1+\cos\Theta) + 3m' \end{cases}, \tag{53}$$

which ensures that $\lim_{A\to 0}\hat{u} = u'$. The supertranslation[17] and the limit are qualitatively depicted in figures 6 and 7. The asymmetric character of the limit with respect to $\Theta = 0, \pi$ is necessary to arrive at a single black hole, and is a direct consequence of the $\cos\Theta/A$ term in (52). When the $A \to 0$ limit is achieved, the poles $x = x_{2,3}$ correspond to

$$A \to 0 \quad\Longrightarrow\quad \begin{cases} \theta = 0 \quad\leftrightarrow\quad \Theta = 0 \quad\leftrightarrow\quad x = \lim_{A\to 0} x_3 = 1 \\ \theta = \pi \quad\leftrightarrow\quad \Theta = \pi \quad\leftrightarrow\quad x = \lim_{A\to 0} x_2 = -1 \end{cases}. \tag{54}$$

This limit is consistent with (39), since, as can be seen in figure 7, the three (upper, lower and left) diamond-shaped blocks vanish, leaving only one block, with the left part being the axial slice $T2_3$, i.e. $x = x_3 = 1 \leftrightarrow \theta = \Theta = 0$, and the right part being $T2_2$, i.e. $x = x_2 = -1 \leftrightarrow \theta = \Theta = \pi$.

Now when the limiting process is properly set up, we continue to examine the behaviour of the news function and the mass aspect. In general, we use (45), (46), (50) and (53) to obtain the $A \to 0$ asymptotic expansion of those quantities.

The $c_{,u}$ expression (32) reduces to

$$\begin{aligned} &\kappa_{\rm ext} = 1: \\ &c_{,u} = -(\cos\theta - 2)\,\mathrm{cotan}^2\frac{\theta}{2}\, m'A + \mathrm{cotan}^2\frac{\theta}{2}\left(2\cos\theta(\cos\theta-2)\left(\frac{u'}{m'} - 2\ln(1-\cos\theta)\right)\right. \\ &\qquad\qquad \left. -\frac{3}{2}\cos^2\theta + \frac{7}{2} + 2\frac{2-\cos\theta}{\sin^2\theta}\right) m'^2A^2 + O(A^3) \\ &\kappa_{\rm in} = 1: \\ &c_{,u} = -(\cos\theta + 2)\tan^2\frac{\theta}{2}\, m'A + \tan^2\frac{\theta}{2}\left(2\cos\theta(\cos\theta+2)\left(\frac{u'}{m'} - 2\ln(1+\cos\theta)\right)\right. \\ &\qquad\qquad \left. -\frac{3}{2}\cos^2\theta + \frac{7}{2} + 2\frac{2+\cos\theta}{\sin^2\theta}\right) m'^2A^2 + O(A^3) \end{aligned} \tag{55}$$

where we have again expressed the result for the case of the conical singularity vanishing either between ($\kappa_{\rm in} = 1$) or outside ($\kappa_{\rm ext} = 1$) of the accelerated black holes, using (45). The

[17] The logarithmic term appears to be unavoidable and a divergence of this type is always present regardless of higher order differences of the limiting scheme. It seems to be a manifestation of the presence of the conical singularity on the respective axial segment; it always diverges on the pole(s) that correspond to the axial segment exhibiting a conical singularity (see (54) and figure 6).

behaviour near the poles is then

$$\kappa_{\rm ext}=1\colon\begin{cases} c_{,u}=(4m'\theta^{-2}+O(\theta^0))A+(8m'^2\theta^{-4}+O(\theta^{-2}))A^2+O(A^3)\\ c_{,u}=\left(\frac{3}{2}m'(\pi-\theta)^2+O((\pi-\theta)^4)\right)A-\left(\frac{3}{2}m'^2+O((\pi-\theta)^2)\right)A^2+O(A^3),\end{cases}$$

$$\kappa_{\rm in}=1\colon\begin{cases} c_{,u}=\left(-\frac{3}{2}m'\theta^2+O(\theta^4)\right)A-\left(\frac{3}{2}m'^2+O(\theta^2))\right)A^2+O(A^3)\\ c_{,u}=\left(-4m'(\pi-\theta)^{-2}+O((\pi-\theta)^0)\right)A\\ \qquad+\left(8m'^2(\pi-\theta)^{-4}+O((\pi-\theta)^{-2})\right)A^2+O(A^3)\end{cases}\tag{56}$$

which is an expected result, consistent with an intuitive approach encouraged by figure 6. In both cases $c_{,u}$ is always regular at one pole regular and diverging at the other, because the supertranslation cancels the symmetry of $\theta\leftrightarrow\pi-\theta$. (By regularity here we mean that the integral $\int_{I_\epsilon}c_{,u}^2\sin\theta\,\mathrm{d}\theta$ exists on some small neighbourhood I_ϵ of a given pole.) As we already know, $\theta=0\leftrightarrow x=1$ and $\theta=\pi\leftrightarrow x=-1$, and since in the C-metric we cannot eliminate the conical singularity on both axes (axial slices), at least one pole must be singular.

We may now continue with the computation of the Bondi mass aspect M. First we check the behaviour of the reduced mass aspect $\hat{M}$. Employing again expansion (50) and supertranslation (53) in equation (27), we find

$$\hat{M}=m'+O(A).\tag{57}$$

The limit of $\hat{M}$ is well behaved with the result being exactly what one would expect from limit (47) of the metric, compared to equation (7).

However, as a more detailed computation shows, the terms of $O(A)$ and higher diverge near both poles at least as $O(\theta^{-3})$ and $O((\pi-\theta)^{-3})$ respectively. Therefore the integral of $\hat{M}$ over the 2-sphere, i.e. the total mass of the system, is defined in the limit itself only, while for an arbitrarily small non-zero A, the integral does not exist. This is not so surprising; we had similar behaviour in the case of the news function $c_{,u}$.

Still, we might have expected that, at least near one pole, the situation could be made regular. This is not true for $\hat{M}$, but as we show below, it can[18] be done in the case of M.

