GENERALIZED SYMMETRIES FOR THE SDIFF(2) TODA EQUATION

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Self-dual Einstein metrics which admit one (rotational) symmetry vector are determined by solutions of the sDiff(2) Toda equation, which has also been studied in a variety of other physical contexts. Non-trivial solutions are difficult to obtain, with considerable effort in that direction recently. Therefore much effort has been involved with determining solutions with symmetries, and also with a specific lack of symmetries. The contact symmetries have been known for some time, and form an infinite-dimensional Lie algebra over the jet bundle of the equation. Generalizations of those symmetries to include derivatives of arbitrary order are often referred to as higher- or generalized-symmetries. Those symmetries are described, with the unexpected result that their existence also requires prolongations to "potentials" for the original dependent variables for the equation: potentials which are generalizations of those already usually introduced for this equation. Those prolongations are described, and the prolongations of the commutators for the symmetry generators are created. The generators so created form an infinite-dimensional, Abelian, Lie algebra, defined over these prolongations.

1 The $sDiff(2)$ Toda equation, and the standard Toda lattice

All self-dual vacuum solutions of the Einstein field equations that admit (at least) one rotational Killing vector are determined by solutions of the $\boldsymbol{sDiff}(2)$ Toda equation, which may be written in various equivalent forms:

$$u_{,q\bar{q}} + e^{u_{,ss}} = 0 \quad \Longleftrightarrow \quad r_{,q\bar{q}} + (e^r_{,s})_{,s} = 0$$
$$\Longleftrightarrow v_{,q\bar{q}} + (e^v)_{,ss} = 0 , \; v \equiv r_{,s} \equiv u_{,ss} . \quad (1)$$

How this comes about was shown by Charles Boyer and myself[1] in 1982. [In fact Plebański and I first wrote the equation down, for complex-valued, self-dual spaces in 1979.[2]] Since that time there has been considerable interest in this equation, in general relativity, and also in some other fields of physics and mathematics. Nonetheless, most currently known solutions describe metrics that also allow a translational Killing vector. Such solutions do not provide much new, real understanding of this equation since they were susceptible to discovery by a much simpler route, as solutions of the 3-dimensional Laplace equation.

To understand how this process occurs, we may begin with the standard **Plebański**[3]

formulation for an $\mathfrak{h}$-space, a heaven, which is a 4-dimensional, complex manifold with a self-dual curvature tensor. He of course showed that such a space is determined by a single function of 4 variables, $\Omega = \Omega(p, \bar{p}, q, \bar{q})$, which must satisfy one constraining pde, and then determines the metric via its second derivatives, as follows:

$$\mathbf{g} = 2(\Omega_{,p\bar{p}} \, dp \, d\bar{p} + \Omega_{,p\bar{q}} \, dp \, d\bar{q}$$
$$+ \Omega_{,q\bar{p}} \, dq \, d\bar{p} + \Omega_{,q\bar{q}} \, dq \, d\bar{q}) ,$$
$$\Omega_{,p\bar{p}}\Omega_{,q\bar{q}} - \Omega_{,p\bar{q}}\Omega_{,q\bar{p}} = 1 . \quad (2)$$

Restricting attention to those complex spaces that allow real metrics of Euclidean signature, there are only two possible "sorts" of Killing vectors, "translations" and "rotations." Noting that the covariant derivative of any Killing tensor must be skew-symmetric, by virtue of Killing's equations, we may make this division more technical by dividing the class of Killing vectors based on this skew-symmetric tensor's anti-self-dual part, which must be constant. For "translational" Killing vectors, this anti-self-dual part vanishes, while it does not for the "rotational" ones. The self-dual case—where the anti-self-dual part vanishes—has been completely resolved.[4,5] (In this case the con-

straining equation for Ω reduces simply to the 3-dimensional Laplace equation.)

We continue by insisting that the space under study admit a rotational Killing vector, $\widetilde{\xi}$, we re-define the variables so that they are adapted to it:

$$\widetilde{\xi} = i(p\partial_p - \bar{p}\partial_{\bar{p}}) \equiv \partial_\phi \,, \qquad \widetilde{\xi}(\Omega) = 0 \,,$$
$$p \equiv \sqrt{r}\,e^{i\phi} \,, \ \bar{p} \equiv \sqrt{r}\,e^{-i\phi} \,, \qquad (3)$$

which changes the constraining equation as follows, construing Ω to now depend on the variables $\{r, \phi, q, \bar{q}\}$:

$$(r\Omega_{,r})_{,r}\Omega_{,q\bar{q}} - r\,\Omega_{,qr}\Omega_{,\bar{q}r} = 1 \,. \qquad (4)$$

It is however often more convenient to rewrite the constraining equation, and the metric, in terms of a new set of coordinates, obtained from the original ones via a Legendre transform based on variables r and $s \equiv r\Omega_{,r}$. Taking $\{s, q, \bar{q}\}$, along with ϕ, as the new coordinates, and $v \equiv \ln r$ as the function of these coordinates that will generate the metric, we find the following new presentation, which show the agreement with the $\boldsymbol{sDiff}(2)$ Toda equation:

$$\mathbf{g} = V\,\boldsymbol{\gamma} + V^{-1}(d\phi + \varpi)^2 \,,$$
$$\boldsymbol{\gamma} \equiv ds^2 + 4\,e^v\,dq \wedge d\bar{q} \,, \qquad (5)$$
$$V \equiv \tfrac{1}{2}v_{,s} \,, \quad \varpi \equiv \tfrac{i}{2}\{v_{,q}dq - v_{,\bar{q}}d\bar{q}\} \,,$$

and we still require that

$$v_{,q\bar{q}} + (e^v)_{,ss} = 0 \,, \qquad (6)$$

and $\quad \underset{\gamma}{*}(d\varpi) = -iV^2 d(2s - 1/V) \,.$

The name I have used for Eq. (1) was first used by Mikhail Saveliev,[6] and also Kanehisa Takasaki and T. Takebe,[7] emphasizing their understanding of its relationship to the algebra of all area-preserving diffeomorphisms of a 2-surface. Saveliev and Vershik used[8] this equation as a non-trivial example of the use of their development of *continuum* Lie algebras, it having the symmetry group which was a limit of $\boldsymbol{A}_n$ as n went to $+\infty$. Takasaki and Takebe created a (double) hierarchy of equations connected with this equation, analogous

to the hierarchy for the KP equation, which contained operator realizations of $\boldsymbol{sDiff}(2)$. The name also emphasized its relationship to certain limits of the 2-dimensional Toda lattice equations:

$$u^a_{,xy} = e^{K^a{}_b u^b} \quad \text{or} \quad v^a_{,xy} = K^a{}_b\,e^{v^b} \,,$$
$$(v^a \equiv K^a{}_b\,u^b) \,, \qquad a, b = 1, 2, \ldots, n \,, \ (7)$$

where $K^a{}_b$ is the Cartan matrix for the Lie algebra which is also the generator of the symmetries of these same Toda equations .

