INFINITE-DIMENSIONAL SYMMETRY ALGEBRAS AS A HELP TOWARD SOLUTIONS OF THE SELF-DUAL FIELD EQUATIONS WITH ONE KILLING VECTOR

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The $\boldsymbol{sDiff}(2)$ Toda equation determines all self-dual, vacuum solutions of the Einstein field equations with one rotational Killing vector. Some history of the searches for non-trivial solutions is given, including those that begin with the limit as $n \to \infty$ of the $\boldsymbol{A}_n$ Toda lattice equations. That approach is applied here to the known prolongation structure for the Toda lattice, hoping to use Bäcklund transformations to generate new solutions. Although this attempt has not yet succeeded, new faithful (tangent-vector) realizations of $\boldsymbol{A}_\infty$ are described, and a direct approach via the continuum Lie algebras of Saveliev and Leznov is given.

1 The $\boldsymbol{sDiff}(2)$ Toda equation, and the standard Toda lattice

All self-dual vacuum solutions of the Einstein field equations that admit (at least) one rotational Killing vector are determined by solutions of the $\boldsymbol{sDiff}(2)$ Toda equation, which may be written in various equivalent forms:

$$u_{,z\tilde{z}} = e^{u_{,tt}} \qquad \Longleftrightarrow \qquad v_{,z\tilde{z}} = (e^v)_{,tt}\ , \qquad v \equiv u_{,tt}\ , \tag{1}$$

as was shown by Charles Boyer and myself in 1982.[1] The angular variable associated with that Killing vector combines with the other three above to constitute a local coordinate chart for the 4-dimensional space, with a Euclidean signature. The equation has a close resemblance to the usual (2-dimensional) Toda lattice equations,

$$u^a_{,xy} = e^{K^a{}_b u^b} \text{ or } v^a_{,xy} = K^a{}_b\, e^{v^b}\ ,\ (v^a \equiv K^a{}_b\, u^b)\ ,\ a, b = 1, 2, \ldots, n\ . \tag{2}$$

where $K^a{}_b$ is the Cartan matrix for the Lie algebra which is also the generator of the symmetries of these same Toda equations . For all finite-dimensional, semi-simple algebras these symmetries allow the determination of Bäcklund transformations which generate new solutions from old ones.

The name I have used for Eq.(1) was first used by Mikhail Saveliev,[2] and also Kanehisa Takasaki and T. Takebe, [3] emphasizing the fact that its symmetry algebra is the set of all area-preserving diffeomorphisms of a 2-surface. This work describes attempts to find limits which not only carry the variables in the Toda lattice case into those for our equation but which also carry over the existence of Bäcklund transformations to our equation with three independent variables.

This equation has been of interest in general relativity, and in several other fields of physics, for almost twenty years; nonetheless, almost all known solutions describe metrics that also allow a translational Killing vector, which make them susceptible to discovery by a much simpler route, so that no real understanding of our equation results. Our approach to these limits begins with the zero-curvature form for the

Toda lattice equations, and the Estabrook-Wahlquist prolongation structure that is an interpretation of it. We will describe that approach next; however, as many groups have considered this equation, along the way we will also give some of the history of these other struggles.

A k-th order pde may be considered[4] as a variety, Y, of a jet bundle, $J^{(k)}(M,N)$, with M the independent- and N the dependent-variables, which can then be prolonged to $J^{(\infty)}$, where arbitrarily many derivatives are allowed. The search for non-local symmetries (pseudopotentials) is made in an additional prolongation, with fibers W, where vertical flows map solution spaces of one pde into another pde, with the pseudopotentials, w^A, as the new dependent variables. Compatibility conditions for the existence of such flows are the zero-curvature conditions:

$$\begin{aligned} 0 = [D_x + \mathbf{F},\, D_y + \mathbf{G}]\big|_{Y^\infty \times W} &\equiv \left[\overline{D}_x + \mathbf{F}, \overline{D}_y + \mathbf{G}\right] \\ &= \left\{\overline{D}_x(G^C) - \overline{D}_y(F^C)\right\} \tfrac{\partial}{\partial w^C} \; + [\mathbf{F}\,,\,\mathbf{G}\,]\,, \end{aligned} \tag{3}$$