Proceeding to find M, we find

$$\begin{aligned}\kappa_{\rm ext}=1\colon\quad & M=m'+\left(3\cos\theta\left(1+2\ln(1-\cos\theta)-\frac{u'}{m'}\right)+5\right)m'^2A+O(A^2)\\ \kappa_{\rm in}=1\colon\quad & M=m'+\left(3\cos\theta\left(1+2\ln(1+\cos\theta)-\frac{u'}{m'}\right)-5\right)m'^2A+O(A^2).\end{aligned}\tag{58}$$

This confirms that in the limit $A\to0$ the Bondi mass equals the Schwarzschild mass. Series expansion near the poles $\theta=0,\pi$ then reveals

$$\kappa_{\rm ext}=1\colon\begin{cases} M=m'+\left(\left(3+6\ln\frac{\theta^2}{2}-3\frac{u'}{m'}+5\right)m'^2+O(\theta^2)\right)A+O(A^2)\\ M=m'+\left(\left(-3-6\ln2+3\frac{u'}{m'}+5\right)m'^2+O((\pi-\theta)^2)\right)A+O(A^2),\end{cases}$$

[18] Of course, the integral of M still does not, and cannot, exist. If it existed, it would, according to equation (37) in [2] and the remark below it, have to be equal to the integral of $\hat{M}$, which is diverging.

$$\kappa_{\rm in} = 1: \begin{cases} M = m' + \left(\left(3 + 6\ln 2 - 3\dfrac{u'}{m'} - 5\right)m'^2 + O(\theta^2)\right)A + O(A^2) \\ M = m' + \left(\left(-3 - 6\ln\dfrac{(\pi-\theta)^2}{2} + 3\dfrac{u'}{m'} - 5\right)m'^2 + O((\pi-\theta)^2)\right)A + O(A^2). \end{cases} \tag{59}$$

This seems rather surprising, since it appears as if the mass aspect M was integrable over the whole 2-sphere. But again, a more detailed analysis reveals that the terms $O(A)^2$ and higher all diverge at the poles where $\kappa \neq 1$, leading to the same qualitative behaviour as in the case of $c_{,u}$. The integral of M over the entire 2-sphere therefore does not exist, as we have expected, unless we restrict to the terms of $O(A)$ or to the limit $A = 0$ itself.

An interesting question arises concerning the relation of the Schwarzschild limit $A \to 0$ and the Minkowski limit $m \to 0$. We will shortly address this issue by investigating the small mass limit of the coordinates used in the Schwarzschild limit. First, we express the function x as $x(w, \Theta)$, using the inverse of the expansion of $\mathrm{Gi}(x)$ (see (A.2)). Assuming axial regularity, i.e. $\kappa = 1 + O(mA)$, this leads to

$$x(w, \Theta) = \mathrm{th}(w - \kappa S) = \frac{\mathrm{ch}\, S\, \mathrm{sh}\, w - \mathrm{sh}\, S\, \mathrm{ch}\, w}{\mathrm{ch}\, S\, \mathrm{ch}\, w - \mathrm{sh}\, S\, \mathrm{sh}\, w} + O(mA) = \frac{\cos\Theta\, \mathrm{ch}\, w + \mathrm{sh}\, w}{\mathrm{ch}\, w + \cos\Theta\, \mathrm{sh}\, w} + O(mA). \tag{60}$$

Comparing this to the Schwarzschild limit formulae for the θ and u' coordinates (see (48) and (49)) we obtain the desired relation:

$$\cos\theta = \frac{\cos\Theta\, \mathrm{ch}\, u'A + \mathrm{sh}\, u'A}{\mathrm{ch}\, u'A + \cos\Theta\, \mathrm{sh}\, u'A} + O(mA). \tag{61}$$

In terms of the complex stereographic coordinate ξ, related to the angle Θ via (17), and the coordinate ζ related in the same way to θ, this translates into a simpler formula:

$$\zeta = \xi \mathrm{e}^{-u'A}, \tag{62}$$

in which we recognize the Lorentz boost along the z axis, with $\beta = -\tanh u'A$ being the velocity. This confirms the intuitive idea that the small mass limit of the Schwarzschild limit $A \to 0$ and the Minkowski limit $m \to 0$ are related by a Lorentz boost along the symmetry axis, with the velocity increasing as u' increases; the Schwarzschild limit corresponds to the observer at rest with respect to the black-hole– particle, which, on the other hand is accelerating along the axis in the Minkowski observer's coordinate frame. They should therefore be related by a boost, with the velocity as a function of the acceleration and time.

Another way to understand this is to realize that different Bondi coordinates must be related by a transformation belonging to the BMS group[19], which, roughly speaking, consists of boosts, rotations and supertranslations. Since in both limits the angular coordinates were adapted to the symmetry axis, they must be related only by a pure boost, or, in general, modulo some additional supertranslation.

6. Bondi time limit u → 0 and u → ±∞

The explicit formulae for the functions $\mathrm{Gj}(x)$ are given in the appendix, equations (D.1) and (D.2). They allow us not only to numerically compute the mass aspect and the total mass as their integral over the entire $\mathcal{J}^+$, but also to obtain some analytic results.

Although the analytical computation of the total mass does not seem feasible in the general case, there exists a well-defined limiting behaviour for the Bondi time $u \to 0^\pm$ and $u \to \pm\infty$.

[19] See, for example [2], section 3.

These limits correspond to the observer located on $\mathcal{J}^+$ approaching the event of the black hole hitting $\mathcal{J}$, or respectively, him moving to the time future or spatial infinity (see figure 4). The total mass can be expressed using (6), which in our case leads to

$$m(u) = \frac{1}{2}\int_0^\pi \hat{M}(u,\Theta)\sin\Theta\,\mathrm{d}\Theta = \int_I J(u,x)\hat{M}(u,x)\,\mathrm{d}x, \qquad I = \begin{cases} u>0 : (x_u, x_3) \\ u<0 : (x_2, x_u) \end{cases} \quad \mathrm{Gj}(x_u) = uAK$$

$$J = \sin\Theta\det\frac{\partial(u,\Theta)}{\partial(u,x)} = -\frac{\sin^2\Theta}{G^{\frac{3}{2}}\mathrm{Gj}\cos\Theta} \underset{(\alpha=0)}{=} -\frac{u^2A^2\kappa^2}{\mathrm{Gj}^2G^{\frac{3}{2}}}\frac{\mathrm{sign}(\mathrm{Gj}\cos\Theta)}{\sqrt{\mathrm{Gj}^2-u^2A^2\kappa^2}} \tag{63}$$

