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Interpretation of twisting type *N* vacuum solutions with cosmological constant

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Abstract

We investigate a new class of twisting type *N* vacuum solutions with nonzero (positive) cosmological constant Λ by studying the equations of geodesic deviations along the privileged radial timelike geodesics, generalizing J Bičák and J Podolský's results on non-twisting type *N* solutions. It is shown that these twisting radiative spacetimes can be interpreted as exact transverse gravitational waves propagating in the de-Sitter universe, with a distinctive feature that all the wave amplitudes are proportional to Λ. Moreover, we demonstrate the cosmic no-hair conjecture in these spacetimes and discuss their Killing horizons.

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1. Introduction

In a recent paper [1], we presented a new class of twisting type *N* vacuum solutions of the Einstein equations with nonzero cosmological constant Λ. These type *N* solutions admit a twisting congruence of shearfree and null geodesics aligned with the unique quadruple principal null direction. They were shown subject to a rather simple-looking second-order nonlinear ODE as imposed by the field equations for type *N*. Various special and series solutions were found or constructed from this ODE. In this paper, we move on to discuss their physical meanings and show that these new exact solutions can serve as models for the behavior of gravitational waves in cosmology.

Certain aspects of these solutions, such as the conformal factor, conformal infinities, periodicity, etc, can be found in the general discussion on algebraically special twisting spacetimes by Hill and Nurowski [2]. Here we focus on the local interpretation associated with the equation of geodesic deviation. Similar analysis has been done for non-twisting type *N* solutions with Λ, i.e. Kundt class and Robinson–Trautman class, by Bičák and Podolský [3] (see also [4, 5]). However, a complication for our solutions is that the nonzero twist is associated with certain non-integrability, and therefore causes a lack of two-dimensional wave surfaces in the spacetimes that always exist in non-twisting solutions. Hence due to extra cross terms in the metric, the coordinate basis adopted in [3] cannot be directly used in our more complicated twisting solutions, and we decide that it would be much more convenient to use non-holonomic bases instead, and therefore we generalize all derivations to such bases. In fact, we find that all major results involving the equations of geodesic deviation remain the same for this generalization.

In the next section, we review the solutions discovered in [1] and present their Levi-Civita connection 1-forms and the only non-vanishing Weyl scalar, Ψ_4. Then in section 3, we write down the general geodesic equations in the non-holonomic basis and use Killing symmetries to simplify them. In sections 4 and 5, a frame for a physical observer and an associated null tetrad basis are constructed along arbitrary timelike geodesics, with respect to which the relative motion of test particles is studied. Conditions are given to determine those geometrically privileged geodesics along which the observer's frame can be parallel-transported. Hence in section 6, these privileged geodesics are identified as the radial geodesics and determined explicitly. Their properties are discussed in section 7, which are all pointing to a Killing horizon. In section 8, we calculate wave amplitudes along radial timelike geodesics and demonstrate that the observer sees gravitational waves decaying exponentially fast, which agrees with the cosmic no-hair conjecture. At the end, we make comments on the appearance of Λ as a proportionality factor in the waves amplitudes, which we find quite unusual.

2. The twisting type N vacuum spacetimes with nonzero Λ

With the real coordinate system $x^\alpha = (x, J, u, r)$, the metric found in [1] can be written as

$$\mathbf{g} = 2(\boldsymbol{\omega}^1\boldsymbol{\omega}^2 + \boldsymbol{\omega}^3\boldsymbol{\omega}^4), \tag{1}$$

where the null tetrad is given by

$$\begin{aligned} \boldsymbol{\omega}^1 &= R\,\mathrm{d}\zeta, & \boldsymbol{\omega}^2 &= R\,\mathrm{d}\bar{\zeta}, \\ \boldsymbol{\omega}^3 &= R\,\lambda, & \boldsymbol{\omega}^4 &= R\left(\mathrm{d}r + W\,\mathrm{d}\zeta + \bar{W}\,\mathrm{d}\bar{\zeta} + H\lambda\right), \end{aligned} \tag{2}$$

with real-valued $P = P(J) > 0$ and $P' \equiv \mathrm{d}P/\mathrm{d}J$ such that

$$\begin{aligned} &\mathrm{d}\zeta = \mathrm{d}x + \frac{\mathrm{i}}{P}\,\mathrm{d}J, \qquad \lambda = \frac{\mathrm{d}u + 2L\,\mathrm{d}x}{-P\partial_J L}, \\ &L = -\mathrm{e}^{-C_1 x}\int \frac{1}{P}\exp\left(\int \frac{P' - 2\Lambda J}{2P}\,\mathrm{d}J\right)\mathrm{d}J, \\ &R = \frac{\sqrt{P}}{2\cos\frac{r}{2}}, \qquad -\pi < r < \pi, \\ &W = \frac{1}{2}\left(\frac{1}{2}P' + \Lambda J + \mathrm{i}C_1\right)(\mathrm{e}^{-\mathrm{i}r} + 1), \qquad H = -\frac{1}{6}\Lambda P\cos r. \end{aligned} \tag{3}$$

