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Difficulties with the SDiff(2) Toda Equation

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Abstract. The SDiff(2) Toda equation appears in several fields of physics. Its algebra of symmetries is the area-preserving diffeomorphisms of a 2-surface, isomorphic to the algebra of Poisson brackets generated by Hamiltonian vector fields in 2 variables. These facts have allowed various researchers to create "zero-curvature representations" and even to create a formal presentation of a general solution; however, none of this seems particularly useful in the search for new, explicit solutions.

A new attempt to find a more direct method to "buy" new solutions from old ones begins with the Bäcklund transformation for the (2-dimensional) Toda lattice built over the Cartan matrix for the Lie algebra $\mathbf{A}_n$ or $\mathbf{A}_n^{(1)}$, and considers the limit as $n \to \infty$. While this limit gives the desired (second-order) pde's, the associated limit of the (first-order) pde's that define the Bäcklund transformation gives only trivial results. Reasons for this are analyzed, and deplored. As well the known "Lax pairs" are analyzed in this language, suggesting that they are not useful for this purpose since the representations of SDiff(2) involved are over the defining manifold rather than over (infinite-dimensional) tangent manifolds in additional (pseudopotential) variables.

1. The Equation Itself, and the Goals

An equation of interest in several fields of theoretical physics is the SDiff(2) Toda equation, for one unknown function of three variables, with differentiation shown by a subscript:

$$u_{,z\tilde{z}} = e^{u_{,tt}} \qquad \Longleftrightarrow \qquad v_{,z\tilde{z}} = (e^v)_{,tt} \ , \qquad v \equiv u_{,tt} \quad . \tag{1.1}$$

One derivation was given by myself and Charles Boyer[**BF**] in 1982. It begins with the Plebański equation for general self-dual vacuum solutions of the Einstein field equations. All such solutions that admit at least one rotational Killing vector may be determined by solutions of this equation. The angular variable associated with that Killing vector combines with the other three above to constitute a local coordinate chart for the 4-dimensional space in question. The name I have chosen has also been used by Mikhail Saveliev[**KS**] and by Kanehisa Takasaki and T.

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Takebe.[**TT**] It emphasizes the resemblance to the usual Toda lattice equations, which involve a family of functions, u^a, of two variables:

$$u^a_{,xy} = e^{K^a{}_b u^b} \qquad \Longleftrightarrow \qquad v^a_{,xy} = K^a{}_b\, e^{v^b} \ , \qquad v^a \equiv K^a{}_b\, u^b \quad , \tag{1.2}$$

where $K^a{}_b$ is the Cartan matrix for some appropriate Lie algebra, which is then a symmetry algebra for the equations.

The second purpose of this name is then to remind us that, in some appropriate limit, the symmetry algebra for our equation is **SDiff**(2), the algebra for the group of area-preserving diffeomorphisms of a 2-surface. In order that such a limit exist, reasonable choices for the original symmetry algebra would be $\mathbf{A}_n$, or $\mathbf{A}_\infty$, or perhaps $\mathbf{A}_n^{(1)}$; here we will simply refer to the algebra as $\mathcal{G}$, deferring an actual choice. When one begins to write the system as a *first-order system*, to effect the prolongation, there is still considerable gauge freedom remaining. We therefore use that freedom, in the manner given by Leznov and Saveliev,[**LS**] to arbitrarily divide the (original) unknown functions into two parts:

$$\begin{gathered} u^a \equiv m^a + n^a \ , \qquad Z^a = m^a_{,x} \ , \quad W^a = -n^a_{,y} \ , \\ a^a \equiv K^a{}_b\, m^b \ , \quad b^a \equiv K^a{}_b\, n^b \implies v^b = a^a + b^a \ . \end{gathered} \tag{1.3}$$

Since many people have already struggled with this equation, toward various goals, it seems important to say what I want from it, and/or why I am not sufficiently happy with the results that already exist? I want a "solution generator," which will "buy" new solutions from old ones. I would say that I want a Bäcklund transform,[**PR**] although the more general description above would satisfy my desires if that were all that was available.