where we have used $\hat{M}$ instead of M to simplify the following calculations. These expressions allow us to compute the various limits with respect to u that we want. We will have to use the first one for $u\to\pm\infty$ and the latter for the $u\to 0$ limit, in order to be able to swap $\int_I$ and $\lim\limits_{u\to\pm\infty,0}$. Also the mass will be finite in the first place in the $u>0$ or $u<0$ case only when we set the conicity parameter κ to have $\kappa_{\mathrm{in}}=1$ or $\kappa_{\mathrm{ext}}=1$ respectively. Then, we obtain the following limiting behaviour for the mass $m(u)$:

$$\begin{aligned} \kappa_{\mathrm{in}}=1: \quad m(u) &= \frac{24x_1^4x_2^4x_3^4}{35A^8(x_3-x_1)^6(x_3-x_2)^6}\,\frac{1}{u^7} + O(u^{-8})\,, \quad \text{as } u\to+\infty \\ \kappa_{\mathrm{ext}}=1: \quad m(u) &= \frac{24x_1^4x_2^4x_3^4}{35A^8(x_2-x_1)^6(x_3-x_2)^6}\,\frac{1}{u^7} + O(u^{-8})\,, \quad \text{as } u\to-\infty. \end{aligned} \tag{64}$$

For the limit of $u\to 0^\pm$ the evaluation of the integral is more complicated. In order to obtain an explicit result, we restrict the supertranslation freedom to the class of $\alpha = \frac{C\sin\Theta}{AK}$. Since this is equivalent to $\mathrm{Gj}\to\mathrm{Gj}' = \mathrm{Gj}+C$, it allows us to use the simplified formulae of $\alpha=0$, which can be explicitly integrated, and then substitute $\mathrm{Gj}\to\mathrm{Gj}+C$. Using this in the series expansion of $J\hat{M}$ in u we obtain the following:

$$\begin{aligned} m(u) &= \int_I J(u,x)M(u,x)\mathrm{d}x = \int_I J(u,x)\hat{M}(u,x)\mathrm{d}x \\ &= \frac{1}{u}\int_{I_{|x_u=x_0}}\left(\frac{3\kappa^2G'^3-24G'-6\kappa^2GG'G''+4\kappa^2G^2G'''}{96A^2G^2} - \frac{\mathrm{Gj}}{4A^2\kappa^2G^{\frac{3}{2}}}\right)\mathrm{d}x + O(1) \\ &= \frac{1}{u}\cdot\left\{\begin{array}{l} \text{for } I=(x_2,x_u) \;\Leftrightarrow\; u<0: \\ -\dfrac{(x_1+x_3-x_2)K_2}{2A^2(x_3-x_2)(x_2-x_1)} - \left[\dfrac{24-3K_2^2G'^2}{96A^2G} + \dfrac{G'''xK_2^2}{24A^2} - \dfrac{C_2^2}{8A^2K_2^2}\right]_{|x=x_0} \\ \text{for } I=(x_u,x_3) \;\Leftrightarrow\; u>0: \\ -\dfrac{(x_1+x_2-x_3)K_3}{2A^2(x_3-x_2)(x_3-x_1)} - \left[\dfrac{24-3K_3^2G'^2}{96A^2G} + \dfrac{G'''xK_3^2}{24A^2} - \dfrac{C_3^2}{8A^2K_3^2}\right]_{|x=x_0} \end{array}\right\} \\ &\quad + O(1), \end{aligned} \tag{65}$$

where x_0 is chosen so that $\mathrm{Gj}(x_0)=0$, and the quantities $C_i=0$ are the constant terms in the expansion of Gj, see (D.1), (D.2), (D.4), while K_i is a conicity parameter corresponding to the physical conicity $\kappa_i=1$ (see (B.2)). Again, the conicity has been set for the conical singularity to disappear on the respective axis segments (either $x=x_2$ or $x=x_3$) in order for the result to be finite in the first place.

To summarize this, we have shown that in the absence of a conical singularity, the Bondi mass behaves as $1/u^7$ for large null time, $u\to\pm\infty$, and as[20] $1/u$ for time close to the event of the black hole reaching $\mathcal{J}^+$ when $u\to\pm 0$.

[20] At least for the $\alpha = \frac{C\sin\Theta}{AK}$ class of supertranslations.

The leading coefficients in the above asymptotic expansions are strictly positive, causing the Bondi mass to be strictly non-increasing, as required by general theorems, demonstrated by, e.g., (6). In the $u \to \pm\infty$ case this is obvious; for $u \to 0^{\pm}$ see the proof in appendix E.

7. Regularity near the poles in the general case

In sections 4 and 5, concerning the small mass and small acceleration limit, we have encountered a rather similar behaviour of the Bondi mass and the news function with respect to the regularity of the corresponding segment of the symmetry axis; the above-mentioned asymptotic quantities were only diverging at the pole if and only if the respective axial segment contained a conical singularity. This suggests that such behaviour may be preserved even in the general case, and we investigate it in this section.

To analyse the Bondi mass (36) and the news function (32) directly is rather involved, in the sense that an analytical expression of M and $c_{,u}$ in terms of the Bondi coordinates $\hat{u}, \Theta$ is extremely complicated[21]. However, as we show below, the situation near the poles $x_{2,3}$ can be investigated analytically, even in the general case.

To obtain the behaviour of the news function $c_{,u}$ and of the Bondi mass M near the poles[22] at $\Theta = 0$ and π, in the full relativistic case, we substitute the expansions for $G(x)$, considering the case that $G(x)$ is a third-order polynomial:

$$\begin{aligned} G(x) = G(x_2 + \xi_2) \quad &\to \quad G(x) = G'(x_2)\xi_2 + G''(x_2)\frac{\xi_2^2}{2} + G'''(x_2)\frac{\xi_2^3}{6} \\ G(x) = G(x_3 - \xi_3) \quad &\to \quad G(x) = -G'(x_3)\xi_3 + G''(x_3)\frac{\xi_3^2}{2} - G'''(x_3)\frac{\xi_3^3}{6} \end{aligned} \tag{66}$$

and analogically for its derivatives, into (32) and (36) respectively. The conicity parameter is expressed in terms of the physical conicity via (B.2). For $c_{,u}$, this leads to

$$c_{,u} = \frac{1}{2\sin^2\Theta}\left(1 - \kappa_i^2\right) + \frac{G_i'''\kappa_i^2}{G_i'\sin^2\Theta}\xi_i^2 + \frac{1}{\sin^2\Theta}\,O(\xi_i^3)\,, \quad i = 2, 3, \quad G_i^{(j)} \equiv G^{(j)}(x_i). \tag{67}$$

Noting that

$$\frac{1}{\sin\Theta} = \frac{1}{AK(\hat{u} - \alpha)}\left(\frac{1}{\sqrt{\xi_{2,3}}} + O(\sqrt{\xi_{2,3}})\right), \tag{68}$$

we see that only[23] the first term in (67) can diverge for $\kappa_{2,3} \neq 1$. In other words this means that on the axial slices $x = x_{2,3}$ we indeed can again, by choosing the axis to be regular either at $x = x_2$ or at $x = x_3$, eliminate the singular behaviour of $c_{,u}$ on $\mathcal{J}^+$, either for $\hat{u} - \alpha > 0$, or $\hat{u} - \alpha < 0$ respectively[24].