Here C_1 is an arbitrary real parameter and the function $P(J)$ must satisfy

$$P'' = -\frac{(P' + 2\Lambda J)^2}{2P} - \frac{2C_1^2}{P} - \frac{10}{3}\Lambda, \tag{4}$$

as required by the vacuum Einstein field equations with real arbitrary constant Λ. By the transformation $J = w/\Lambda$, $P = g(w)/\Lambda > 0$, with $\Lambda \neq 0$, the above equation can be put into a Λ-independent form

$$g'' = -\frac{(g' + 2w)^2}{2g} - \frac{2C_1^2}{g} - \frac{10}{3}. \tag{5}$$

These ODEs allow solutions for either $\Lambda > 0$ or $\Lambda < 0$, though the construction of the explicit general solution, if possible, still remains an open problem. (See [1] for examples of special solutions and series solutions.)

Now we introduce the non-holonomic null basis $\{\mathbf{e}_\alpha\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ dual to the tetrad (2):

$$
\begin{aligned}
\mathbf{e}_1 &= \frac{1}{R}\,(\partial - W\partial_r), \qquad \mathbf{e}_2 = \frac{1}{R}\,(\bar\partial - \bar W\partial_r),\\
\mathbf{e}_3 &= \frac{1}{R}\,(\partial_0 - H\partial_r), \qquad \mathbf{e}_4 = \frac{1}{R}\,\partial_r,
\end{aligned} \tag{6}
$$

where we define

$$
\begin{aligned}
&\partial = \partial_\zeta - L\partial_u, \qquad \partial_\zeta = \tfrac{1}{2}\,(\partial_x - \mathrm{i}P\partial_J),\\
&\partial_0 = \mathrm{i}(\bar\partial L - \partial\bar L)\partial_u = -P(\partial_J L)\partial_u.
\end{aligned} \tag{7}
$$

In particular, the vector field $\mathbf{e}_4$ is tangent to a twisting congruence of shearfree null geodesics, and is also aligned with the quadruple principal null direction of the metric. To ensure a nonzero twist along this congruence (also for the basis $\{\mathbf{e}_\alpha\}$ to be valid), it is required that

$$
\mathrm{i}(\bar\partial L - \partial\bar L) = -P(\partial_J L) \neq 0. \tag{8}
$$

With all said, we present components of the Levi-Civita connection 1-forms $\mathbf{\Gamma}^\lambda{}_\mu = \Gamma^\lambda{}_{\mu\nu}\boldsymbol{\omega}^\nu$ calculated from Cartan's structure equations $\mathrm{d}\boldsymbol{\omega}^\lambda + \mathbf{\Gamma}^\lambda{}_\mu \wedge \boldsymbol{\omega}^\mu = 0$ for the null tetrad (2):

$$
\begin{aligned}
&\Gamma_{121} = -\overline{\Gamma}_{122} = \frac{\mathrm{i}}{8R}[(\mathrm{e}^{-\mathrm{i}r}+1)P' + 2(\mathrm{e}^{-\mathrm{i}r}-1)(\Lambda J + \mathrm{i}C_1)],\\
&\Gamma_{123} = -\frac{\mathrm{i}}{12R}\Lambda P(2+\cos r), \qquad \Gamma_{124} = -\frac{\mathrm{i}}{2R},\\
&\Gamma_{231} = \frac{1}{12R}\Lambda P\left(\mathrm{i}\mathrm{e}^{-\mathrm{i}r} - \tan\frac{r}{2} + 2\mathrm{i}\right),\\
&\Gamma_{233} = \frac{\mathrm{i}}{24R}\Lambda P(\mathrm{e}^{-\mathrm{i}r}+1)(P' + 2\Lambda J - 2\mathrm{i}C_1),\\
&\Gamma_{241} = \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}r/2}}{2R\cos\frac{r}{2}}\\
&\Gamma_{341} = \overline{\Gamma}_{342} = -\frac{\mathrm{i}}{8R}[(\mathrm{e}^{-\mathrm{i}r}-1)P' + 2(\mathrm{e}^{-\mathrm{i}r}+1)(\Lambda J + \mathrm{i}C_1)],\\
&\Gamma_{343} = \frac{1}{12R}\Lambda P(2+\cos r)\tan\frac{r}{2}, \qquad \Gamma_{344} = \frac{1}{2R}\tan\frac{r}{2},\\
&\Gamma_{232} = \Gamma_{234} = \Gamma_{242} = \Gamma_{243} = \Gamma_{244} = 0.
\end{aligned} \tag{9}
$$

