KILLING-VECTOR REDUCTIONS FOR COMPLEX-VALUED, TWISTING, TYPE-N VACUUM SOLUTIONS

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1 Simpler Description of the Local Metric

The goal of understanding general classes of solutions of Petrov Type N, with non-zero twist, is one that is still not realized. The use of $\mathfrak{h}\mathfrak{h}$-spaces to forge a different path toward this goal was developed to a reasonable form in 1992.[1] This work pushes that path a step further, in a direction that has interested the more standard analysis for some time: to look for solutions that admit one Killing and also one homothetic vector, to simplify the task.[2]

A general $\mathfrak{h}\mathfrak{h}$-space is a complex-valued solution of the Einstein vacuum field equations that admits (at least) one congruence of null strings, i.e., a foliation by completely null, totally geodesic two-dimensional surfaces.[3] Those solutions with algebraically-degenerate, real Petrov type have two distinct such congruences. To describe them we use coordinates $\{p, v, y, u\}$, where p is an affine, null coordinate along one null string, v specifies local wave surfaces, and y and u are transverse coordinates, in those surfaces. The metric is determined by x and λ, functions of $\{v, y, u\}$, which must satisfy three quasilinear pde's, involving two gauge functions,

$$\Delta = \Delta(x,y) \text{ such that } \Delta_1 \neq 0 = \Delta_3 , \quad \gamma = \gamma(v,u) \text{ such that } \gamma_1 \neq 0 = \gamma_2 . \quad (1)$$

The equations may be most easily presented by first introducing a non-holonomic basis for the derivatives in these three variables:

$$\partial_1 \equiv \partial_v , \quad \partial_2 \equiv \partial_y , \quad \partial_3 \equiv \partial_u + a\partial_v, \text{ with } a \equiv -x_u/x_v , \quad (2)$$

where ∂_3 is actually the derivative with respect to u holding the function $x = x(v, y, u)$ constant, instead of v, which is then construed as $v = v(x, y, u)$. (Sometimes $F = F(x, y, u) \equiv v_x \, \lambda[v(x, y, u), y, u]$ is also useful.) The twist of the solution is then proportional to a_2. The constraining pde's then have the following form:

$$\begin{aligned} \lambda_{22} = \Delta\lambda , \qquad \lambda_{33} + 2a_1\lambda_3 + a_{31}\lambda = \gamma\lambda , \\ a_2(\lambda_{23} + \lambda_{32}) + a_{22}\lambda_3 + a_{32}\lambda_2 + \tfrac{1}{2}a_{322}\lambda = 0 . \end{aligned} \quad (3)$$

The only known non-trivial solution is that due to Hauser[4]

$$\begin{aligned} a = y + u , \quad \Delta = 3/(8x) , \quad \gamma = 3/(8v) , \quad x + v = \tfrac{1}{2}(y+u)^2 , \\ \lambda = (y+u)^{3/2} f(t) , \quad it + 1 \equiv 4v/(y+u)^2 , \quad f \text{ a hypergeometric function.} \end{aligned} \quad (4)$$

A null tetrad can be given in terms of these quantities, and the associated non-zero components of the curvature:

$$
\begin{aligned}
&\mathbf{g} = \underset{\sim}{\omega}^1 \otimes \underset{\sim}{\omega}^2 + \underset{\sim}{\omega}^2 \otimes \underset{\sim}{\omega}^1 + \underset{\sim}{\omega}^3 \otimes \underset{\sim}{\omega}^4 + \underset{\sim}{\omega}^4 \otimes \underset{\sim}{\omega}^3 , \quad \text{with } \underset{\sim}{\omega}^1 \equiv p\, du , \\
&\underset{\sim}{\omega}^2 \equiv Z\, dy + a_1 \underset{\sim}{\omega}^3 , \ \underset{\sim}{\omega}^3 \equiv dv - a\, du , \ \underset{\sim}{\omega}^4 \equiv dp + E\, du - Q\, \underset{\sim}{\omega}^3 , \\
&\text{where } E \equiv \lambda(\lambda\, a_{32} + 2\lambda_3\, a_2) , \ Z \equiv p/\lambda^2 + a_2 , \ Q \equiv p/\lambda^2 + \lambda^2(\lambda_2/\lambda)_3 , \\
&\text{and } \ 2R_{1313} = 2\gamma_1/p = C^{(1)} , \quad 2R_{2323} = 2(\lambda^2/Z)\Delta_1 = \overline{C}^{(1)} .
\end{aligned}
\tag{5}
$$

These constraining pde's are unchanged under any one of the following coordinate transformations.[1] In each of them the function denoted by a capital letter is arbitrary but invertible:

Transf. I: $\{v, y, u\} \rightarrow \{\overline{v}, y, u\}$, with $\overline{v} = \overline{V}(v, u)$, and F, x, γ, Δ as scalars;

Transf. II: $\{x, y, u\} \rightarrow \{\overline{x}, y, u\}$, with $\overline{x} = \overline{X}(x, y)$, and λ, v, γ, Δ as scalars;

Transf. III: $\{v, y, u\} \rightarrow \{v, \overline{y}, u\}$, with $y = Y(\overline{y})$, and x, γ as scalars, while λ scales as $\lambda = \sqrt{Y_{\overline{y}}}\,\overline{\lambda}$, and Δ has an additional term: $\Delta = \overline{\Delta} + \{1/\sqrt{Y_{\overline{y}}}\}_{,yy}$.