The hoped-for virtue of the relationship with the Toda lattice lay in the fact that the Toda lattice equations, when based on any finite-dimensional, semi-simple algebra, has symmetries which allow the determination of Bäcklund transformations which generate new solutions from old ones. We first use the gauge freedom in the original equations to divide into two parts the unknown functions $v^a \equiv a^a + b^a$. Then for the case when the Lie algebra is $\boldsymbol{sl}(n+1)$, one finds that the explicit first-order pde's for the Bäcklund transformation as the following, where the $w^a = w^a(x, y)$ are the "other" set of dependent variables, i.e., the "pseudopotentials" involved in the transformation:

$$\{w^a - a^a\}_{,x} = -e^{w^a + b^a} + e^{w^{a-1} + b^{a-1}} \,,$$
$$\{w^a + b^a\}_{,y} = e^{-w^a + a^a} - e^{-w^{a+1} + a^{a+1}} \,. \ (8)$$

The zero-curvature conditions, for the difference of the two cross-derivatives, then generate exactly the original Toda equations, Eqs. (7), in the variables v^a, as desired, and expected. Moreover, if one adds the two cross-derivatives, inserts the form for $b^a_{,xy}$ from the Toda equations, **and** changes to the new, translated pseudopotentials, $\ell^k \equiv w^k - w^{k+1} + a^{k+1} + b^k$, then these new dependent variables are also required to satisfy the Toda equations, although for $\boldsymbol{sl}(n)$, since there are only $n-1$ of them:

$$\ell^a_{,xy} = 2e^{\ell^a} - e^{\ell^{a-1}} - e^{\ell^{a+1}} = \left(K_{n-1}\right)^a{}_b\,e^{\ell^b} \,,$$
$$a, b = 1, \ldots, n-1 \,. \qquad (9)$$

2 The continuous limit of the Toda lattice equations

Following the straight-forward existence of both soliton-type solutions and Bäcklund transformations of the Toda lattice equations, we, earlier, studied limits of these equations to the continuous case, with the intent of course that these limits would carry over to the existence of Bäcklund transformations for our equation with three independent variables. To accomplish the change from discrete indices to functions of a (new) continuous variable, we begin with a new function, $V = V(z, \overline{z}, s)$, that depends on a third continuous variable, s, which varies, say, from 0 to β. We then superpose on these values for s a lattice of n points, a distance δ apart, fill in the space between the lattice points by taking the limit as $n \to \infty$, with β fixed, which is the same as taking the limit as $\delta \to 0$, and following earlier work of Park,[9] re-scale the other continuous variables so that appropriate differences of the exponentials of the v^a's will create second derivatives with respect to s:

$$V(z, \overline{z}, s)\Big|_{s=a\delta} \equiv v^a(z/\delta, \overline{z}/\delta) ,$$
$$a = 1, \ldots, n , \quad \delta = \beta/(n-1) , \quad (10)$$
$$\implies V(z, \overline{z}, s) \equiv \lim_{\delta \to 0} v^{[s/\delta]}(z/\delta, \overline{z}/\delta) ,$$

where the square brackets indicate the integer part of the quotient within them. This works well, in fact, although the u^b's need a scaling of their own, to create their second derivatives:

$$U(z, \overline{z}, s)\Big|_{s=a\delta} \equiv \delta^2 \{u^a(z/\delta, \overline{z}/\delta)\} . \quad (11)$$

Assuming sufficient continuity of our functions, we may now take limits of the Toda equations, which give the desired results:

$$U_{,z\overline{z}} = e^{-U_{,ss}} , \quad V_{,z\overline{z}} = -\partial_s^2 e^V . \quad (12)$$

However, when we take the same limits on the prolongation equations themselves, agreeing to treat the gauged parts, a^a and b^a, the same as their sum v^a, and also the pseudopotentials w^a, we acquire the following limiting forms for the "proposed" Bäcklund transformations:

$$(W - A)_{,z} = -\partial_s e^{+(W+B)} ,$$
$$(W + B)_{,\overline{z}} = -\partial_s e^{-(W-A)} . \quad (13)$$

However, the integrability conditions of these equations are not what was desired:

$$V_{,z\overline{z}} = -\partial_s e^V \partial_s V = -\partial_s^2 e^V , \quad (14)$$
$$\text{and} \quad L_{,z\overline{z}} = -\partial_s \left\{ e^V \partial_s L \right\} , \quad (15)$$
$$L \equiv 2W + B - A .$$

The first of these equations is of course what we expect; but the second is not. This particular pair of equations is just the system of equations that LeBrun[10] requires to determine his "weak heavens," which have only self-dual conformal curvature, and therefore a possibly non-zero matter tensor. There is quite a lot of interesting work on the complete resolution of this pair of equations; perhaps we ought to look at it as a system and "try again"? Nonetheless, it certainly does not create a Bäcklund equation for the original problem.

At this point no progress has been made toward the advertised goal, namely a method to "buy" some new solutions to the original $\boldsymbol{sDiff}(2)$ Toda equation from old, previously-known ones, or, stated differently, how to obtain families of metrics that are solutions to the stated problem in general relativity using ones that were already known. It is also worth noting that the approach via limits seemed desirable and interesting because the Estabrook-Wahlquist method of finding prolongations, pseudopotentials, Bäcklund transformations, etc.— which is our method of choice—has yet to find a truly effective generalization to problems with more than two independent variables. It had indeed been hoped that a solution to this problem would allow an understanding of the 3-variable problem sufficient to create a good generalization; unfortunately this has not (yet) occurred.

Before continuing, this is perhaps a good point to review some of the studies of this equation that have been made by several other groups. We have of course already mentioned Mikhail Saveliev and A.M. Vershik and their theory of continuum Lie algebras.[8] where the elements in the algebra might be labelled by continuous variables, as "indices," instead of the more usual discrete indices. Their continuum approach to, say, the Lie algebra $\boldsymbol{A}_\infty$, would use "test functions" from some appropriate function space to label the elements of the algebra instead of discrete labels. The result would then be the following where $X_0(f)$ are elements of the (Abelian) Cartan subalgebra, i.e., with grade 0, while $X_{\pm 1}(f)$ are elements of the first and minus-first grades, often referred to as E_i and F_j:

$$[X_0(f), X_{\pm 1}(g)] = \pm X_{\pm 1}((fg)') , \quad (16)$$

$$[X_{+1}(f), X_{-1}(g)] = X_0(fg) . \quad (17)$$

This approach enabled them to write down a form for a "general solution" for an initial-value problem for our equation, involving choices of functions of two variables; unfortunately, at least as we see it, this form is rather too formal, and has not yet been made practically useful.