2. Prolongation Structures for the (2-dimensional) Toda Lattice

Our (geometric) approach to finding new solutions, via non-local symmetries, begins[**FM**] by thinking of a k-th order partial differential equation as a variety, Y, of a finite jet bundle, $J^{(k)}(M,N)$, with M the independent- and N the dependent-variables, which is then prolonged to a fiber space (over $J^{(\infty)}$) with fibers W, where vertical flows map solution spaces of one pde into another pde, which has as its dependent variables the quantities w^A, that coordinatize the fibers. The compatibility conditions for such flows to exist are *"zero-curvature conditions"* for the fiber space, required to vanish when restricted to the infinite prolongation, $Y^\infty \times W$, of the fiber space over the original pde. We may express these conditions by first lifting $\{\partial_x, \partial_y\}$, a basis for the tangent vectors over M, to the total derivative operators, $\{D_x, D_y\}$ over the infinite jet, which still commute, and then prolonging them further into the fibers W.

This act requires the addition of some vertical vector fields, $\mathbf{F} = \sum F^A(\partial/\partial w^A)$ and $\mathbf{G} = \sum G^A(\partial/\partial w^A)$. As the total derivatives are of the form $D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \dots$, the additional terms denoted by, for instance, $\mathbf{F}$, should have the same interpretation with respect to the fiber variables, i.e., we eventually want to think of the w^A also as functions of the independent variables that will satisfy some pde's. This means that the satisfaction of the prolongation requirements should allow us to interpret F^A as w^A_x and G^A as w^A_y, where, for instance F^A originally are functions of all the jet variables and the fiber variables. Such an interpretation of course requires that the prolonged total derivatives should commute, at least on the restricted manifold:

$$0 \;=\; [D_x + \mathbf{F}\,,\,D_y + \mathbf{G}\,]\Big|_{Y^\infty\times W} = \left\{\overline{D}_x(G^C) - \overline{D}_y(F^C)\right\}\frac{\partial}{\partial w^C} \;+\; [\mathbf{F}\,,\,\mathbf{G}\,]\;, \tag{2.1}$$

where the overbar indicates the restriction in question. As the construction gives **F** and **G** the "form" of the components of a connection, these equations are referred to as "zero-curvature" requirements for non-local symmetries for the original pde. This is a slight generalization of the usual approach since they are still only elements of an **abstract** Lie algebra of vector fields, with neither coordinates, nor even the number of those coordinates yet determined. Nonetheless, at least in two variables, the general solution for **F** and **G** describes all possible Bäcklund transformations for this pde.

Solutions to the prolongation equations, for **F** and **G**, have two requisites, namely the determination of the (smallest) Lie algebra which satisfies the requirements for the commutators of these connections, and an acceptable decision for the number of pseudopotentials to be involved in a realization of this algebra. The constraints in question are induced by the requirement that the restricted zero-curvature equation vanishes. For now, we follow the conventional choice for a Lie algebra that satisfies the constraints, namely the same Lie algebra that defines the system of pde's itself. If one then also follows the thoughts of Leznov and Saveliev, who insist that **F** and **G** should be a linear combination of elements chosen from the Cartan subalgebra, $\mathcal{G}_0$, and its simple roots, $\mathcal{G}_{\pm 1}$, spanned by $\{\mathbf{h}_a\}_1^n$, $\{\mathbf{e}_b\}_1^n$ and $\{\mathbf{f}_c\}_1^n$, respectively, the unique solution is the following:

$$\begin{aligned}
\mathbf{F} &= Z^a\,\mathbf{h}_a + e^{n^b(\mathrm{ad}\,\mathbf{h}_b)}\sum_c \mathbf{e}_c \;\;=\;\; Z^a\,\mathbf{h}_a + e^{b^c}\mathbf{e}_c\;,\\
\mathbf{G} &= W^a\,\mathbf{h}_a + e^{-m^b(\mathrm{ad}\,\mathbf{h}_b)}\sum_c \mathbf{f}_c \;\;=\;\; W^a\,\mathbf{h}_a + e^{a^c}\mathbf{f}_c\;.
\end{aligned} \tag{2.2}$$