Now for the Bondi mass aspect M, we again use the above-mentioned approach and arrive at

$$\begin{aligned} M = {} & \frac{1}{4}\frac{\epsilon_i\kappa_i\left(\kappa_i^2 - 1\right)}{A\sqrt{|G_i'|}\,\sin^3\Theta}\frac{1}{\sqrt{\xi_i}} + \frac{1}{4}\frac{\kappa_i^2 - 1}{\sin^4\Theta}(\alpha'\sin\Theta\cos\Theta - \alpha''\sin^2\Theta + u - \alpha) \\ & + \frac{3}{16}\frac{\kappa_i\left(\kappa_i^2 - 1\right)}{A\sin^3\Theta}\frac{G_i''}{|G_i'|^{\frac{3}{2}}}\sqrt{\xi_i} + \frac{\epsilon_i\kappa_i}{384A}\frac{G_i'G_i'''\left(72\kappa_i^2 - 40\right) - 15G_i''^2\left(\kappa_i^2 - 1\right)}{|G_i'|^{\frac{5}{2}}\sin^3\Theta}\xi_i^{\frac{3}{2}} \end{aligned}$$

[21] This is so because we would have to express x as a function of Θ and $\hat{u}$, using the inverse of Gj(x), and then substitute it into G and its derivatives.
[22] See (39), (54) for x_i correspondence.
[23] Assuming that $\lim_{\Theta\to(0,\pi)}\alpha \neq u$, which is where the black holes approach $\mathcal{J}^+$.
[24] See also the conformal diagrams in figures 2–4.

$$
\begin{aligned}
&- \frac{1}{4}\frac{\kappa_i^2 G_i'''}{G_i' \sin^4\Theta}(\alpha' \sin\Theta\cos\Theta - \alpha'' \sin^2\Theta + u - \alpha)\, \xi_i^2 \\
&- \left[\frac{A G_i'''}{\sqrt{|G_i'|}\sin^5\Theta}(\alpha' \sin\Theta + \cos\Theta(u-\alpha))^2 + \frac{1}{48A}\frac{G_i''' G_2''}{|G_i'|^{\frac{5}{2}}\sin^3\Theta}\right]\xi_i^{\frac{5}{2}} \;+\; O\big(\xi_i^3\big),
\end{aligned}
\tag{69}
$$

where $\epsilon_i = +1, -1$ for $i = 2, 3$. Noting (68) we see that also in this case, for $\kappa_{2,3} = 1$, the mass aspect M does not diverge near the poles.

This in fact even holds for any $\alpha(\Theta)$, which is at least[25] $\mathcal{C}^2$ on a neighbourhood of $\Theta = 0$ and $\Theta = \pi$, as can be seen by analysing the Θ-dependent terms in the numerator:

$$
A(\Theta) \equiv \alpha' \sin\Theta\cos\Theta - \alpha'' \sin^2\Theta + u - \alpha\ , \qquad B(\Theta) \equiv \alpha' \sin\Theta + \cos\Theta(u-\alpha).
$$

For the situation near $\Theta = 0$,

1. we first assume that α is bounded at $\Theta = 0$, and rewrite A and B as

$$
\begin{aligned}
A(\Theta) &= 5(\alpha\sin\Theta)'\cos\Theta + \alpha\sin^2\Theta - (\alpha\sin^2\Theta)'' + u - 4\alpha\ , \\
B(\Theta) &= (\alpha\sin\Theta)' + (u - 2\alpha)\cos\Theta.
\end{aligned}
\tag{70}
$$

   Now, there is a particulary nice limiting property of the expressions $(\alpha\sin\Theta)'$ and $(\alpha\sin^2\Theta)''$ at $\Theta_0 = 0$:

$$
\begin{aligned}
(\alpha\sin\Theta(\Theta))'_{|\Theta=\Theta_0} &= \lim_{\epsilon\to 0}\frac{\sin(\Theta_0+\epsilon)\,\alpha(\Theta_0+\epsilon) - \sin(\Theta_0)\,\alpha(\Theta_0)}{\epsilon} \\
&\underset{\Theta_0=0}{=} \lim_{\epsilon\to 0}\frac{\sin(\epsilon)\,\alpha(\epsilon)}{\epsilon} = \alpha(0) \\
(\alpha\sin^2\Theta(\Theta))''_{|\Theta=\Theta_0=0} &= \lim_{\epsilon\to 0}\frac{\sin^2 2\epsilon\ \alpha(2\epsilon) - 2\sin^2\epsilon\ \alpha(\epsilon) - \sin^2 0\ \alpha(0)}{\epsilon^2} = 2\alpha(0)\ ,
\end{aligned}
\tag{71}
$$

   which, together with (70), ensures the finiteness of A and B at $\Theta = 0$.