From different directions, R. S. Ward,[11,12,13] and K. Takasaki[7] have created objects they refer to as Lax pairs for this equation, using Poisson brackets instead of the usual commutators. However, the Lax pairs involved to not seem to involve pseudopotentials, i.e., realizations of the group of symmetry involving new dependent variables. We have therefore been unable to use them to generate Bäcklund transformations, although they surely do generate an infinite hierarchy of associated equations, in the spirit of the KP hierarchy.

Although Boyer and myself showed that the metrics did not admit just two, rotational symmetry vectors, it is certainly true that there are metrics which admit an entire $\boldsymbol{sl}(2)$ of such vectors. These have originally been found by Atiyah,[14] originally studying monopole solutions of the Yang-Mills equations; this was then elaborated in some detail by Olivier.[15] The advantage of such a large symmetry group is that the pde's are reduced to ordinary differential equations in just one independent variable. A somewhat different approach, via (3-dimensional) Einstein-Weyl spaces, has brought Tod,[16] and co-workers,[17,18] to looking at other reductions to simply ordinary differential equations. Olivier's equations involve elliptic functions as solutions, while Tod's go one step higher and involve Painlevé transcendents.

Yet another approach is to simply look for ansätze that give non-trivial results. Plebański and myself already put forward some very simple ansätze for this equation, which did indeed demonstrate all possible heavenly Petrov types; nonetheless, they were not particularly inspired. In fact interesting ansätze are rather difficult to acquire, since too much symmetry is easily acquired, reducing the problem to one that also contains a self-dual Killing vector. However, one plausible ansatz is obtained by requiring the second term in the v-form of the equation to be independent of s. This imposes the condition that e^v be a second-order polynomial in s, and creates a problem that can be resolved by simply solving some Liouville equations. This generates the result

$$\begin{aligned} e^{v_C} &= +[s + G(q)][s + H(\bar{q})]\alpha \\ &= -[s + G(q)][s + H(\bar{q})]\frac{A'(q)B'(\bar{q})}{(A + B)^2} . \end{aligned} \tag{18}$$

where α is the general solution of the Liouville equation, and the four functions shown are arbitrary functions of one variable. It should of course be pointed out that the equation has conformal symmetry so that two of these functions could be absorbed into new definitions of the original independent variables. By different methods this solution has

been published some 3 different times in the last few years. Calderbank and Tod[19] found it (first) by imposing restrictions on the associated Einstein-Weyl spaces. Martina, Sheftel and Winternitz[20] also found it by asking a question related to an allusion above, namely by looking specifically for solutions without excess invariances.

3 Generalized Symmetries of the Equation

Eventually, following some helpful comments by M. Dunajski, we began an alternative attempt to find a method to obtain new solutions, which will indeed be the principal point of this talk. Unable to find proper prolongation algebras for the equation, we changed tactics and looked at the question of finding the algebra of generalized symmetries over the infinite jet over the pde. We begin our hunt for the generalized symmetries, as usual by considering the pde, in the form with $v = v(q, \bar{q}, s)$, as a variety in the second jet bundle, with coordinates $\{q, \bar{q}, s, v, v_q, v_{\bar{q}}, v_s, v_{qq}, v_{qs}, v_{ss}, v_{\bar{q},s}, v_{\bar{q},\bar{q}}\}$ and *co-coordinate* defining the surface, $v_{q\bar{q}}$, determined in terms of the coordinates from the pde. We then prolong that bundle to the infinite jet, where the co-coordinates are chosen to be all "derivatives" of v that involve at least one q and also one $\bar{q}$, resolved from the equations created from all possible derivatives of the original pde. On this infinite jet we use the usual total derivative operators in each direction, of the form, for example,

$$\overline{D}_q = \partial_q + v_q \partial_v + v_{qq} \partial_{v_q} + \widetilde{v_{,q\bar{q}}} \partial_{v_{\bar{q}}} + v_{,qs} \partial_{v_s}$$
$$+ v_{,qqq} \partial_{v_{qq}} + v_{,qqs} \partial_{v_{qs}} + v_{qss} \partial_{v_{ss}}$$
$$+ v_{q\bar{q}s} \partial_{v_{\bar{q}s}} + v_{q\overline{qq}} \partial_{v_{\overline{qq}}} + \ldots \ , \quad (19)$$

where the overbar on the derivative operator reminds us that this generic operator has been restricted to live on the variety which defines the pde. Because of this, we then use the "over-tilde" to indicate that this coefficient is to be determined from the constraint

equations defining the co-coordinates. We may then look for generators, φ_v, of symmetries involving any (finite) number of derivatives of the original variables, which must satisfy the standard equation, which we take from Vinogradov's approach:[21]

$$\left\{ \overline{D}_q \overline{D}_{\bar{q}} + e^v \left[\overline{D}_s \overline{D}_s + 2 v_s \overline{D}_s \right. \right.$$
$$\left. \left. + (v_{ss} + \Omega_s^2) \right] \right\} \varphi_v = 0 \ . \quad (20)$$

We were rather perplexed when explicit computations showed that there were no such objects involving derivatives higher than first order. (We actually did expect a symmetry algebra built on $\boldsymbol{sDiff}(2)$.) These first-order ones were of course just the ordinary (Lie) symmetries for the equation, published at various times before:[22]

$$\varphi_v = A(q) \, v_q + \bar{A}(\bar{q}) \, v_{\bar{q}} + (\alpha \, s + \beta) v_s$$
$$+ A_{,q}(q) + \bar{A}_{,\bar{q}}(\bar{q}) - 2\alpha \ , \quad (21)$$

with the two arbitrary functions of 1 variable, $A(q)$ and $\overline{A}(\bar{q})$. When q and $\bar{q}$ are restricted to be complex conjugates of one another, originating from the original geometry of a Euclidean signature metric, then this pair coordinate and define the well-understood conformal transformations of that underlying 2-space. We record here their commutators:

$$\{\varphi_\alpha, \varphi_\beta\} = \varphi_\beta \ , \quad \{\varphi_A, \varphi_{\bar{A}}\} = 0 \ ,$$
$$\{\varphi_\alpha, \varphi_A\} = \{\varphi_\beta, \varphi_A\} = 0 \ ,$$
$$\{\varphi_\beta, \varphi_{\bar{A}}\} = \{\varphi_\alpha, \varphi_{\bar{A}}\} = 0 \ , \quad (22)$$
$$\{\varphi_{A_1}, \varphi_{A_2}\} = \varphi_{A_1 A_{2,q} - A_2 A_{1,q}} \ ,$$
$$\{\varphi_{\bar{A}_1}, \varphi_{\bar{A}_2}\} = \varphi_{\bar{A}_1 \bar{A}_{2,\bar{q}} - \bar{A}_2 \bar{A}_{1,\bar{q}}} \ .$$