One then requires a choice for the realization. A very reasonable approach to the number of pseudopotentials takes it as the same number as the original dependent variables v^a, thereby making it reasonably simple to hope to find an auto-Bäcklund transformation. A choice of presentation of this standard solution may then be written:

$$\begin{aligned}
\mathbf{F} &= \left\{K^a{}_b Z^b - e^{w^a+b^a} + e^{w^{a-1}+b^{a-1}}\right\}\partial_{w^a}\;,\\
\mathbf{G} &= \left\{K^a{}_b W^b + e^{-w^a+a^a} - e^{-w^{a+1}+a^{a+1}}\right\}\partial_{w^a}\;.
\end{aligned} \tag{2.3}$$

The first-order pde's for the explicit Bäcklund transformation are then simply

$$\{w^a - a^a\}_{,x} = -e^{w^a+b^a} + e^{w^{a-1}+b^{a-1}}\;,\;\{w^a + b^a\}_{,y} = e^{-w^a+a^a} - e^{-w^{a+1}+a^{a+1}}\;. \tag{2.4}$$

The zero-curvature conditions, where we subtract the extended derivatives of **F** and **G**, are then exactly the original Toda equations, Eqs. (1.2), in the variables v^a, as desired. Moreover, if instead one adds the two derivatives, and inserts the form for $b^a_{,xy}$ from the Toda equations, the result is a system of pde's that must be satisfied by the pseudopotential variables:

$$\begin{aligned}
w^a_{,xy} &= a^a_{,xy}\; - e^{v^a} + e^{v^{a-1}} + e^{\ell^a} - e^{\ell^{a-1}}\;,\\
&\qquad \text{where } \ell^a_{,xy} \equiv w^a_{,xy} - w^{a+1}_{,xy} + a^{a+1}_{,xy} + b^a_{,xy}\;.
\end{aligned} \tag{2.5}$$

The Bäcklund transformation so created does indeed determine a mapping between the pde's satisfied by the v^a's and the pde's satisfied by the w^a's. However, it is surely **not** in an optimal form since the new pde's contain not only the new variables but also the old ones. This difficulty can be resolved by using, instead, the new variables $\{\ell^a\}_1^{n-1}$ that occur here in a very natural way, although there are in fact only $n-1$ of them. (This is not a problem in the limit as n goes off to infinity.) We find the very simple result that these new variables must also satisfy the Toda equations, but for $\mathbf{sl}(n)$ instead of $\mathbf{sl}(n+1)$:

$$\ell^a_{,xy} = 2e^{\ell^a} - e^{\ell^{a-1}} - e^{\ell^{a+1}} = \big(\underset{n-1}{K}\big)^a{}_b\, e^{\ell^b} \quad , \quad a, b = 1, \ \dots \ , n-1 \ . \tag{2.6}$$

3. Continuous Limit of the Toda Lattice Equations

We now consider limits on the Bäcklund transformation equations, as we change from discrete indices to functions of a continuous variable. We use the system of functions v^a to define a new function that also depends on a continuous variable, t, which varies from, say, 0 to β. We superpose on this domain a lattice of n points, a distance δ apart, and then fill in the space between the lattice points by taking the limit as $n \to \infty$, with β fixed, which is the same as taking the limit as $\delta \to 0$:

$$\begin{gathered} V(z,\tilde{z},t)\Big|_{t=a\delta} \equiv v^a(z/\delta, \tilde{z}/\delta) \ , \quad a = 1, \dots , n \ , \quad \delta = \beta/(n-1) \, , \\ \Longrightarrow V(z,\tilde{z},t) \equiv \lim_{\delta\to 0} v^{[t/\delta]}(z/\delta, \tilde{z}/\delta) \, , \end{gathered} \tag{3.1}$$

where the square brackets indicate the integer part of the quotient within them. Following Park's example,[**P**] we have re-scaled the original two independent variables, to have appropriate differences of the exponentials of the v^a's to create second derivatives, with respect to t. However, the u^b's need a scaling of their own, to create their second derivatives:

$$U(z,\tilde{z},t)\Big|_{t=a\delta} \equiv \delta^2 \left\{u^a(z/\delta, \tilde{z}/\delta)\right\} . \tag{3.2}$$