2. In the alternative case, of diverging α, $\lim_{\Theta\to 0}\alpha(\Theta) = \pm\infty$, and we can rewrite A and B in terms of $\beta = 1/\alpha$:

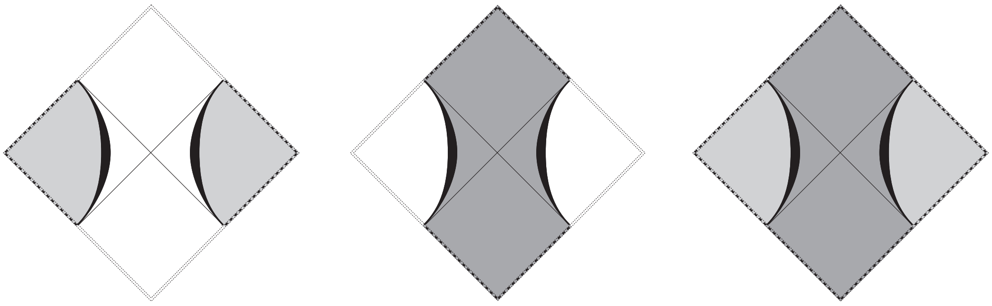
$$
\begin{aligned}
A(\Theta) &= \frac{(\beta\sin^2\Theta)''}{\beta^2} - 2\frac{(\beta\sin\Theta)'^2}{\beta^3} - \frac{(\beta\sin\Theta)'}{\beta^2}\cos\Theta + \frac{\sin^2\Theta}{\beta} + u, \\
B(\Theta) &= -\frac{(\beta\sin\Theta)'}{\beta^2} + u\cos\Theta.
\end{aligned}
\tag{72}
$$

   Using the same idea as in the previous case, we see from (71) that both A and B diverge at worst as $1/\beta$. But this does not spoil the finiteness of (69), since A and B only occur as $\frac{A(\Theta)}{\sin^4\Theta}$ and $\frac{B^2(\Theta)}{\sin^5\Theta}$ (see (36)), and according to (68), these $1/\sin^n(\Theta)$ factors are more than enough to compensate for the divergence of the excessive α in the numerator.

The case of $\Theta \to \pi$ is completely analogous and leads to the same conclusion.

Therefore, as illustrated in figure 8, even in the general C-metric case, the conclusions of sections 4 and 5 hold; the Bondi mass and the news function cannot be made regular if the corresponding axial segment contains a conical singularity, and conversely, if the axial segment is regular, those quantities are regular and integrable at the corresponding pole.

[25] So its second derivative in (69) is defined.



(a) κ_{in}=1 (b) κ_{ext}=1 (c) $\kappa_{ext}\neq 1$, $\kappa_{in}\neq 1$, note that $\kappa_{in}>\kappa_{ext}$

Figure 8. Schematic conformal diagrams for the three possible cases of the conical singularity location and its influence on the $\mathcal{J}^+$ regularity. See also conformal diagrams of four prototypes of a general boost-rotation symmetric spacetime given in figures 3–6 in [8].

8. Conclusions

We have successfully applied the method of Tafel and co-workers to obtain the news function and the Bondi mass aspect of the C-metric. These quantities have been obtained without any approximation, for the general allowed values of all C-metric parameters including the axial conicity. Our news function (32), for the special case of the conicity parameter $\kappa = 1$, is the same as the one presented in [17] for the special case without rotation, electric and magnetic charge and without the NUT parameter. Also, our $m \to 0$ limit (38) of the news function, for the case $\kappa_{\text{ext}} = 1$, is equivalent to the formula presented in [12], equations (144) and (145) (the general κ case is not discussed there). The overall form of the news function and of the mass factor is compatible with the general expressions for a boost-rotation symmetric spacetime presented in [10–12, 18, 20]. More specifically, if the news function (32) is used to obtain the function $\mathcal{K}(w)$ of [12], equation (98), from which the mass aspect change $M_{,u}$ can be computed via [12] (118–120), the result is the same as presented here in (35). Apart from those general formulae, however, there has, to the authors' knowledge, not been any published explicit result for the Bondi mass aspect $M(u, \Theta)$ or $M_{,u}(u, \Theta)$ of the C-metric.

In sections 4 and 5 the behaviour of the news function and of the mass aspect has been investigated under the small mass limits $m \to 0$ and the small acceleration limit $A \to 0$. The regularity of M and c near poles $\Theta = 0, \pi$ has been found to reflect the presence of the conical singularity on the associated axial segment; this has been confirmed in section 7 for a general C-metric.

In section 6 the total Bondi mass of the C-metric has been explicitly integrated for special limiting values of the Bondi time and found to be, in the regular cases when the integral is defined, proportional to $1/u^7$ for large null time, $u \to \pm\infty$ (63) and behaving as $1/u$ for time close to the event of the black hole reaching $\mathcal{J}^+$ when $u \to \pm 0$ (65). The leading coefficients in these asymptotic expansions in u have been explicitly expressed, and shown to be strictly positive, causing the Bondi mass to be strictly non-increasing, as required by general theorems.

Section 7, as noted above, then contains the analysis of the regularity of the news function $c_{,u}$ and of the mass aspect M in the general case. This means that no approximation or limit of the original C-metric is assumed, and also the Bondi time $\hat{u}$ includes freedom given by a general supertranslation $\alpha(\Theta)$. The results are completely analogous to the limiting cases of sections 4 and 5.

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Appendix A. Some details of the computations

To obtain the Bondi mass aspect, the news function and to perform the asymptotic calculations on $\mathcal{J}^+$ in general, the partial derivatives corresponding to the coordinate transformation $(\hat{u}, \Theta) \leftrightarrow (w, x)$ are often needed. Here, according to equations (25) and (18), the complete transformation Jacobian for the general case $\alpha \equiv \alpha(\Theta, \Phi) \neq 0$ is given:

$$J = \begin{pmatrix} \dfrac{\partial \hat{u}}{\partial w} & \dfrac{\partial \hat{u}}{\partial x} \\ \dfrac{\partial \Theta}{\partial w} & \dfrac{\partial \Theta}{\partial x} \end{pmatrix} = \frac{\sin\Theta}{KG} \begin{pmatrix} \dfrac{\cos\Theta}{AK} G\mathrm{Gj} + G\dfrac{\partial\alpha}{\partial\Theta} & , & \dfrac{1}{A\sqrt{G}} - \dfrac{\cos\Theta}{AK}\mathrm{Gj} - \dfrac{\partial\alpha}{\partial\Theta} \\ G & & -1 \end{pmatrix}$$

$$J^{-1} = \begin{pmatrix} \dfrac{\partial w}{\partial \hat{u}} & \dfrac{\partial w}{\partial \Theta} \\ \dfrac{\partial x}{\partial \hat{u}} & \dfrac{\partial x}{\partial \Theta} \end{pmatrix} = \frac{K}{\sin\Theta} \begin{pmatrix} A\sqrt{G} & , & 1 - \dfrac{\cos\Theta}{K}\sqrt{G}\mathrm{Gj} - A\sqrt{G}\dfrac{\partial\alpha}{\partial\Theta} \\ AG^{\frac{3}{2}} & , & -\dfrac{\cos\Theta}{K}G^{\frac{3}{2}}\mathrm{Gj} - AG^{\frac{3}{2}}\dfrac{\partial\alpha}{\partial\Theta} \end{pmatrix} \qquad \text{(A.1)}$$