The lack of any genuine symmetries at higher order than the first jet was eventually resolved by the introduction of "potentials" into the jet bundle. This is perhaps not truly surprising since our original presentation of our equation gave not only the v-form of the equation, currently being considered, but also two forms involving two different potential functions, r and u, defined such that $r_{,s} \equiv v$

and $u_{,ss} \equiv v$, believing them to be "equivalent" equations. Indeed, when we prolonged our jet bundle for v in these "integral directions," such symmetry generators did in fact appear. We intend now to first write down generators at the next two levels. However, to simplify the discussion, we note that, modulo the well-understood conformal transformations depending on arbitrary functions, the Lie symmetries could be thought of as being generated by the two transformations in s and simply by v_q and $v_{\overline{q}}$. Therefore, when discussing the generalized ones we will also think of them modulo those conformal symmetries and therefore as generated by two sequences of generators, beginning with v_q and $v_{\overline{q}}$, respectively, which we will label as Q_1 and $\overline{Q}_1$. By explicit calculation we found the next two pairs, and display those three pairs here:

$$Q_1 = v_q = (r_q)_s = e^{-v}(e^v)_q \ ,$$
$$\overline{Q}_1 = v_{\overline{q}} \ ,$$
$$Q_2 = r_{qq} + r_q \, v_q = [u_{qq} + \tfrac{1}{2}(r_q)^2]_s$$
$$= e^{-v}[e^v \, r_q]_q \ ,$$
$$\overline{Q}_2 = r_{\overline{q}\overline{q}} + r_{\overline{q}} \, v_{\overline{q}} = [u_{\overline{q}\overline{q}} + \tfrac{1}{2}(r_{\overline{q}})^2]_s$$
$$= e^{-v}[e^v \, r_{\overline{q}}]_{\overline{q}} \ , \quad (23)$$
$$Q_3 = e^{-v}[e^v(u_{qq} + (r_q)^2)]_q \ ,$$
$$\overline{Q}_3 = e^{-v}[e^v(u_{\overline{q}\overline{q}} + (r_{\overline{q}})^2)]_{\overline{q}} \ .$$

It is probably important to emphasize at this point that, for instance, Q_1 satisfies the linearization equation, Eq. (23), just as it stands. However, Q_2 does not, because it involves a potential for the pde. In principle, one could imagine two different sorts of generalizations to that equation which might be appropriate for Q_2. The first option would say that, since it involves the potential r, we should just start the problem over again and look for symmetries of the defining pde for r, and expect that this is what we should obtain. This is definitely **not true**. That pde suffers exactly the same deficit of generalized symmetries, in its own right, as did our original, equivalent equation for v. The second option would say that, since this symmetry generator involves a potential, and a prolongation of the jet bundle in that direction, then we must also prolong the total derivative operators that appear in Eq. (23). This, indeed, is the correct choice, if we replace those total derivative operators, $\overline{D}_i$, by new ones, $_1\widehat{D}_i$, prolonged with appropriate additional terms, then Q_2 will indeed satisfy that prolonged version of the equation. We denote these prolonged operators also with a pre-script 1 since there will be more:

$$_1\widehat{D}_q - \overline{D}_q = r_q\partial_r + r_{qq}\partial_{r_q} + r_{qqq}\partial_{r_{qq}} + \ldots$$
$$+ \widetilde{r_{q\overline{q}}}\partial_{r_{\overline{q}}} + \widetilde{r_{q\overline{q}\overline{q}}}\partial_{r_{\overline{q}\overline{q}}} + \ldots \ ,$$
$$_1\widehat{D}_{\overline{q}} - \overline{D}_{\overline{q}} = r_{,\overline{q}}\partial_r + r_{\overline{q}\overline{q}}\partial_{r_{\overline{q}}} + r_{\overline{q}\overline{q}\overline{q}}\partial_{r_{\overline{q}\overline{q}}} + \ldots$$
$$+ \widetilde{r_{q\overline{q}}}\partial_{r_q} + \widetilde{r_{qq\overline{q}}}\partial_{r_{qq}} + \ldots \ , \quad (24)$$
$$_1\widehat{D}_s - \overline{D}_s = v\partial_r + v_{\overline{q}}\partial_{r_{\overline{q}}} + v_{\overline{q}\overline{q}}\partial_{r_{\overline{q}\overline{q}}} + \ldots$$
$$+ v_q\partial_{r_q} + v_{qq}\partial_{r_{qq}} + \ldots \ .$$

The same sort of thing happens, again, of course, when we attempt to find the generalized symmetry, Q_3, which involves u in its definition. We again prolong the underlying jet bundle and also the total derivative, this time creating the quantities $_2\widehat{D}_i$, with coefficients that involve the various q- and $\overline{q}$-derivatives of u, but no mixed ones, and no s-derivatives, either, for they are expressible in terms of the r's which are already in the prolonged bundle. As before Q_3 is a symmetry only for this re-definition of the requirements.

4 Commutators for the Symmetry Generators

The first pair of generalized symmetries required a potential at the level of a first "integral"; the the second pair needed a potential at the second level of integration. These were easy because they were already understood. On the other hand, one needs yet a third level of integration to acquire another pair of generators. The question immediately arises as to how to choose these higher levels of potentials. This question is of course closely

related to similar questions that occur in the study of the KP equation, for example, where the standard (Japanese school) approach involves an infinite hierarchy of dependent variables all satisfying more- and more-involved equations as one climbs upward in the hierarchy. Therefore we used as a guide the hierarchical approach to this equation taken by Takasaki and Takebe, for which we now give a (very) brief description. They created[7] a pair of hierarchies that are associated with the $sDiff(2)$ Toda equation, which involve 4 infinite sequences of functions dependent on our 3 independent variables, which we may label as u_i, v_i, $\widehat{u}_i$ and $\widehat{v}_i$, as i goes from 0 to $+\infty$. The first pair are involved with the quantities that will create symmetries in the q variables, while the second pair are involved with symmetries in the $\bar{q}$ variables. They are then inserted as coefficients into two series in powers of a "spectral" variable, λ, which act as generating functions for the entire sequence of pde's, written in a Poisson-bracket format, as follows, where we write only the ones for the q variables, with the $\bar{q}$ ones being completely analogous:

$$\mathcal{L} \equiv \lambda + u_0 + u_1\lambda^{-1} + u_2\lambda^{-2} + u_3\lambda^{-3} + \dots$$
$$= \lambda + \sum_0^\infty u_i\lambda^{-i} , \ u_0 \equiv r_q = v_{qs} , \quad (25)$$

$$\mathcal{M} \equiv q\mathcal{L} + s + \sum_1^\infty v_n\mathcal{L}^{-n} \qquad (26)$$
$$= q\lambda + s + q\,u_0 + (qu_1 + v_1)\lambda^{-1} + \dots .$$