We suppose the new functions are sufficiently continuous to expand in a Taylor series about the point $t = a\delta$. Suppressing the dependence on $z \equiv x\delta$ and $\tilde{z} \equiv y\delta$, we may write

$$v^{a\pm 1} \ \longrightarrow \ V(t\pm\delta) = V(t) \pm \delta\{V_{,t}(t)\} + \tfrac{1}{2}\delta^2\{V_{,tt}(t)\} + O(\delta^3) \ , \tag{3.3}$$

which give us the desired differences, including $K^a{}_b\, v^b \equiv -v^{a+1} + 2v^a - v^{a-1}$:

$$\begin{aligned} v^{a+1} - v^a \ &\longrightarrow \ V(t+\delta) - V(t) = \delta\{V_{,t}(t)\} + \tfrac{1}{2}\delta^2\{V_{,tt}(t)\} + O(\delta^3) \ , \\ v^a - v^{a-1} \ &\longrightarrow \ V(t) - V(t-\delta) = \delta\{V_{,t}(t)\} - \tfrac{1}{2}\delta^2\{V_{,tt}(t)\} + O(\delta^3) \ , \\ -K^a{}_b\, v^b \ &\longrightarrow \ V(t+\delta) - 2V(t) + V(t-\delta) = \delta^2\{V_{,tt}(t)\} + O(\delta^3) \ . \end{aligned} \tag{3.4}$$

We may now take limits of the Toda equations, which give the desired, and expected, results:

$$U_{,z\tilde{z}} = e^{-U_{,tt}} \ , \quad V_{,z\tilde{z}} = -\partial_t^2 e^V \ . \tag{3.5}$$

However, we also want to take the same limits on the prolongation equations themselves. Rewriting the prolongation equations in terms of the w^a's in Eqs. (2.4), we have the equations and their integrability conditions:

Looking at these equations, we *agree* to treat the gauged parts, a^a and b^a, the same as their sum v^a, and also the pseudopotentials w^a, which gives us the following limiting forms:

$$\begin{gathered}
W(z,\tilde{z},t)\Big|_{t=a\delta} \equiv w^a(z/\delta,\tilde{z}/\delta)\ , \\
A(z,\tilde{z},t)\Big|_{t=a\delta} \equiv a^a(z/\delta,\tilde{z}/\delta)\ , \quad B(z,\tilde{z},t)\Big|_{t=a\delta} \equiv b^a(z/\delta,\tilde{z}/\delta)\ ; \\
\Longrightarrow \quad (W-A)_{,z} = -\partial_t e^{W+B}\ , \qquad (W+B)_{,\tilde{z}} = -\partial_t e^{-(W-A)}\ ,
\end{gathered} \tag{3.6}$$

where this last line is then the limit of the original Bäcklund transformation equations. Their integrability conditions are then

$$\begin{aligned}
(W-A)_{,z\tilde{z}} =&\ -\partial_t\{e^{W+B}(-\partial_t e^{-(W-A)})\} = -\partial_t\{e^V\partial_t(W-A)\}\,, \\
(W+B)_{,\tilde{z}z} =&\ -\partial_t\{e^{-(W-A)}\partial_t e^{(W+B)}\} \\
=&\ -\partial_t\{e^{-(W-A)}e^{(W+B)}\partial_t(W+B)\} = -\partial_t\{e^V\partial_t(W+B)\}\ ,
\end{aligned} \tag{3.7}$$

where the sum of A and B that appears creates the V shown. Adding and subtracting these two equations we finally acquire

$$\begin{aligned}
V_{,z\tilde{z}} = -\partial_t e^V\partial_t V =&\ -\partial_t^2 e^V\ , \\
\text{and } (2W+B-A)_{,z\tilde{z}} =&\ -\partial_t\Big\{e^V\partial_t(2W+B-A)\Big\}\ .
\end{aligned} \tag{3.8}$$

The first of these equations is of course what we expect; however, the second is surely not what we expect/want. In particular, since $B-A$ is independent of $B+A=V$, then the second equation is simply *linear* in the unknown function, $2W+B-A$. Of course in the symmetric gauge choice, where $a^a=\frac{1}{2}v^a=b^a$, this would just be $2W$. [It is amusing that this pair is just the system LeBrun finds to determine his "weak heavens," but then we would have to treat this as a system, and again try to determine its Bäcklund transformation.]