$$\det J = -\frac{\sin^2\Theta}{AK^2G^{\frac{3}{2}}}.$$

In sections 4, 5 on limits, we used the mA series of various expressions used in the general calculations. For reference, the most important formulae are included here:

$$\mathrm{Gi} = \operatorname{arctanh} x + \left(\frac{1}{1-x^2} + \ln(1-x^2)\right) mA + \left(-\frac{x(8x^4 - 25x^2 + 15)}{2(1-x^2)^2} + \frac{15}{4}\ln\frac{1-x}{1+x}\right) m^2A^2 + O(m^3A^3) \qquad \text{(A.2)}$$

$$\mathrm{Gj} = \frac{x}{\sqrt{1-x^2}} + \frac{3x^2-2}{(1-x^2)^{\frac{3}{2}}}mA + \left(\frac{x(23x^4 - 35x^2 + 15)}{2(1-x^2)^{\frac{5}{2}}} - \frac{15}{2}\arcsin x\right) m^2A^2 - \frac{35x^8 - 280x^6 + 560x^4 - 448x^2 + 128}{2(1-x^2)^{\frac{7}{2}}} m^3A^3 + O(m^4A^4) \qquad \text{(A.3)}$$

$$x = \frac{\mathrm{Gj}}{\sqrt{1+\mathrm{Gj}^2}} + \frac{2-\mathrm{Gj}^2}{1+\mathrm{Gj}^2}mA + \left(15\arcsin\frac{\mathrm{Gj}}{\sqrt{1+\mathrm{Gj}^2}} + 5\mathrm{Gj}^3 - 27\mathrm{Gj}\right)\frac{m^2A^2}{2(1+\mathrm{Gj}^2)^{\frac{3}{2}}} + O(m^2A^2) \qquad \text{(A.4)}$$

$$\kappa_{\text{ext}} = K(1 - 2mA) + O(m^2A^2), \qquad \kappa_{\text{in}} = K(1 + 2mA) + O(m^2A^2)$$
$$\longleftrightarrow \quad K = \kappa_{\text{in}}\left(1 - 2mA + \frac{15}{2}m^2A^2 - 32m^3A^3\right) + O(m^4A^4) \tag{A.5}$$
$$= \kappa_{\text{ext}}\left(1 + 2mA + \frac{15}{2}m^2A^2 + 32m^3A^3\right) + O(m^4A^4),$$

where x_i are the roots of the $G(x)$, K is the conicity parameter and κ_{in}, κ_{ext} are the conicities of the corresponding axial segments, see also appendix B. Interestingly, the conicity parameter K can, in the light of (A.5), be interpreted as an average of the external and internal physical conicity, up to the second order of A.

Appendix B. Conicity of the *C*-metric

For a general axially symmetric 2-space, $g_2 = g_{RR}\,\mathrm{d}R^2 + g_{\phi\phi}\,\mathrm{d}\phi^2$, with the coordinate R such that the axis is located at $R = 0$, the definition of the conicity leads to the formula

$$\kappa = \lim_{\text{'distance to axis'}\to 0} \frac{\text{'circumference'}}{2\pi \times \text{'distance to axis'}} = \lim_{R\to 0} \left.\frac{\frac{\partial}{\partial R}\sqrt{g_{\phi\phi}}}{\sqrt{g_{RR}}}\right|_{R=0}. \tag{B.1}$$

In the case of the *C*-metric (10), we find that

$$\kappa_{1,2,3} = \frac{K}{2}|G'|_{|x=x_{1,2,3}}, \tag{B.2}$$

so the K parameter can really adjust κ on a specific segment of the symmetry axis. For the domain discussed here[26], the axis segment between the particles lies at $x = x_3$ and the segment outside lies at $x = x_2$. We will therefore use $\kappa_{\text{ext}} \equiv \kappa_2$ and $\kappa_{\text{in}} \equiv \kappa_3$ as synonyms in the text.

Appendix C. Horizon area

The Schwarzschild limit $A \to 0$ was done holding the horizon area constant. Using a straightforward integration, we find

$$\mathcal{A} = \int_{\partial S_2} \sqrt{g}\,\mathrm{d}S_2 = \int_0^{2\pi}\int_{x_2}^{x_3} \sqrt{g_{xx}g_{\varphi\varphi}}\,\mathrm{d}x\,\mathrm{d}\varphi = \frac{2\pi K}{A^2}\frac{x_3 - x_2}{(x_2 - x_1)(x_3 - x_1)}. \tag{C.1}$$

Performing an expansion in A leads to

$$\begin{aligned} \kappa_{\text{ext}} = 1: \quad & M_0^2 \equiv \frac{\mathcal{A}}{16\pi} = m^2 + 2m^3A + 26m^4A^2 + 69m^5A^3 + O(A^4) \\ \kappa_{\text{in}} = 1: \quad & M_0^2 \equiv \frac{\mathcal{A}}{16\pi} = m^2 - 2m^3A + 26m^4A^2 - 69m^5A^3 + O(A^4) \end{aligned} \tag{C.2}$$

where M_0 is the mass of a Schwarzschild black hole with the same horizon area $\mathcal{A}$ as the accelerated black hole of the *C*-metric. Inverting this series, we finally obtain (45)

$$\begin{aligned} \kappa_{\text{ext}} = 1: \quad & m = m' - m'^2A - \frac{21}{2}m'^3A^2 + \frac{71}{2}m'^4A^3 + O(A^4) \\ \kappa_{\text{in}} = 1: \quad & m = m' + m'^2A - \frac{21}{2}m'^3A^2 - \frac{71}{2}m'^4A^3 + O(A^4). \end{aligned} \tag{C.3}$$

[26] Segment $x = x_1$ does not lie in our spacetime; see the beginning of section 2 and also figure 1.