These series must satisfy the following Poisson brackets (analogous to commutator brackets), which generate infinite sequences of relationships, equating powers of λ:

$$\{\mathcal{B}, \mathcal{L}\} = \mathcal{L}_{,q} , \ \{\widehat{\mathcal{B}}, \mathcal{L}\} = \mathcal{L}_{,\bar{q}} ,$$
$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L} , \quad (27)$$
$$\{\mathcal{B}, \mathcal{M}\} = \mathcal{M}_{,q} \iff$$

$$\mathcal{L} - \lambda = \sum_1^\infty \{-v_{n,q} + \lambda v_{n,s}\}\mathcal{L}^{-n} , \quad (28)$$

$$\{\widehat{\mathcal{B}}, \mathcal{M}\} = \mathcal{M}_{,\bar{q}} \iff$$

$$-\widehat{\mathcal{B}} = \sum_1^\infty \{+v_{n,\bar{q}} + \widehat{\mathcal{B}}\,v_{n,s}\}\mathcal{L}^{-n} . \quad (29)$$

The quantities $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are the following short, finite series, while the Poisson bracket is in the variables s and the logarithm of the spectral variable, $p \equiv \ln\lambda$:

$$\mathcal{B} = \lambda + u_0 , \quad \widehat{\mathcal{B}} = \frac{e^v}{\lambda} , \qquad (30)$$
$$\{A, B\} \equiv \lambda A_{,\lambda}B_{,s} - \lambda B_{,\lambda}A_{,s} .$$

Comparing powers of λ gives several infinite sequences of pde's, involving the u_j's, the v_k's, $u_0 = r_q$ and v. As expected the earliest members of these sequences repeat the original pde's, while we acquire more identifications, such as $u_1 = u_{qq}$ and $v_1 = u_q$. More specifically, they give the s- and $\bar{q}$-derivatives of the u_j's in terms of derivatives of lower-order u_k's, and the s- and q-derivatives of the v_m's in terms of the u_k's and lower-order v_n's. These can be arranged to determine a hierarchy of equations.

Since all the higher-numbered quantities in these hierarchies are essentially potentials for the lower ones, there were not unique choices for the desired extension. It turned out that Takasaki's quantities v_j seem to be a very good choice, however. We therefore have taken a renormalization of them for an infinite sequence of potentials in the q-direction, and a similar choice involving the $\hat{v}_k$'s for potentials in the $\bar{q}$-direction. There does not seem to be a single choice appropriate to both directions at once, as there was in the beginning when we were using r and u. However, this does not increase, noticeably, the total size of the prolonged jet bundle. The reason for this is that while, for instance, for u as a potential, we had to also append all its q-derivatives and all its $\bar{q}$-derivatives (but not the mixed ones, nor the ones involving s-derivatives), for these new objects we need only one set, and not the others. More particularly, since v_1 may be identified with u_q, we really begin with v_2. Labelling the elements of this sequence of potentials by x_i,

and the corresponding $\overline{q}$-type potentials by y_j, we have the following:

$$x_2 \equiv \tfrac{1}{2}v_2 \implies \begin{cases} x_{2,s} = u_{qq} + \tfrac{1}{2}(r_q)^2, \\ x_{2,\overline{q}} = -r_q\,e^v. \end{cases} \quad (31)$$

$$y_2 \equiv \tfrac{1}{2}\hat{v}_2 \implies \begin{cases} y_{2,s} = u_{\overline{q}\overline{q}} + \tfrac{1}{2}(r_{\overline{q}})^2, \\ y_{2,q} = -r_{\overline{q}}\,e^v. \end{cases} \quad (32)$$

As can be seen we already know both the s-derivatives and the $\overline{q}$-derivatives of x_2, so that only the infinite sequence of q-derivatives, and the $\overline{q}$-derivatives of y_2, must be appended to the list of coordinates for the prolonged infinite jet bundle.

We now note a few more of the q-forms of these potentials, and then explain reasons why they are good choices:

$$x_3 : \begin{cases} x_{3,s} = x_{2,q} + r_q u_{qq} + \tfrac{1}{3}(u_q)^3 \,, \\ x_{3,\overline{q}} = -[u_{,qq} + (r_q)^2]\,e^v \,, \end{cases} \quad (33)$$

$$x_4 : \begin{cases} x_{4,s} = x_{3,q} + r_q x_{2,q} + u_{qq}(r_q)^2 \\ \qquad + \tfrac{1}{2}(u_{qq})^2 + \tfrac{1}{4}(r_q)^4 \,, \\ x_{4,\overline{q}} = -[x_{2,q} + 2r_q u_{qq} + (r_q)^3]\,e^v \,. \end{cases} \quad (34)$$

Now, of course, after having added appropriate dimensions to the jet bundle as already discussed, and after having additional appropriate terms to the yet-again-prolonged total derivative operators, these potentials must be suitable to generate additional generalized symmetries for our equation: indeed we may write them in terms of these quantities, one new generator for each new potential:

$$Q_4 = e^{-v}\{e^v[x_{2,q} + 2r_q u_{qq} + (r_q)^3]\}_q \,, (35)$$

$$\overline{Q}_4 = e^{-v}\{e^v[y_{2,\overline{q}} + 2r_{\overline{q}} u_{\overline{q}\overline{q}} + (r_{\overline{q}})^3]\}_{\overline{q}} \,, (36)$$

$$Q_5 = e^{-v}\{e^v[x_{3,q} + 2r_q x_{2,q} + (u_{qq})^2 \\ \qquad + 3u_{qq}(r_q)^2 + (r_q)^4]\}_q \,, \quad (37)$$

$$Q_6 = e^{-v}\{e^v[x_{4,q} + 2r_q x_{3,q} \\ \qquad + (2u_{qq} + 3r_q^2)x_{2,q} + 3(u_{qq})^2 r_q \\ \qquad + 4u_{qq}(r_q)^3 + (r_q)^5]\}_q \,. \quad (38)$$

$$\cdots$$

Each of these satisfies the appropriate prolongation of the Vinogradov equation for symmetry generators. They have quite interesting structure, and we presume that a recursion process may be defined for them, although this has not yet been found. On the other hand, they do have several other rather unexpected properties. Each of the generalized symmetry generators can be written as a perfect s-derivative; moreover, each of them may also be written in two different ways in terms of second derivatives of the correspondingly numbered new potential:

$$-e^{-v}\,\overline{D}_q\overline{D}_{\overline{q}}\,x_j = Q_j = \overline{D}_s\{\overline{D}_s(x_j)\} \,; \quad (39)$$