where the $\overline{D}_a$ are the total derivative operators, restricted to $Y \subset J^{(\infty)}$, while the vertical vector fields $\mathbf{F} = \sum\limits_A F^A(x;u;w)\big(\frac{\partial}{\partial w^A}\big) \;\; \rightarrow \;\; w^A_{,x}\big(\frac{\partial}{\partial w^A}\big)$, and the similar $\mathbf{G}$, prolong D_x, and D_y into the fibers, in the spirit of connections. They lie in a subalgebra of the Lie algebra of vector fields, with the coordinates, w^A, and even their number, yet unknown. However, at least in two independent variables, the general solution for $\mathbf{F}$ and $\mathbf{G}$ determines all possible Bäcklund transformations for this pde. Generically, the zero-curvature constraint determines their dependence on the jet variables,with coefficients that are purely vertical vector fields, $\{\mathbf{W}_\alpha\}$, along with **only some** of the commutation relations among the $\mathbf{W}_\alpha\}$, so that the resulting Lie algebras are usually infinite dimensional! Considerable interest in these infinite-dimensional algebras has been expressed by several groups, including Estabrook, Omote and Hoenselaers.[5]

Our plan is to consider Eq.(1) as some limit of the Toda lattice equations, Eqs.(2), built on the algebra sl(n+1,$\mathbb{C}$) $\equiv A_n$ since this form easily creates the second derivatives needed with respect to the third independent variable. Therefore we begin with a description of the zero-curvature equations for those Toda lattice equations, Eqs.(2), where we follow closely the format of Leznov and Saveliev.[6] This says that we should first use the gauge freedom which the first-order form of those equations has, to arbitrarily divide the dependent variables into two parts, and then to agree to span $\mathbf{F}$ and $\mathbf{G}$ in only the Cartan subalgebra of $\mathbf{A}_n$ and its simple roots:

$$\begin{aligned} &u^a \equiv m^a + n^a\,, \qquad v^b = a^a + b^a\,, \;\Longrightarrow\; a^a \equiv K^a{}_b\, m^b\,, \quad b^a \equiv K^a{}_b\, n^b\,, \\ &\quad \mathbf{F} = (m^a_{,x})\,\mathbf{h}_a + e^{n^b(\mathrm{ad}\,\mathbf{h}_\mathrm{b})}\textstyle\sum_c \mathbf{e}_c \;\;=\;\; +(m^a_{,x})\,\mathbf{h}_a + e^{b^c}\mathbf{e}_c\,, \\ &\mathbf{G} = \;-(n^a_{,y})\,\mathbf{h}_a + e^{-m^b(\mathrm{ad}\,\mathbf{h}_\mathrm{b})}\textstyle\sum_c \mathbf{f}_c \;\;=\;\; -(n^a_{,y})\,\mathbf{h}_a + e^{a^c}\mathbf{f}_c\,. \end{aligned} \tag{4}$$

A very reasonable choice for a realization of this algebra is to use the same number of w^A's as v^a's, for A_n, hoping to simplify the search for Bäcklund transformations. This leads to a unique answer, which may, for instance, be presented as follows:

$$\mathbf{F} = \left\{a^a_{,x} - e^{w^a+b^a} + e^{w^{a-1}+b^{a-1}}\right\}\partial_{w^a}\,,\; \mathbf{G} = \left\{-b^a_{,y} + e^{-w^a+a^a} - e^{-w^{a+1}+a^{a+1}}\right\}\partial_{w^a}\,, \tag{5}$$

which is equivalent to the following first-order pde's, which indeed constitute a Bäcklund transformation between the two sets:

$$\{w^a - a^a\}_{,x} = -e^{w^a + b^a} + e^{w^{a-1} + b^{a-1}}, \; \{w^a + b^a\}_{,y} = \; e^{-w^a + a^a} - e^{-w^{a+1} + a^{a+1}} \; . \quad (6)$$

There are of course two sets of integrability conditions for this system. The first are those obtained by subtracting the extended total derivatives, which give back just the original Toda equations, in the variables $v^a \equiv a^a + b^a$, as expected. The second set are obtained by adding those two (total) derivatives. This results in a system in the fiber variables, w^a, which still has the original dependent variables mixed inside. The equations, however, easily indicate a much better choice for (new) fiber variables, $\ell^a \equiv w^a - w^{a+1} + a^{a+1} + b^a$. Both sets of new pde's are given below, where we see that the variables ℓ^a also are required to satisfy the Toda lattice equations, as was to be hoped:

$$\begin{aligned} w^a_{,xy} &= a^a_{,xy} \;\; -e^{v^a} + e^{v^{a-1}} + e^{\ell^a} - e^{\ell^{a-1}} \; , \\ &\text{where} \quad \ell^a \equiv w^a - w^{a+1} + a^{a+1} + b^a \; , \text{ and} \\ \ell^a_{,xy} &= 2e^{\ell^a} \; -e^{\ell^{a-1}} - e^{\ell^{a+1}} = K^a{}_b \, e^{\ell^b} \; . \end{aligned} \quad (7)$$