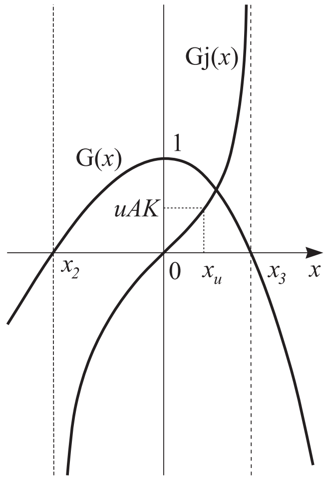

Figure D1. Plot of the $G(x)$ and $\mathrm{Gj}(x)$ functions.

Appendix D. Explicit formula for the Gj function

The function $\mathrm{Gj}(x)$ is a crucial part of the formula for the Bondi time $\hat{u}$ and propagates into other results as well. While there is no problem with its qualitative description (see figure D1) or numerical computation, an analytical expression would certainly be helpful as well. Fortunately, it appears to be possible to express the defining integral

$$\mathrm{Gj}(x) = \int \frac{\mathrm{d}x}{G(x)^{\frac{3}{2}}} = \frac{1}{a^{\frac{3}{2}}} \int \frac{\mathrm{d}x}{\big((x-x_1)(x-x_2)(x_3-x)\big)^{\frac{3}{2}}} \equiv \frac{1}{a^{\frac{3}{2}}} \int \frac{\mathrm{d}x}{(\xi_1\xi_2\zeta_3)^{\frac{3}{2}}}, \qquad \begin{array}{l} \xi_i = x - x_i \\ \zeta_i = -\xi_i, \end{array}$$

in an explicit form using elliptic functions. Here, we present two equivalent forms[27], differing only by an integration constant:

$$\begin{aligned} \mathrm{Gj}_{(2)}(x) = {} & -b d_{21}^{\frac{1}{2}}\left(d_{21}^2 + d_{32}^2 + d_{31}^2\right) \mathrm{E}(z_2, k_2) \;+\; b d_{21}^{\frac{1}{2}}(d_{31} + d_{21})\, \mathrm{F}(z_2, k_2) \\ & +\; b\frac{\xi_1\xi_2 d_{21}^2 + \xi_2\xi_3 d_{32}^2 + \xi_3\xi_1 d_{31}^2}{\sqrt{\xi_1\xi_2\zeta_3}} \;+\; C_2 \\ \text{with } z_2 = {} & \sqrt{\frac{\xi_2}{d_{32}}} = \sqrt{\frac{x - x_2}{d_{32}}},\;\; k_2 = \mathrm{i}\sqrt{\frac{d_{32}}{d_{21}}}\,, \end{aligned} \tag{D.1}$$

$$\begin{aligned} \mathrm{Gj}_{(3)}(x) = {} & \;\; b d_{32}^{\frac{1}{2}}\left(d_{32}^2 + d_{31}^2 + d_{21}^2\right) \mathrm{E}(z_3, k_3) \;+\; b d_{32}^{\frac{1}{2}}(d_{32} - d_{21})\, \mathrm{F}(z_3, k_3) \\ & +\; b\frac{\xi_1\xi_2 d_{21}^2 + \xi_2\xi_3 d_{32}^2 + \xi_3\xi_1 d_{31}^2}{\sqrt{\xi_1\xi_2\zeta_3}} \;+\; C_3 \\ \text{with } z_3 = {} & \sqrt{\frac{\zeta_3}{d_{31}}} = \sqrt{\frac{x_3 - x}{d_{32}}},\;\; k_3 = \sqrt{\frac{d_{31}}{d_{32}}}\,, \end{aligned} \tag{D.2}$$

and we have used the abbreviations $\dfrac{2}{b} = a^{\frac{3}{2}} d_{21}^2 d_{31}^2 d_{32}^2, \quad d_{ij} = x_i - x_j,$

[27] The third obvious one related to x_1 could be obtained as the last cyclic permutation of the indices of x_i.

also with $\mathrm{F}(z,k)$ and $\mathrm{E}(z,k)$ being the incomplete elliptic integrals of the first and second kind respectively[28]. The two different forms are useful for series expansion as $x\to x_2$ from the right and $x\to x_3$ from the left, with the advantage of ξ_1, ξ_2, ζ_3 being always positive. In the case of the gauge (11*b*) we also have additional relations:

$$2mA = a, \quad x_1 + x_2 + x_3 = -\frac{1}{a}, \quad x_1x_2 + x_1x_3 + x_2x_3 = 0, \quad x_1x_2x_3 = \frac{1}{a}. \tag{D.3}$$

Expressions (D.1), (D.2) have a notable property of having no additional constant term (besides $C_{2,3}$) in the expansion series near x_3 and x_2 respectively, i.e. we have

$$\begin{aligned} \mathrm{Gj}_{(2)}(x) &= -\frac{2}{a^{\frac{3}{2}} d_{21}^{\frac{3}{2}} d_{32}^{\frac{3}{2}} \sqrt{\xi_2}} + C_2 + \frac{3(d_{21} - d_{32})}{a^{\frac{3}{2}} d_{21}^{\frac{5}{2}} d_{32}^{\frac{5}{2}}} \sqrt{\xi_2} + O\big(\xi_2^{\frac{3}{2}}\big), \\ \mathrm{Gj}_{(3)}(x) &= \frac{2}{a^{\frac{3}{2}} d_{32}^{\frac{3}{2}} d_{31}^{\frac{3}{2}} \sqrt{\zeta_3}} + C_3 + \frac{3(d_{32} + d_{31})}{a^{\frac{3}{2}} d_{32}^{\frac{5}{2}} d_{31}^{\frac{5}{2}}} \sqrt{\zeta_3} + O\big(\zeta_3^{\frac{3}{2}}\big), \end{aligned} \tag{D.4}$$

where C_2 and C_3 are exactly the same as in (D.1), (D.2).

Sometimes it is also convenient to choose the integration constant so we have $\mathrm{Gj}(0) = 0$:

$$\tilde{\mathrm{Gj}}(x) = \mathrm{Gj}(x) - \mathrm{Gj}(0). \tag{D.5}$$

In general, the choice of this constant is tantamount to the supertranslation by $\alpha = \frac{C \sin\Theta}{AK}$ (see (25)), i.e. to the corresponding redefinition of $\hat{u}$. Note that this is still compatible with our choice of $\hat{u}$, namely preserving $\hat{u} = 0$ on $\mathcal{J}^+$ where the black hole reaches it (see figure 4).