In the discrete case, instead, a re-formulation of this equation turned out to be the Toda lattice equations again, for the new dependent variables, ℓ^a. In the current case, however, the limit of that particular combination of pseudopotentials and original dependent variables that satisfies the same pde as the original variables, namely ℓ^a, is the same as that of v^a:

$$L(t) = W(t) + B(t) - \{W(t) - A(t)\} = V(t)\ . \tag{3.9}$$

The two satisfy the same equations, **but**, unfortunately they are not different since the limit $W(t)$ of the pseudopotential variables simply cancels out of the combination. A way of looking at "why" this happens is to note that for the finite lattice case the difference was $w^a - w^{a+1}$. What can one do about this, if anything?

4. Some other Work Toward Finding a Solution Generator

Various other groups have considered this equation, and even provided interesting results. Foremost in my mind is that provided by Mikhail Saveliev and A.M. Vershik.[**SV**] They used this equation as a beginning place to guide them toward a general theory of continuum Lie algebras. In fact they published several papers explaining the form of the "general solution" to the equation as determined by choices of initial conditions. Unfortunately, from my viewpoint, this form is very complicated, and not at all workable for obtaining manageable and interesting, specific, new solutions.

In a different direction, in Britain, R. S. Ward,[**W**] and separately, in Japan, K. Takasaki and various coworkers[**TT**] have created objects they refer to as Lax pairs for this equation, although in the process they have changed the usual matrix/vector field commutator in the Lax equation into a Poisson bracket for functions. As well, their approach has generated an infinite hierarchy of associated equations—in the spirit of the KP hierarchy of equations; nonetheless their methods also do not appear to me to allow for the determination of interesting, specific, new solutions. I have some specific thoughts on their approach that I will describe below.

There are also some groups that have, instead, been looking directly at the properties of the full Plebański equation, before the reduction—via one rotational Killing vector—to our equation was made. Obviously they could also be useful for the goals I have in mind. Early work was by Boyer and Plebański,[**BP**] looking for nonlinear superposition principles underlying the solution space for the Plebański equation. Their viewpoint on the equation amounts to lifting curves of solutions into an infinite jet bundle. It may be that they are at least very close to what I want. They give an algorithm that could perhaps be followed to generate new solutions from old. For a case that begins with simple pp-waves, they actually do it. Can it be generalized? I don't know, but we are working on that.

Then Boyer and Pavel Winternitz[**BW**] found the complete set of 1-, 2-, and 3-dimensional subgroups of the group of symmetries of the Plebański equation. The reduction to the SDiff(2) Toda equation is clear and other lower-dimensional reductions exist; however, this approach has not yet been used to acquire previously-unknown solutions.

More recently various twistorial researchers have considered these problems from both sides. Firstly, they have created quite a clever approach to describing a Poisson-bracket based Lax pair for Grant's form[**G**] of the Plebański equation, which allows Ian Strachan[**S**] to describe an infinite hierarchy of symmetries of this equation in a very straightforward way. On the other end of the complications they have also been rather involved in finding classes of solutions to the equation when it has been reduced to a system of ode's by insisting on a **sl**(2) group of symmetries.[**T**]

5. (ZS) Zero-Curvature Relations, (EW) Prolongation Structures, and the (T/W) Poisson Bracket

It is worth giving some explanation of why the currently-known "Lax pairs" are interesting, but are not useful toward the goals of finding specific new solutions. Takasaki's version of the "Lax pair" for this equation is written in terms of a pair of functions of 4 variables, with the usual commutator being replaced by a Poisson bracket in two of those variables:

$$g_{,\tilde{z}} - f_{,z} + \{f, g\} = 0\ . \tag{5.1}$$

The Poisson bracket uses the (phase-space) variables t and $p \equiv \ln \lambda$. Three of these variables, $\{z, \tilde{z}, t\}$, come from the original pde, and thus should be expected, although obviously they have not been all treated in the same manner. However, the fourth is an extra variable that has been added, although it is not a pseudopotential, or fiber, variable; Takasaki refers to λ as a "spectral parameter." In fact, it ends up playing two different roles. Since it is a "parameter," instead of an independent variable, they feel justified in specifying in advance the dependence of f and g on λ. This creates a polynomial in λ, so that the vanishing of the commutator engenders three different equations, which must then be combined to re-create the original pde. However, quite a different role is its use in "modifying" the usual vector-field (or matrix) commutator, in the zero-curvature form, to a Poisson bracket in t and p. While this is perhaps unexpected, from a physical point of view, the purpose of our pde is to determine a potential for a spacetime with a metric that always admits one Killing vector. Therefore the metric itself will be independent of the fourth coordinate, and it is reasonable that the potential will have rather simple dependence on that variable.