Therefore, we may write, for each j, the corresponding "linear" pde:

$$x_{j,q\overline{q}} + e^v\,x_{j,ss} \,. \quad (40)$$

That the symmetry generators may be written in terms of second derivatives of potentials is, *after the fact,* not too surprising, for reasons which will be explained shortly. On the other hand, that each of those potentials satisfies this linear equation similar to the Le-Brun monopole equation was certainly unexpected. (The statement that it is linear is of course slightly misleading insofar as the q_j's are potentials for the unknown function v that also appears within the equation.)

A last comment relevant to the choice of potentials for this problem returns to Takasaki's approach to the hierarchy of dependent functions and corresponding pde's that they satisfy. In this hierarchical approach it is usual to also introduce an additional infinite sequence, of independent variables, on which the various functions may depend. As the equations in the hierarchy may be satisfied simultaneously they constitute distinct, commuting flows over the solution manifold, so that these new independent variables may be thought of as the flow parameters along the curves described by the flows. Takasaki and Takebe refer to these additional independent variables by q_m and $\overline{q}_m$, for $m = 1..\infty$, and give generalizations of the Poisson-bracket equations above that apply to them. It is then also common in such descriptions to determine a τ-function that depends on the entire infinite set of independent variables, and allows one to determine

all the other dependent variables from that. In their description, the various derivatives of *the logarithm of* that τ-function, with respect to the variables q_i and $\bar{q}_j$ are just these quantities v_i and $\hat{v}_j$; i.e., $\partial(\log \tau)/\partial_{q_i} \propto v_i \propto x_i$, making it seem rather more reasonable that these functions would indeed to "potentials" to describe the desired properties of the solution space.

To return to the symmetries now, and consider why it is not too surprising that these expressions may be written as perfect s-derivatives, we must first re-visit the notion that they are generators for symmetries. If the symmetries were written in terms of their associated vector fields, over the jet bundle, then we would expect to consider the standard commutators, i.e., Lie brackets, of two of them, and insist that they close onto themselves. Since we are describing the symmetries in terms of their generators, instead of their vector fields, there must be an associated mapping of the generators that accomplishes the same thing, i.e., a realization of the Lie bracket in the underlying, abstract algebra. Our approach to this realization is via the universal linearization operator, which was also used to create the (Vinogradov) equation that must be satisfied by a symmetry generator. We define a linear operator for functions on the infinite jet bundle and then restrict it to the variety defined by some system of pde's, F, which are resolved by some system of functions $u^\nu = u^\nu(x^a)$:

$$\Im_\phi \equiv \left\{ \phi^\nu \partial_{u^\nu} + \{\overline{D}_a(\phi^\nu)\}\partial_{u_a^\nu} \right.$$
$$\left. + \{\overline{D}_a \overline{D}_b(\phi^\nu)\}\partial_{u_{ab}^\nu} + \ldots \right\}$$
$$\equiv \sum_{\sigma=0}^{(\infty)} \left\{\overline{D}_{(\sigma)}(\phi^\nu)\right\}\partial_{u_{(\sigma)}^\nu}, \quad (41)$$

where the sum is over all "multi-indices." The Vinogradov equation, which determines a system defining a symmetry ϕ^ν of F, is simply that $\Im_\phi(F) = 0$. In general, given two such solutions, i.e., two such symmetries, then they determine a third solution, possi-

bly just 0, that we refer to as the commutator of the two solutions because the vector field that it generates is the vector field commutator of the two vector fields generated by the initial pair of symmetries. This commutator is specified by a Poisson-bracket sort of relationship: given two symmetry generators, ϕ and ψ, then the one they determine is η, given by

$$\eta^\mu = \{\phi, \psi\}^\mu \equiv \Im_\phi(\psi^\mu) - \Im_\psi(\phi^\mu) . \quad (42)$$

For symmetry generators on the variety over our jet bundle defined by the pde and all its derivatives, the coordinates in use are the sets $\{v, v_q, v_{qq}, \ldots\}$, $\{v_{\bar{q}}, v_{\overline{qq}}, \ldots\}$, $\{v_s, v_{ss}, \ldots\}$, $\{v_{sq}, \ldots\}$, and $\{v_{s\bar{q}}, \ldots\}$. Therefore, for a function, say Q, defined over this variety, we can say that a more explicit form for the $\Im$ operator will be

$$\Im_Q = \left\{ Q\partial_v + \{\overline{D}_q(Q)\}\partial_{v_q} + \{\overline{D}_q^2(Q)\}\partial_{v_{qq}} \right.$$
$$+ \cdots$$
$$+ \{\overline{D}_{\bar{q}}(Q)\}\partial_{v_{\bar{q}}} + \{\overline{D}_{\bar{q}}^2(Q)\}\partial_{v_{\overline{qq}}} + \cdots$$
$$+ \{\overline{D}_s(Q)\}\partial_{v_s} + \{\overline{D}_s\overline{D}_q(Q)\}\partial_{v_{sq}}$$
$$+ \{\overline{D}_s\overline{D}_{\bar{q}}(Q)\}\partial_{v_{s\bar{q}}} + \{\overline{D}_s^2(Q)\}\partial_{v_{ss}}$$
$$\left. + \cdots \right\} . \quad (43)$$

However, for our situation there are not very many interesting places to apply this since we have only just Q_1 and $\overline{Q}_1$ as symmetry generators defined on the variety over the original jet bundle. All the other symmetry generators require this potentialization of the bundle, described above. As already noted, this potentialization requires a prolongation of the total derivative operator. However, it **also requires** a prolongation of the linearization operator, since the functions involved are now defined over a rather larger space.