Appendix E. Bondi mass change positivity in $u \to \pm 0$ limit

In order to prove positivity of the first term in the series in u, around $u = 0$, of the Bondi mass $m(u)$ for all $\alpha \sim \sin\Theta$ supertranslations, we have to prove that the following quantity,

$$m_2 \equiv -\frac{(x_1 + x_3 - x_2)K_2}{2A^2(x_3 - x_2)(x_2 - x_1)} - \left[\frac{24 - 3K_2^2 G'^2}{96A^2 G} + \frac{G''' x K_2^2}{24A^2} - \frac{C_2^2}{8A^2K_2^2}\right]_{|x=x_0}, \tag{E.1}$$

is positive[29] for all x_0 in the open interval $I \equiv (x_2, x_3)$. First, we prove that the function

$$h = \frac{1}{4A^2\sqrt{G}}\left[\frac{1}{4}K_2^2 G G' G'' - \frac{1}{6}K_2^2 G^2 G''' - \frac{1}{8}K_2^2 G'^3 + G'\right] - \frac{C_2}{4A^2K_2^2}$$

is positive for all $x_0 \in I$. Realizing that $C_2 = -\tilde{\mathrm{Gj}}_{(2)}(x_0)$ where $\tilde{\mathrm{Gj}}_{(2)}$ is $\mathrm{Gj}_{(2)}$ with $C_2 = 0$ (D.1), we see that together with (B.2) and (D.4) this gives us the limiting value $h_{|x=x_0=x_2} \equiv h(x_2) = 0$. Now, to prove that $h > 0$ it is sufficient[30] to show that $\frac{\mathrm{d}h}{\mathrm{d}x}_{|x=x_0} > 0$. But this is obvious, since

$$\frac{\mathrm{d}h}{\mathrm{d}x}_{|x=x_0} = \frac{1}{A^2K_2^2G^{\frac{3}{2}}}\left(\frac{1}{2} + \frac{1}{4}K_2^2 G G'' - \frac{1}{8}K_2^2 G'^2\right)^2.$$

To conclude the proof, we realize that $h = G^{\frac{3}{2}}\frac{\mathrm{d}m_2}{\mathrm{d}x}$. Since $m_2(x_2) = 0$ (see (65)), this means that $m_2 > 0$ for all $x_0 \in (x_2, x_3)$. The proof of the second case for the regular axis segment $x = x_3$ is completely analogous; the function h is the same, only now $h(x_3) = 0$. Hence $\frac{\mathrm{d}h}{\mathrm{d}x}_{|x=x_0} > 0$ implies $h < 0$ on I, which together with $m_3(x_3) = 0$ again implies $m_3 > 0$ on I.

[28] $\mathrm{F}(z,k) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$, $\mathrm{E}(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\mathrm{d}t$.

[29] Being zero at one of the boundary points and, as can also be shown, diverging at the other.

[30] In fact, it still suffices if $h_{,x}$ is zero at a finite number of points, as in our case, where this happens at one of the boundary points.

References

[1] Tafel J and Pukas S 2000 Comparison of the Bondi–Sachs and Penrose approaches to asymptotic flatness *Class. Quantum Grav.* **17** 1559–70

[2] Tafel J 2000 Bondi mass in terms of the Penrose conformal factor *Class. Quantum Grav.* **17** 4397–408

[3] Tafel J and Natorf W 2004 Asymptotic flatness and algebraically special metrics *Class. Quantum Grav.* **21** 5397–407

[4] Cornish F H J and Uttley W J 1994 The interpretation of the *C*-metric. The vacuum case *Gen. Rel. Grav.* **27** 439–54

[5] Kinnersley W and Walker M 1970 Uniformly accelerating charged mass in general relativity *Phys. Rev.* D **2** 1359–70

[6] Sládek P 2002 Global structure of rotation and boost-rotation symmetric spacetimes *Master's Thesis* Institute of Theoretical Physics, Charles University, Prague

[7] Griffiths J B, Krtouš P and Podolský J 2006 Interpreting the *C*-metric *Class. Quantum Grav.* **23** 6745–66

[8] Bičák J and Schmidt B 1989 Asymptotically flat radiative space-times with boost-rotation symmetry: the general structure *Phys. Rev.* D **40** 1827–53

[9] Ashtekar A and Dray T 1981 On the existence of solutions to Einstein's equation with non-zero Bondi news *Commun. Math. Phys.* **79** 581–9

[10] Bičák J 1968 Gravitational radiation from uniformly accelerated particles in general relativity *Proc. R. Soc.* A **302** 201–24

[11] Bičák J 1985 On exact radiative solutions representing finite sources *Galaxies, Axisymmetric Systems and Relativity (Essays Presented to W B Bonnor on his 65th Birthday)* ed M A H MacCallum (Cambridge: Cambridge University Press) pp 99–114

[12] Bičák J and Pravdová A 1998 Symmetries of asymptotically flat electrovacuum space-times and radiation *J. Math. Phys.* **39** 6011–39

[13] Penrose R 1963 Asymptotic properties of fields and space-times *Phys. Rev. Lett.* **10** 66–8

[14] Penrose R 1965 Zero rest-mass fields including gravitation: asymptotic behaviour *Proc. R. Soc.* A **284** 159–203

[15] Bondi H, van der Burg M G J and Metzner A W K 1962 Gravitational waves in general relativity: VII. Waves from axi-symmetric isolated systems *Proc. R. Soc.* A **269** 21–52

[16] Sachs R K 1962 Gravitational waves in general relativity: VIII. Waves in asymptotically flat space-time *Proc. R. Soc.* A **270** 103–26

[17] Farhoosh H and Zimmerman R L 1979 Stationary charged *C*-metric *J. Math. Phys.* **20** 2272

[18] Bičák J 1987 Radiative properties of space-times with the axial and boost symmetries *Gravitation and Geometry; A Volume in Honour of Ivor Robinson* ed W Rindler and A Trautman (Naples: Bibliopolis) pp 55–69

[19] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)

[20] Pravda V and Pravdová A 2000 Boost-rotation symmetric spacetimes—review *Czech. J. Phys.* **50** 333–76

[21] Bičák J 1990 Is there a news function for an infinite cosmic string? *Astron. Nachr.* **311** 189–92

[22] Bičák J and Schmidt B 1989 On the asymptotic structure of axisymmetric radiative spacetimes *Class. Quantum Grav.* **6** 1547–59