Those unlisted components can be obtained by either $\mathbf{\Gamma}_{\mu\nu} = -\mathbf{\Gamma}_{\nu\mu}$ or complex conjugation on (9) which interchanges the indices $1 \leftrightarrow 2$ and leaves 3 and 4 unchanged, e.g., $\Gamma_{214} = \overline{\Gamma}_{124}$, $\Gamma_{211} = \overline{\Gamma}_{122}$. The only non-vanishing Weyl scalar is

$$
\begin{aligned}
\Psi_4 = C_{3232} &= -\frac{\Lambda}{3}\left[\Lambda JP' - \frac{2}{3}\Lambda P + 2\Lambda^2 J^2 - 4C_1^2 - 2\mathrm{i}C_1\left(P' + 3\Lambda J\right)\right]\mathrm{e}^{-\mathrm{i}r/2}\cos^3\frac{r}{2}\\
&= -\frac{\Lambda}{3}\left[wg' - \frac{2}{3}g + 2w^2 - 4C_1^2 - 2\mathrm{i}C_1\left(g' + 3w\right)\right]\mathrm{e}^{-\mathrm{i}r/2}\cos^3\frac{r}{2},
\end{aligned} \tag{10}
$$

which, being proportional to Λ, requires $\Lambda \neq 0$ for type N solutions. In addition, the metric has the following two Killing vectors [6]

$$
\partial_u, \quad \partial_x - C_1 u\,\partial_u, \tag{11}
$$

which we will use to simplify the geodesic equations.

3. Geodesic equations

Using the non-holonomic basis (6), we consider a freely falling test particle (observer) with the 4-velocity

$$u^\alpha = (u^1, u^2, u^3, u^4), \qquad u^1 = \overline{u^2},$$

along an arbitrary timelike geodesic such that

$$\mathbf{u}\cdot\mathbf{u} = 2(u^1u^2 + u^3u^4) = \epsilon \tag{12}$$

with $\epsilon = -1$ (also, $\epsilon = 0$ if one considers null geodesics). From the fact that $\mathbf{u} = u^\alpha \mathbf{e}_\alpha = \dot{x}^\mu \partial_{x^\mu}$ as expressed in non-holonomic and coordinate bases, we obtain

$$\begin{aligned}
\dot{x} &= \frac{1}{2R}\,(u^1 + u^2), \qquad \dot{J} = -\frac{\mathrm{i}P}{2R}\,(u^1 - u^2),\\
\dot{u} &= -\frac{1}{R}[L(u^1 + u^2) + P(\partial_J L)u^3],\\
\dot{r} &= -\frac{1}{R}\,(Wu^1 + \bar{W}u^2 + Hu^3 - u^4).
\end{aligned} \tag{13}$$

with $\cdot \equiv \mathrm{d}/\mathrm{d}\tau$ and τ the proper time. The geodesic equations read

$$\begin{aligned}
0 &= \frac{\mathrm{d}u^1}{\mathrm{d}\tau} + \Gamma_{2\mu\nu}u^\mu u^\nu, \qquad 0 = \frac{\mathrm{d}u^2}{\mathrm{d}\tau} + \Gamma_{1\mu\nu}u^\mu u^\nu,\\
0 &= \frac{\mathrm{d}u^3}{\mathrm{d}\tau} + \Gamma_{4\mu\nu}u^\mu u^\nu, \qquad 0 = \frac{\mathrm{d}u^4}{\mathrm{d}\tau} + \Gamma_{3\mu\nu}u^\mu u^\nu,
\end{aligned} \tag{14}$$

from which one can verify, using (9), that $\frac{\mathrm{d}}{\mathrm{d}\tau}(u^1u^2 + u^3u^4) = 0$ (hence, the length (12) is constant along geodesics). For a Killing vector ξ^μ, the product $u_\mu\xi^\mu$ is a conserved quantity along geodesics with the 4-velocity u^μ. Hence from (11), we find

$$\begin{aligned}
C_2 &= \frac{R}{-P\partial_J L}(Hu^3 + u^4),\\
C_3 &= R[u^1 + u^2 + (W + \bar{W})u^3] + 2C_2L - C_2C_1u(\tau),
\end{aligned} \tag{15}$$

where C_2 and C_3 are real constants. One can show that the above two simpler equations are indeed first integrals of the system (14). However, with an implicit $P(J)$ generally satisfying (4), these geodesic equations are still very complicated to solve directly for $x^\mu(\tau)$ without additional assumptions.