A convenient approach to determine how this should be done is accomplished by first retreating somewhat, and "deriving" the expression involving the linearization operator, using the Frechét (or Gateaux) derivative on function spaces. If σ is a function over some

space of functions, and ϕ and ψ are functions in that space, then we may of course talk about $\sigma(\phi)$ or $\sigma(\phi + \epsilon\psi)$. The Frechét derivative, of σ in the direction ψ is then the function on the function space, $\sigma'[\psi]$, which is that part of $\sigma(\phi + \epsilon\psi)$ that is linear in ϵ, divided by ϵ: $\sigma(\phi + \epsilon\psi) = \sigma(\phi) + \epsilon\sigma'[\psi](\phi) + O(\epsilon^2)$. Applying this now to functions over our infinite jet, we usually think of σ as depending on every one of the coordinates on the jet, $\{x^a, v, v_a, v_{ab}, v_{abc}, \ldots\}$. Now, when we imagine translating v by some very small amount, in some direction, say by $\epsilon\psi$, determined by ψ, some other function on the jet, we need to have a method to determine how this translation affects each of the other jet coordinates. To do this we consider an operator that "creates" the appropriate jet variable from q, such as $q_{ab} = \overline{D}_{x^a}\overline{D}_{x^b}q$, and then let that operate on the translated version of q:

$$\sigma(q, q_a, q_{ab}, q_{abc}, \ldots) \longrightarrow$$
$$\sigma(q + \epsilon\psi, q_a + \epsilon\overline{D}_a\psi, q_{ab} + \epsilon\overline{D}_a\overline{D}_b\psi,$$
$$q_{abc} + \epsilon\overline{D}_a\overline{D}_b\overline{D}_c\psi, \ldots)$$
$$= \sigma(q, q_a, \ldots) + \epsilon\Big\{(\psi)\,\sigma_q + (\overline{D}_a\psi)\,\sigma_{q_a}$$
$$+ (\overline{D}_a\overline{D}_b\psi)\,\sigma_{q_{ab}} + \ldots\Big\} + O(\epsilon^2)$$
$$\equiv \quad \sigma(q, \ldots) + \epsilon \mathfrak{Z}_\psi(\sigma) + \ldots \quad , \quad (44)$$

where the last equality follows from the definition of the linearization operator given above. This allows us to see that the Frechét derivative, of a function over a function space, is the complete analogue of the linearization operator acting on functions over a jet bundle. However, our "philosophical" understanding of the one concept can assist us in determining the correct prolongation of the other. This is especially because our introduction of various potentials into the larger jet bundle requires us to now consider functions that also depend on some of these "integrals" of jet coordinates. The simplest case is just the one where our function, like the symmetry generator Q_2, depends on the first integral, r, of the original dependent variable

v, i.e., $v = \overline{D}_s(r)$, or $r \equiv \overline{D}_s^{-1}(v)$. Therefore the prolongation of the linearization operator $\mathfrak{Z}_\psi$, which we would denote by $\widehat{\mathfrak{Z}}_\psi$, should have an extra term $\{\overline{D}_s^{-1}\psi\}\partial_r$. However, the existence of this new coordinate, v, also generates its ordinary derivatives as well; therefore, we also need new coordinates on the jet of the form $D_q r, D_{qq} r$, etc., while $D_s r$ would simply be v, and therefore not generate any new jet coordinate. The prolongation $\widehat{\mathfrak{Z}}$ may then be expressed in the following way, where we use Q_2 as a reasonable example function for it, and we use the pre-script 1 to indicate that this is simply the first of several prolongations that we will have to make:

$$_1\widehat{\mathfrak{Z}}_{Q_2} = \mathfrak{Z}_{Q_2} + \{\overline{D}_s^{-1}Q_2\}\,\partial_r$$
$$+ \{\overline{D}_q\overline{D}_s^{-1}Q_2\}\,\partial_{r_q} + \{\overline{D}_q^2\overline{D}_s^{-1}Q_2\}\,\partial_{r_{qq}}$$
$$+ \ldots$$
$$+ \{\overline{D}_{\overline{q}}\overline{D}_s^{-1}Q_2\}\,\partial_{r_{\overline{q}}} + \{\overline{D}_{\overline{q}}^2\overline{D}_s^{-1}Q_2\}\,\partial_{r_{\overline{qq}}}$$
$$+ \ldots \quad . \quad (45)$$

Going on toward the symmetry Q_3, we have introduced yet a new potential, $u = \overline{D}_s^{-2}(v)$, and, as discussed earlier, all of its **unmixed** q- and $\overline{q}$-derivatives. We may then use the same approach as above for the next prolongation of the linearization operator:

$$_2\widehat{\mathfrak{Z}}_{Q_3} = {}_1\widehat{\mathfrak{Z}}_{Q_3} + \{\overline{D}_s^{-2}Q_3\}\,\partial_u$$
$$+ \{\overline{D}_q\overline{D}_s^{-2}Q_3\}\,\partial_{u_q} + \{\overline{D}_q^2\overline{D}_s^{-2}Q_3\}\,\partial_{u_{qq}}$$
$$+ \ldots$$
$$+ \{\overline{D}_{\overline{q}}\overline{D}_s^{-2}Q_3\}\,\partial_{u_{\overline{q}}} + \{\overline{D}_{\overline{q}}^2\overline{D}_s^{-2}Q_3\}\,\partial_{r_{\overline{qq}}}$$
$$+ \ldots \quad . \quad (46)$$

Proceeding onward with the strings of potentials that we need, from Section III, the next one is considerably more complicated, being nonlinear: $x_2 = \overline{D}_s^{-1}(u_{qq} + \frac{1}{2}(r_q)^2)$. This time the explicit operator that acts on v to create x_2 is nonlinear. Nonetheless, we create it, replace v by $v + \epsilon\Psi$, and then find the first-term in ϵ in this process; i.e., we linearize it by finding the first functional derivative:

$$x_2 = \overline{D}_s^{-1}\Big\{\overline{D}_s^{-2}\overline{D}_q^2 v + \frac{1}{2}\Big[\overline{D}_s^{-1}\overline{D}_q v\Big]^2\Big\}$$

$$\equiv x_2(v) , \qquad (47)$$

$$\implies x_2(v + \epsilon\psi) - x_2(v)$$
$$= \epsilon\left\{ \overline{D}_s^{-3}\overline{D}_q^2\psi + \overline{D}_s^{-1}\left[r_q\overline{D}_s^{-1}\overline{D}_q\psi \right] \right\}$$
$$+ O(\epsilon^2)$$
$$\equiv \epsilon X_2(\psi) + O(\epsilon^2) . \qquad (48)$$

We also recall that this potential only needs us to introduce the set of all of its q-derivatives, as additional jet variables, all other "derivatives" already being functions of coordinates on the bundle. However, as compensation for this, at this level, we also need to introduce the potential $y_2 = \overline{D}_s^{-1}(u_{\overline{q}q} + \frac{1}{2}(r_{\overline{q}})^2)$, and all of its $\overline{q}$-derivatives.