I find it informative to put their approach back into the EW approach already outlined above, by the use of Hamiltonian vector fields generated by their functions. Recall that an arbitrary Hamiltonian vector field lives on the tangent space to some appropriate phase space, is generated by some (non-constant) function of the coordinates on that space, say $f = f(q,p) \equiv f(q^i)$, and has the following standard properties:

$$\begin{aligned} \mathcal{H}_f &\equiv f_{,q}\partial_p - f_{,p}\partial_q = J^{ij} f_{,i}\partial_{,j} \ , \\ \mathcal{H}_{af+g} &= a\mathcal{H}_f + \mathcal{H}_g \ , \\ \mathcal{H}_f(g) &= \{f,g\} \ , \qquad [\mathcal{H}_f, \mathcal{H}_g] = \mathcal{H}_{\{f,g\}} \ , \end{aligned} \tag{5.2}$$

Here J^{ij} are the components of the symplectic 2-form, $\mathbf{J}$, for that phase space; in our two dimensions it may be taken as the (2-dimensional) Levi-Civita symbol, ϵ^{ij}. We use $\mathcal{H}_f$ for our earlier $\mathbf{F}$ and $\mathcal{H}_g$ for $\mathbf{G}$, and we choose t and p for our phase-space variables. We may then re-formulate their language in our own:

$$\begin{aligned} [D_x + \mathcal{H}_f \, , \, D_y + \mathcal{H}_g] &= \mathcal{H}_{g,x} - \mathcal{H}_{f,y} + [\mathcal{H}_f, \mathcal{H}_g] \\ &= \mathcal{H}_{g,x} - \mathcal{H}_{f,y} + \mathcal{H}_{\{f,g\}} = \mathcal{H}_{g,x - f,y + \{f,g\}} \ . \end{aligned} \tag{5.3}$$

As our algebra has been divided out by the set of all constant functions, we may claim that a Hamiltonian vector field vanishes if and only if its generating function vanishes. Therefore, the requirement of "zero-curvature" is the same as the vanishing of the generating function for the (single) Hamiltonian vector field on the far right hand side of Eq. (5.3):

$$[D_x + \mathcal{H}_f \, , \, D_y + \mathcal{H}_g] = 0 \iff g_{,x} - f_{,y} + \{f,g\} = 0 \ . \tag{5.4}$$

Takasaki's explicit λ-dependence, for his f and g, is the following, where the other quantities, $\{a, \overline{a}, b, \overline{b}\}$, depend on the remaining three variables:

$$f = -\lambda a - b \ , \qquad g = -\lambda\overline{a} - \overline{b} \ . \tag{5.5}$$

The three equations that result do re-create our original SDiff(2) Toda equation.

The very important difference between this and the desired prolongation equations is the lack of fiber variables, to be used to generate another pde. We recall that the zero-curvature equations restrict the prolongations $\mathbf{F}$ and $\mathbf{G}$ to lie in some sub-algebra of all the vector fields over a fiber W. If we now take $\{\mathbf{A}^\mu\}$ as a set of vector fields that constitute a basis for that algebra, we could write out

$$F^A \frac{\partial}{\partial w^A} = \mathbf{F} \equiv \mathcal{F}_\mu \mathbf{A}^\mu \ , \tag{5.6}$$

where the $\mathcal{F}_\mu$ depend on the original jet variables, only. The dependence on the pseudopotentials is in the basis vector-fields $\mathbf{A}^\mu$. (Recall Eqs. (2.2).) In Takasaki's analogues of a "Lax pair," the $\mathbf{A}^\mu$ are vector fields directly over the independent variable t and the spectral parameter. We need, instead to create a related approach, but with a realization of **SDiff**(2) "lifted" to some additional (pseudopotential) variables. We had hoped this limiting procedure might accomplish this; unfortunately our efforts so far, as already presented, have failed.

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