4. Geodesic deviation

The equation of geodesic deviation reads

$$\frac{D^2Z^\mu}{\mathrm{d}\tau^2} = -R^\mu{}_{\alpha\beta\gamma}u^\alpha Z^\beta u^\gamma, \tag{16}$$

where $\mathbf{u} = \mathrm{d}\mathbf{x}/\mathrm{d}\tau = u^\alpha\mathbf{e}_\alpha$, $\mathbf{u}\cdot\mathbf{u} = -1$ as introduced before, and $\mathbf{Z}(\tau)$ is the displacement vector. Note that (16) is still valid despite the use of non-holonomic bases [7]. Following the construction[1] in [3], we set up the observer's frame $\{\mathbf{e}_{(\alpha)}\}$ along the geodesic with $\mathbf{e}_{(4)} = \mathbf{u}$ and spacelike orthonormal vectors $\{\mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}\}$ in the local hypersurface orthogonal to $\mathbf{u}$, i.e. $\mathbf{e}_{(\alpha)}\cdot\mathbf{e}_{(\beta)} = g_{\mu\nu}e^\mu_{(\alpha)}e^\nu_{(\beta)} = \eta_{(\alpha)(\beta)} = \mathrm{diag}(1, 1, 1, -1)$. The dual basis is $\mathbf{e}^{(4)} = -\mathbf{u}$ and $\mathbf{e}^{(i)} = \mathbf{e}_{(i)}$, $i = 1, 2, 3$. Then we can project (16) onto the observer's frame:

$$\ddot{Z}^{(\alpha)} \equiv \mathbf{e}^{(\alpha)}\cdot\frac{D^2\mathbf{Z}}{\mathrm{d}\tau^2} = e^{(\alpha)}_\mu\frac{D^2Z^\mu}{\mathrm{d}\tau^2} = -R^{(\alpha)}{}_{(4)(\beta)(4)}Z^{(\beta)} \tag{17}$$

[1] We will use the indices {1, 2, 3, 4} for basis elements instead of {0, 1, 2, 3} adopted in [3], i.e. changing 0 to 4, and changing the ordering.

with $Z^{(\beta)} = \mathbf{e}^{(\beta)} \cdot \mathbf{Z} = e^{(\beta)}_{\mu} Z^{\mu}$ and $R_{(\alpha)(4)(\beta)(4)} = e^{\mu}_{(\alpha)} u^{\nu} e^{\gamma}_{(\beta)} u^{\delta} R_{\mu\nu\gamma\delta}$. Now we introduce the second null basis $\{\mathbf{e}_{\hat{\alpha}}\} = \{\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}}, \mathbf{e}_{\hat{4}}\} = \{\mathbf{m}, \bar{\mathbf{m}}, \mathbf{l}, \mathbf{k}\}$ associated with the observer's frame:

$$\begin{aligned} \mathbf{m} &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_{(1)} + \mathrm{i}\mathbf{e}_{(2)}), \qquad & \bar{\mathbf{m}} &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_{(1)} - \mathrm{i}\mathbf{e}_{(2)}), \\ \mathbf{l} &= \tfrac{1}{\sqrt{2}}(\mathbf{u} - \mathbf{e}_{(3)}), & \mathbf{k} &= \tfrac{1}{\sqrt{2}}(\mathbf{u} + \mathbf{e}_{(3)}). \end{aligned} \tag{18}$$

One can check that all derivations in section II of [3] also hold for our non-holonomic basis $\{\mathbf{e}_{\alpha}\}$. Here we only quote those results relevant to our purpose. To begin with, all test particles should be synchronized so that $Z^{(4)} = 0$ (they always stay in the same local hypersurface). For type N spacetimes, the rest of (17) reads

$$\begin{aligned} \ddot{Z}^{(1)} &= \left(\frac{\Lambda}{3} - \mathcal{A}_{+}\right) Z^{(1)} + \mathcal{A}_{\times} Z^{(2)}, \\ \ddot{Z}^{(2)} &= \left(\frac{\Lambda}{3} + \mathcal{A}_{+}\right) Z^{(2)} + \mathcal{A}_{\times} Z^{(1)}, \\ \ddot{Z}^{(3)} &= \frac{\Lambda}{3} Z^{(3)} \end{aligned} \tag{19}$$

with wave amplitudes of the two polarization modes given by

$$\mathcal{A}_{+} = \tfrac{1}{2}\,\mathbf{Re}\,\hat{\Psi}_4, \qquad \mathcal{A}_{\times} = \tfrac{1}{2}\,\mathbf{Im}\,\hat{\Psi}_4, \qquad \hat{\Psi}_4 = C_{\alpha\beta\gamma\delta} l^{\alpha} \bar{m}^{\beta} l^{\gamma} \bar{m}^{\delta}. \tag{20}$$