2 Prior Work on This Equation: Some History

Before discussing the limits of the Toda lattice prolongation above, we would like to describe some of the earlier work done on this problem. In our opinion, the foremost contributions have been provided by Mikhail Saveliev and A.M. Vershik.[7] They used this equation as a guide[2] toward a theory of continuum Lie algebras; for $\boldsymbol{A}_\infty$, the usual commutation relations take the following form, where we give them both with continuous "indices," so that Kronecker deltas are replaced by Dirac deltas, and with functional labels:

$$\begin{gathered} [X_0(s), X_{\pm 1}(t)] = \pm\{\delta_{,tt}(s-t)\} \, X_{\pm 1}(t) \; , \quad [X_{+1}(s), X_{-1}(t)] = \delta(s-t) \, X_0(t), \\ [X_0(f), X_{\pm 1}(g)] = \pm \, X_{\pm 1}((fg)') \; , \quad [X_{+1}(f), X_{-1}(g)] = X_0(fg) \; . \end{gathered} \quad (8)$$

This approach led them to write down a form for a "general solution" for an initial-value problem for our equation; unfortunately, at least as we see it, this form is rather too formal, and not practically useful. From a slightly different direction, R. S. Ward,[8] and separately, K. Takasaki and various coworkers[9] have created objects they refer to as Lax pairs for this equation, but with Poisson brackets instead of the usual commutators. Although those pairs do not appear to involve pseudopotentials, it is true that they generate an infinite hierarchy of associated equations, in the spirit of the KP hierarchy. Perhaps more thoughts on their approach will be described below.

At the same time, other researchers were considering general solutions for the full Plebański equation, before the one Killing vector reduction to Eq.(1). Early work was by C. Boyer and J.F. Plebański,[10] who lifted curves of solutions into an infinite jet bundle based on “twistor constructions.” They created a program to determine nonlinear superposition principles for the Plebański equation, and gave an example where the superposition of a pair of simple pp-waves gives something quite different. Whether this program can be generalized is as yet unknown, since we believe no work has been done on this approach since that time. However, Boyer and Winternitz[11] did consider all the 1-, 2-, and 3-dimensional subgroups of the group of symmetries of the Plebański equation, and located various reductions, including our one Killing vector case; however, again these have not yet been used to look for previously-unknown solutions.

Lastly, various twistorial researchers have considered these problems. Firstly, they have created quite a clever approach to describing a Poisson-bracket based Lax pair for Grant's form[12] of the general Plebański equation. This form allows Ian Strachan[13] to describe an infinite hierarchy of symmetries of this equation in a very straightforward way, that is an extension of the earliest work on symmetries by Boyer and Plebański.[14] Along with Tod[15] and Dancer[16], they have also been involved in finding various classes of solutions for Eq.(1), when it is reduced to ode's using an sl(2) symmetry group, so that there are three rotational Killing vectors. An early beginning to this is the solution due to Michael Atiyah.[17] A rather different approach comes from quantum field theory via work of Bakas.[18]

3 Continuous Limit of the Toda Lattice Equations

We may now describe our own efforts at performing the limit from the discrete indices of the Toda lattice problem in Section 2 to functions of a (new) continuous variable, t, which varies from, say, 0 to β. We superpose on this range a lattice of n points, a distance δ apart, and take the limit as $n \to \infty$, with β fixed:

$$\begin{aligned} V(z,\tilde{z},t)\Big|_{t=a\delta} &\equiv v^a(z/\delta,\tilde{z}/\delta)\,, \Longrightarrow V(z,\tilde{z},t) \equiv \lim_{\delta\to 0} v^{[t/\delta]}(z/\delta,\tilde{z}/\delta)\,, \\ \text{and}\;\; U(z,\tilde{z},t)\Big|_{t=a\delta} &\equiv \;\delta^2\left\{u^a(z/\delta,\tilde{z}/\delta)\right\}\,, \end{aligned} \tag{9}$$

where the square brackets indicate the integer part of the quotient inside, and, following Park,[19] we have re-scaled the original two independent variables and the other set of dependent variables, to create the proper scaling for second derivatives, with respect to t.