Transf. IV: $\{v, y, u\} \rightarrow \{v, y, \overline{u}\}$, with $u = U(\overline{u})$, and x, Δ as scalars, while λ scales as $\lambda = \sqrt{U_{\overline{u}}}\,\overline{\lambda}$, and γ has an additional term: $\gamma = \overline{\gamma} + \{1/\sqrt{U_{\overline{u}}}\}_{,uu}$.

We refer to $\gamma = \gamma(u, v)$ and $\Delta = \Delta(x, y)$ as gauge functions since transformations I and II would allow them to be replaced by v and x, respectively. However, we will save that freedom for now.

2 Killing's Equations

We reduce the generality of the pde's by insisting that the metric allow some symmetries. An arbitrary homothetic vector, $\widetilde{V}$, constrains the metric and curvature:

$$
\mathcal{L}_{\tilde{V}}\, g_{\alpha\beta} \equiv V_{(\alpha;\beta)} = 2\chi_0\, g_{\alpha\beta} , \qquad \mathcal{L}_{\tilde{V}}\, \underset{\sim}{\Gamma}^{\alpha}{}_{\beta} = 0 = \mathcal{L}_{\tilde{V}}\, \underset{\sim}{\Omega}^{\alpha}{}_{\beta} . \tag{6}
$$

When put together with the pde's for the metric functions, Eqs.(3), these require any prospective homothetic vector to be determined by two functions, $K = K(u)$ and $B = B(v, u)$, as follows:

$$
\widetilde{V} = +(2\chi_0 - B_{,v})p\partial_p + \frac{(\partial_u + a\partial_v - a_{,v})(B - aK)}{a_{,y}}\, \partial_y + K\partial_u + B\partial_v , \tag{7}
$$

along with various constraints on λ, a, Δ, and γ, relative to K and B. We may however use our coordinate freedom(s) to simplify those equations.

$$
\begin{aligned}
&\text{Under Transformation I,} \quad \overline{v} = \overline{V}(v, u) \implies \overline{K} = K , \quad \overline{B} = K\overline{V}_{,u} + B\overline{V}_{,v} ; \\
&\text{under Transformation IV,} \quad \overline{u} = \overline{U}(u) \implies \overline{K} = \overline{U}_{,u} K , \quad \overline{B} = B .
\end{aligned}
\tag{8}
$$

Therefore we may always choose coordinates so that $B = 0$ and K is a constant, say $+1$, and then ask for the constraints on $\{\lambda, a, \Delta, \gamma\}$. We now do this for one true Killing vector, which takes the form $\partial_u - \partial_y$, and gives us the (known) result[1] that a and λ must depend only on v and $s \equiv y + u$, while $\gamma = \gamma(v)$ and $\Delta = \Delta(x)$. (This is the usual transverse Killing vector allowed in this problem.)

A second (homothetic) symmetry vector, $\widetilde{H}$, will have the generic form given in Eq.(7), with its own functions B and K, constrained by the fact that the commutator of two homothetic vectors is a Killing vector,[5] which requires that

$$\partial_u B = 0 = \partial_u^2 K \ . \tag{9}$$

By using the translation and scaling freedom for v and u still remaining in transformations IV and I, we acquire the following form for our homothetic vector:

$$\widetilde{H} = (2\chi_0 - \mu_0) p\partial_p + s\partial_s + \mu_0 v \partial_v \ , \qquad s \equiv y + u \ , \tag{10}$$

and "scaling" equations for each of our dependent variables:

$$\begin{aligned} \widetilde{H}(a) = (\mu_0 - 1)\, a \ , &\qquad \widetilde{H}(\lambda) = (\chi_0 + 1 - \mu_0)\, \lambda \ ; \\ \widetilde{H}(\gamma) = -2\,\gamma \ , &\qquad \widetilde{H}(\Delta) = -2\,\Delta \ . \end{aligned} \tag{11}$$

These allow all the original constraint equations, Eqs.(3), to be rewritten in terms of functions of a single variable, $t \equiv v/s^{\mu_0}$. Calculations for an optimal presentation for those equations are not yet fully completed, and will be presented elsewhere.

References

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