At the next level, we proceed similarly:

$$x_3 = \overline{D}_s^{-2}\left\{ \overline{D}_s^{-2}\overline{D}_q^3 v + 2(\overline{D}_s^{-1}\overline{D}_q v) \right.$$
$$(\overline{D}_s^{-1}\overline{D}_q^2 v) + (\overline{D}_q v)[\overline{D}_s^{-2}\overline{D}_q^2 v$$
$$\left. + (\overline{D}_s^{-1}\overline{D}_q v)^2] \right\} \equiv x_3(v) , \qquad (49)$$

$$\implies x_3(v + \epsilon\psi) - x_3(v)$$
$$= \epsilon\overline{D}_s^{-2}\left\{ \overline{D}_s^{-2}\overline{D}_q^3\psi + 2r_q\overline{D}_s^{-2}\overline{D}_q^2\psi \right.$$
$$+ 2r_{qq}\overline{D}_s^{-2}\overline{D}_q\psi$$
$$+ v_q\left[\overline{D}_s^{-2}\overline{D}_q^2\psi + 2r_q\overline{D}_s^{-2}\overline{D}_q\psi \right.$$
$$\left.\left. + (u_{qq} + (r_q)^2)\overline{D}_q\psi \right] \right\} + O(\epsilon^2)$$
$$\equiv \epsilon X_3(\psi) + O(\epsilon^2) . \qquad (50)$$

For the analogous operations based on the y_j variables, and relevant to the $\overline{Q}_j$ symmetry generators, using $\overline{q}$-derivatives, we will use the notation Y_j, analogous to the X_j operators above. Then we may explicitly write down the form of the further prolongations of the linearization operator:

$$_3\widehat{\mathfrak{Z}}_\psi = {}_2\widehat{\mathfrak{Z}}_\psi + X_2(\psi)\partial_{x_2} + \{\overline{D}_q X_2(\psi)\}\partial_{x_{2,q}}$$
$$+ \{\overline{D}_q^2 X_2(\psi)\}\partial_{x_{2,qq}} + \dots$$
$$+ Y_2(\psi)\partial_{y_2} + \{\overline{D}_{\overline{q}} Y_2(\psi)\}\partial_{y_{2,\overline{q}}}$$
$$+ \{\overline{D}_{\overline{q}}^2 Y_2(\psi)\}\partial_{y_{2,\overline{qq}}} + \dots , \qquad (51)$$

$$_4\widehat{\mathfrak{Z}}_\psi = {}_3\widehat{\mathfrak{Z}}_\psi + X_3(\psi)\partial_{x_3} + \{\overline{D}_q X_3(\psi)\}\partial_{x_{3,q}}$$
$$+ \{\overline{D}_q^2 X_3(\psi)\}\partial_{x_{3,qq}} + \dots$$
$$+ Y_3(\psi)\partial_{y_3} + \{\overline{D}_{\overline{q}} Y_3(\psi)\}\partial_{y_{3,\overline{q}}}$$

$$+ \{\overline{D}_{\overline{q}}^2 Y_3(\psi)\}\partial_{y_{3,\overline{qq}}} + \dots . \qquad (52)$$

Having now prolonged the linearization operator to include these new, nonlinear potentials as well, the calculation of a commutator of two symmetries requires that both of those symmetries admit these various integrals themselves: a first, and a second, integral, with respect to s, and then accept rather more complicated things in the cases for x_2 and x_3. This is surely not the case for every function on the jet bundle; however, we are really only interested in applying them to the various symmetry generators, such as Q_j and $\overline{Q}_j$. This of course now, **finally**, tells us why we should have not have been surprised to learn that each of the symmetry generators could be conceived of as a perfect s-derivative, and even a perfect second s-derivative. Since this is true, let us use the symbol η_j for the first s-integrals of the symmetry generators, x_j, and we can write out some preliminary general cases of the action of the higher-level operators on the symmetry generators, and also some examples:

$$Q_j = \overline{D}_s(\eta_j) \equiv \overline{D}_s^2(x_j) , \quad (53)$$

$$X_2(Q_j) = \overline{D}_s^{-1}\{x_{j,qq} + r_q\eta_{j,q}\} , \quad (54)$$

$$X_3(Q_j) = \overline{D}_s^{-2}\left\{ x_{j,qqq} + 2(r_q\eta_{j,q})_q \right.$$
$$+ v_q(x_{j,qq} + 2r_q\eta_{j,q})$$
$$\left. + (u_{qq} + (r_q)^2)Q_{j,x} \right\} ; \quad (55)$$

$$\dots ,$$
$$X_2(Q_1) = x_{2,q} ,$$
$$X_2(Q_2) = x_{3,q} - \tfrac{1}{2}(u_{qq})^2 ,$$
$$X_2(Q_3) = x_{4,q} - u_{qq}x_{2,q} , \quad \dots , \quad (56)$$
$$X_3(Q_1) = x_{3,q} , \quad \dots . \quad (57)$$

This structure is important in understanding that everything displayed is actually what was wanted, namely generalized symmetries. In addition, we have succeeded in understanding how to "drag along" the prolongations of various important operators when we prolong the standard infinite jet bundle. Nonetheless, there are two rather

distressing difficulties with it. All the displayed symmetries are part of a commuting hierarchy; i.e., they commute as vector fields, so that all the commutators are in fact zero. This structure then asserts that it has been created correctly, but it does not help you in finding more details, since it contains no *recursion operators*. One must retreat to the generating equations again, presumably, to determine algorithms that concisely show what the form of the n-th symmetry generator is. Or, there may be some other approach to finding details of recursion operators.

A second, current difficulty is that the intended purpose of calculating the symmetry generators is to use them as a tool, or guide, to methods to reduce the difficulty of determining solutions of the original pde. For example, taking the first pair of symmetry generators, v_q and $v_{\overline{q}}$, we may use the vanishing of one of them alone, or a linear relation between them to lower the dimension of the searched-for solution space. [Any of those requirements are equivalent, under the conformal symmetries of the equation.] One then hopes that similar use of the generalized symmetries would also help solve the original pde. So far, we have been unable to find anything new there. It is true that one can locate, again, the solution of the form of a quadratic polynomial in s for e^v, but this is not particularly exciting. Lastly, in the study of the KP hierarchy there are methods to determine solutions that originate in nice behavior of the τ-function. When looked at fairly carefully none of those methods for the Toda lattice equations have reasonable limits for this problem. As well, Takasaki's τ-function appears to be simply our potential function, u, which satisfies an equation equivalent to the original pde, which also does not help.

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