Assuming sufficient continuity to expand them in Taylor series, we may write

$$\begin{aligned} v^{a+1} - v^a &\longrightarrow \delta\{V_{,t}(t)\} + \tfrac{1}{2}\delta^2\{V_{,tt}(t)\} + O(\delta^3)\,, \\ v^a - v^{a-1} &\longrightarrow \delta\{V_{,t}(t)\} - \tfrac{1}{2}\delta^2\{V_{,tt}(t)\} + O(\delta^3)\,, \\ -K^a{}_b v^b = v^{a+1} - 2v^a + v^{a-1} &\longrightarrow \delta^2\{V_{,tt}(t)\} + O(\delta^3)\,. \end{aligned} \tag{10}$$

The limits of the Toda lattice equations then give the expected results, namely Eq.(1) To determine the limits of the prolongation equations, for $\mathbf{F}$ and $\mathbf{G}$, we treat the divisions of $v^a = a^a + b^a$, in the same way, and also do this for the pseudopotentials, w^a. This scaling of the pseudopotentials may or not be optimal;

however, as far as we can tell, any other scaling causes singular limits to occur! Then the limits of the prolongation equations, Eqs.(5), are

$$(W-A)_{,z} = -\partial_t e^{W+B}\ , \qquad (W+B)_{,\bar z} = -\partial_t e^{-(W-A)}\ . \tag{11}$$

and their (separate) integrability conditions are

$$\begin{aligned}(W-A)_{,z\bar z} &= -\partial_t\{e^{W+B}(-\partial_t e^{-(W-A)})\} = -\partial_t\{e^V\partial_t(W-A)\}\,,\\ (W+B)_{,\bar z z} &= \ \ -\partial_t\{e^{-(W-A)}\partial_t e^{(W+B)}\} \ = -\partial_t\{e^V\partial_t(W+B)\}\,.\end{aligned} \tag{12}$$

Adding and subtracting these gives the forms we want:

$$V_{,z\bar z} = -\partial_t e^V\partial_t V = -\partial_t^2 e^V\ ,\ (2W+B-A)_{,z\bar z} = -\partial_t\big\{e^V\partial_t(2W+B-A)\big\}\ . \tag{13}$$

The first is of course required; however, the second is surely not what was wanted. Since $B-A$ is independent of $B+A=V$, the second equation is simply *linear* in $2W+B-A$. Looking back at the definition of ℓ^a, in Eq(7), we may easily see the "reason" for the problem: the limit does not distinguish between indices that differ by just 1, so that the limit of ℓ^a, is the same as that for v^a:

$$L(t) = W(t)+B(t)-\{W(t)-A(t)\} = V(t)\ , \tag{14}$$

This means that both V(t) and L(t) do indeed satisfy the **same** equation, but because they are the same function. We do not yet know how to find the desired solution to this problem; possible "errors" in the current approach might be that (a) we began where the limit of the realization sends the center to zero, or (b), the behavior imposed on the w^a's is too well behaved, inappropriate to the Dirac delta's in the continuum algebra approach described in Eqs.(8). We try, now, to describe some possible approaches to avoidance of these problems.

4 Tangent Vector Realizations for $\boldsymbol{A}_n^{(1)}$

The algebra ***sDiff*(2)** usually is chosen to include a central term; perhaps we need it that way for our problem. Therefore we might need to begin with $\boldsymbol{A}_n^{(1)}$ instead of just $\boldsymbol{A}_n$ to get the version of ***sDiff*(2)** that's needed. However, all faithful representations of $\boldsymbol{A}_n^{(1)}$ are infinite-dimensional. Pressley and Segal[20] use the Grassmannian of all infinite subspaces of a Hilbert space to create realizations for $\boldsymbol{A}_n^{(1)}$ in terms of appropriate operators, via functional analysis. Our approach, however, requires a realization in an infinite-dimensional tangent bundle! Taking ideas originally from Takasaki,[21] and Sato, for $\mathbb{Z}\times\mathbb{Z}$ matrices, with finitely-many non-zero diagonals, over $\mathbb{C}^\infty$ without convergence problem, we have moved that matrix construction to local sections of the tangent bundle, and find an apparently-new realization there. We use local coordinates, for the Grassmannian bundle there, along with a (1-dimensional) determinant fiber, namely $\big\{{w_\alpha}^j \mid \alpha = 0,1,2,\ldots\ \ j=-1,-2,\ldots\big\}$, and ζ, and denote the realization by Ω, so that the Chevalley generators for $\boldsymbol{A}_n^{(1)}$ are realized as follows, where we use $\overline{n}$ to represent $n+1$:

$$
\begin{aligned}
\Omega(h_i) &= \sum_{\beta=0}^{+\infty}\sum_{j=-1}^{-\infty}\Big\{ {w_{\overline{n}\beta+i-1}}^j \partial_{{w_{\overline{n}\beta+i-1}}^j} - {w_{\overline{n}\beta+i}}^j \partial_{{w_{\overline{n}\beta+i}}^j} \\
&\qquad -{w_\beta}^{\overline{n}j+i-1}\partial_{{w_\beta}^{\overline{n}j+i-1}} + {w_\beta}^{\overline{n}j+i}\partial_{{w_\beta}^{\overline{n}j+i}} \Big\} , \\
\Omega(h_0) &= \zeta\partial_\zeta - \sum_{\beta=0}^{+\infty}\sum_{j=-1}^{-\infty}\Big\{ {w_{\overline{n}\beta}}^j \partial_{{w_{\overline{n}\beta}}^j} - {w_{\overline{n}(\beta+1)-1}}^j \partial_{{w_{\overline{n}(\beta+1)-1}}^j} \\
&\qquad -{w_\beta}^{\overline{n}j}\partial_{{w_\beta}^{\overline{n}j}} + {w_\beta}^{\overline{n}(j+1)-1}\partial_{{w_\beta}^{\overline{n}(j+1)-1}} \Big\} , \\
\Omega(e_i) &= \sum_{\beta=0}^{+\infty}\sum_{j=-1}^{-\infty}\Big\{ {w_{\overline{n}\beta+i-1}}^j \partial_{{w_{\overline{n}\beta+i}}^j} - {w_\beta}^{\overline{n}j+i}\partial_{{w_\beta}^{\overline{n}j+i-1}} \Big\} , \\
\Omega(e_0) &= \sum_{\beta=0}^{+\infty}\sum_{j=-1}^{-\infty}\Big\{ {w_{\overline{n}(\beta+1)-1}}^j \partial_{{w_{\overline{n}(\beta+1)}}^j} - {w_\beta}^{\overline{n}j}\partial_{{w_\beta}^{\overline{n}j-1}} \Big\} + \partial_{{w_0}^{-1}} , \\
\Omega(f_i) &= \sum_{\beta=0}^{+\infty}\sum_{j=-1}^{-\infty}\Big\{ {w_{\overline{n}\beta+i}}^j \partial_{{w_{\overline{n}\beta+i-1}}^j} - {w_\beta}^{\overline{n}j+i-1}\partial_{{w_\beta}^{\overline{n}j+i}} \Big\} , \\
\Omega(f_0) &= \sum_{\beta=0}^{+\infty}\sum_{j=-1}^{-\infty}\Big\{ {w_{\overline{n}(\beta+1)}}^j \partial_{{w_{\overline{n}(\beta+1)-1}}^j} - {w_\beta}^{\overline{n}j-1}\partial_{{w_\beta}^{\overline{n}j}} - {w_\beta}^{-1}{w_0}^j\partial_{{w_\beta}^j} \Big\} \\
&\qquad + {w_0}^{-1}\zeta\partial_\zeta . \\
&\text{and}\quad \Omega(c) = \zeta\partial_\zeta .
\end{aligned}
\tag{15}
$$

With this realization, our prolongation structure in Eqs.(4) is the following, where we use ${q_\alpha}^k \equiv \ln({w_\alpha}^k)$ and $[\alpha] \equiv \alpha \bmod n+1$, $[k] \equiv k \bmod n+1$, etc.:

$$
\begin{aligned}
\mathbf{F} = F_\zeta\,\zeta\partial_\zeta + {F_\alpha}^k\partial_{{w_\alpha}^k} , &\qquad \mathbf{G} = G_\zeta\,\zeta\partial_\zeta + {G_\alpha}^k\partial_{{w_\alpha}^k} , \\
F_\zeta = m^0_{,x} , &\qquad G_\zeta = e^{B^0 - {q_0}^{-1}} , \\
{F_\alpha}^k = \left(m^{[\alpha+1]} - m^{[\alpha]} - m^{[k+1]} + m^{[k]}\right)_{,x} &+ \delta^0_\alpha\delta^k_{-1}\, e^{B^0 - {q_0}^{-1}} \\
+(1-\delta^0_\alpha)\, e^{B^{[\alpha]} + {q_{\alpha-1}}^k - {q_\alpha}^k} &- (1-\delta^k_{-1})\, e^{B^{[k+1]} + {q_\alpha}^{k+1} - {q_\alpha}^k} , \\
{G_\alpha}^k = -\left(n^{[\alpha+1]} - n^{[\alpha]} - n^{[k+1]} + n^{[k]}\right)_{,y} &- e^{A^0 + {q_\alpha}^{-1} + {q_0}^k - {q_\alpha}^k} \\
+e^{A^{[\alpha+1]} + {q_{\alpha+1}}^k - {q_\alpha}^k} &- e^{A^{[k]} + {q_\alpha}^{k-1} - {q_\alpha}^k} .
\end{aligned}
\tag{16}
$$

The derivatives $({F_\alpha}^k)_{,y}$ and $({G_\alpha}^k)_{,x}$ have been calculated and of course their differences return the original Toda equations, in the v^μ variables, as required. However, their sums constitute the "new" pde's to be considered, involving the infinitely many $\partial_x\partial_y({q_\alpha}^k)$'s, and are long and complicated, although they are only needed in the limit as $n \to \infty$. This line of work has yet to be completed. The current status of work on the other alternative is also being pursued and will be described as well.

5 Zero-Curvature Equations and Poisson Brackets

To motivate all this, we first insert some extra discussion about the Poisson-bracket-based "Lax pairs" used by Ward and also Takasaki, whose version we follow here. Using a pair of functions of 4 variables, instead of vector fields, with the usual commutator replaced by a Poisson bracket in t and $p \equiv \ln\lambda$, their zero-curvature form is

$$
g_{,\tilde{z}} - f_{,z} + \{f, g\} = 0 . \tag{17}
$$

They use an a priori form for the λ-dependence, namely $f = -\lambda a - b$, and $g = -\lambda^{-1}\overline{a} - \overline{b}$, where λ is simultaneously a phase-space variable, in the Poisson bracket, and also a spectral parameter, which generates a polynomial in λ for the constraint above. The vanishing of that polynomial does re-generate Eq.(1):

$$\lambda^{+1}\{a_{\overline{z}} + a\overline{b}_t\} + \lambda^{-1}\{\overline{a}_z - \overline{a}b_t\} + \left\{b_{\overline{z}} - \overline{b}_z + a\overline{a}_t + \overline{b}aa_t\right\} = 0 \quad . \tag{18}$$

However, this may actually be put into our language by using their functions to generate *Hamiltonian vector fields* in the tangent space to this phase space. Generically such fields are generated by (non-constant) functions on the underlying manifold and satisfy the following requirements:

$$\mathcal{H}_f \equiv f_{,t}\partial_p - f_{,p}\partial_t \;, \mathcal{H}_{af+g} = \; a\mathcal{H}_f + \mathcal{H}_g \;, \mathcal{H}_f(g) = \{f, g\} \,, \; [\mathcal{H}_f, \mathcal{H}_g] = \mathcal{H}_{\{f,g\}} \;, \tag{19}$$

In these terms Takasaki's equation takes the following zero-curvature form:

$$\begin{aligned} [D_x + \mathcal{H}_f \,,\, D_y + \mathcal{H}_g] &= \mathcal{H}_{g,x} \; - \mathcal{H}_{f,y} + [\mathcal{H}_f, \mathcal{H}_g] \\ &= \mathcal{H}_{g,x} \; - \mathcal{H}_{f,y} + \mathcal{H}_{\{f,g\}} = \mathcal{H}_{g,x - f,y + \{f,g\}} \;, \end{aligned} \tag{20}$$
$$\text{so that } [D_x + \mathcal{H}_f \,,\, D_y + \mathcal{H}_g] = 0 \Longleftrightarrow g_{,x} - f_{,y} + \{f, g\} = 0 \,.$$

This form will show us that the "Lax pair" has no pseudopotentials involved in it. Writing, for instance, the prolongation vector $F^A \frac{\partial}{\partial w^A} = \mathbf{F} \equiv \mathcal{F}_\mu \mathbf{W}^\mu$, where the $\mathcal{F}_\mu$ depend on the original jet variables, only, the dependence on the pseudopotentials should come from the $\mathbf{W}^\mu$. However, here they are simply vector fields directly over t and p, which is the *defining realization* for ***sDiff*(2)**, rather than one that has been "lifted" to new variables. This has led us to reconsider the problem with singular limits of the pseudopotentials, and to attempt to follow Saveliev's continuum-algebra approach to ***sDiff*(2)** which has $\delta''(s - t)$ as roots for the Cartan subalgebra elements.

Closely related work has been done by Fairlie, Fletcher and Zachos[22], who have defined limits for $\boldsymbol{A}_n$ that also have distributional roots when $n \to \infty$. For $\boldsymbol{A}_n$ they use a trigonometric basis : $J_{\vec{n}} \equiv J_{n,m}$, where

$$[J_{\vec{n}}, J_{\vec{m}}] = -2i \, \sin\left(\frac{2\pi}{n}(\vec{n} \times \vec{m})\right) J_{\vec{n}+\vec{m}} \;. \tag{21}$$

We will also look at this limit in terms of Hamiltonian vector fields, over a phase space, with the 2-surface chosen as a torus, where $\{\phi_{\vec{n}} \equiv e^{i\vec{n}\cdot\vec{q}} \equiv e^{i(nq+mp)} \mid n, m \in \mathbb{Z}\}$ is a basis for functions. Then $\mathcal{H}_{\vec{n}}$ is a basis for Hamiltonian vector fields

$$\begin{aligned} \mathcal{H}_f \equiv \textstyle\sum_{\vec{n}} f_{\vec{n}} \mathcal{H}_{\vec{n}} \;, \quad \mathcal{H}_{\vec{n}} \equiv \mathcal{H}_{\phi_{\vec{n}}} = \quad & i\kappa e^{i\vec{n}\cdot\vec{q}}\,(n_1\partial_p - n_2\partial_q) \;, \\ \Longrightarrow \mathcal{H}_{\vec{n}}(\phi_{\vec{m}}) = \{\phi_{\vec{n}}, \phi_{\vec{m}}\} = (\vec{n} \times \vec{m})\,\phi_{\vec{n}+\vec{m}} \;, \; & [\mathcal{H}_{\vec{n}}, \mathcal{H}_{\vec{m}}] = (\vec{n} \times \vec{m})\,\mathcal{H}_{\vec{n}+\vec{m}} \;, \end{aligned} \tag{22}$$

and, as abstract Lie algebras, they are isomorphic to the limit of the trigonometric basis above:

$$\mathcal{H}_{\vec{n}} \;\sim\; L_{\vec{n}} \equiv \lim_{n\to\infty} \frac{in}{4\pi}\, J_{\vec{n}} \;. \tag{23}$$

Our prolongation process involves sums over an appropriate choice of Cartan subalgebra and its simple roots. In this formulation, possible choices for those elements are

$$\mathfrak{h}(t) \equiv \frac{1}{4\pi} \sum_{j=-\infty}^{+\infty} j e^{itj} \mathcal{H}_{j,0} \, , \; \mathfrak{e}(t) \equiv \frac{1}{4\pi i} \sum_{j=-\infty}^{+\infty} e^{itj} \mathcal{H}_{j,+1} \, , \; \mathfrak{f}(t) \equiv \frac{1}{4\pi i} \sum_{j=-\infty}^{+\infty} e^{itj} \mathcal{H}_{j,-1} \, , \tag{24}$$

with t a continuous coordinate on the torus. Then the commutators of these objects do satisfy the commutators in Eqs.(8):

$$\begin{aligned} [\mathfrak{h}(t), \mathfrak{e}(r)] = -\delta''(t-r)\, \mathfrak{e}(r) \quad , \quad [\mathfrak{h}(t), \mathfrak{f}(r)] = +\delta''(t-r)\, \mathfrak{f}(r) \; , \\ [\mathfrak{e}(t), \mathfrak{f}(r)] \;=\; \delta(t-r)\, \mathfrak{h}(r) \; . \end{aligned} \tag{25}$$

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