Here $\hat{\Psi}_4$ is calculated in the second null basis (18), hence different from, but related to Ψ_4 in (10). Assuming that the observer's frame $\{\mathbf{e}_{(\alpha)}\}$ is parallel-transported along $\mathbf{u}$, i.e. $D\mathbf{e}^{(\alpha)}/\mathrm{d}\tau = 0$, then we have $\ddot{Z}^{(\alpha)} = D^2(\mathbf{e}^{(\alpha)} \cdot \mathbf{Z})/\mathrm{d}\tau^2 = \mathrm{d}^2 Z^{(\alpha)}/\mathrm{d}\tau^2$, which makes (19) easier to solve.

5. Parallel-transported frames

Given a radiative spacetime with a principal null direction $\mathbf{k}$ and an observer's 4-velocity $\mathbf{u}$, we can construct the observer's frame $\{\mathbf{e}_{(\alpha)}\}$ according to (18) together with the following proposition .

Proposition 1 ([3])**.** *Let* $\mathbf{u}$ *be the observer's 4-velocity and* $\mathbf{k}$ *be the null vector (principal null directions) that satisfy* $\mathbf{k} \cdot \mathbf{u} = -\frac{1}{\sqrt{2}}$ *. Then there is a unique spacelike vector* $\mathbf{e}_{(3)} = \sqrt{2}\mathbf{k} - \mathbf{u}$*. Another null vector* $\mathbf{l}$ *is given by* $\mathbf{l} = \sqrt{2}\mathbf{u} - \mathbf{k}$ *such that* $\mathbf{l} \cdot \mathbf{k} = -1$*. The only remaining freedoms are rotations in the transverse plane* $(\mathbf{e}_{(1)}, \mathbf{e}_{(2)})$ *perpendicular to* $\mathbf{e}_{(3)}$*.*

With the null basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, the metric (1) takes the very simple form

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{21}$$

We take the vector field $\mathbf{k}$ to be aligned with the quadruple principal null direction $\mathbf{e}_4$, i.e.

$$k^{\mu} = (0, 0, 0, k^4),$$

and recall that $u^{\mu} = (u^1, u^2, u^3, u^4)$, $u^1 = \overline{u^2}$ in the basis $\{\mathbf{e}_{\alpha}\}$. The interpretation null basis described in (18) and proposition 1 has the form

$$\begin{aligned} m^{\mu} &= \left(0, -1, 0, \frac{u^1}{u^3}\right), \qquad \bar{m}^{\mu} = \left(-1, 0, 0, \frac{u^2}{u^3}\right), \\ l^{\mu} &= \left(\sqrt{2}u^1, \sqrt{2}u^2, \sqrt{2}u^3, \sqrt{2}u^4 + \frac{1}{\sqrt{2}u^3}\right), \\ k^{\mu} &= \left(0, 0, 0, -\frac{1}{\sqrt{2}u^3}\right), \end{aligned} \tag{22}$$

which is unique up to rotations $\mathbf{m} \to e^{i\theta}\mathbf{m}$ and trivial reflections. The corresponding orthonormal frame in the basis $\{\mathbf{e}_\alpha\}$ is given by

$$\begin{aligned}
e^\mu_{(1)} &= \frac{1}{\sqrt{2}}\left(-1, -1, 0, \frac{u^1+u^2}{u^3}\right),\\
e^\mu_{(2)} &= \frac{1}{\sqrt{2}}\left(-i, i, 0, \frac{u^1-u^2}{iu^3}\right),\\
e^\mu_{(3)} &= -\left(u^1, u^2, u^3, u^4+\frac{1}{u^3}\right),\\
e^\mu_{(4)} &= u^\mu = (u^1, u^2, u^3, u^4).
\end{aligned} \tag{23}$$

One can check that these expressions are consistent with equations (18) and (19) of [3] when applied to (21) with matrix elements rearranged accordingly.

In general, the frames $\{\mathbf{e}_{(\alpha)}\}$ and $\{\mathbf{e}_{\hat{\alpha}}\}$ cannot be parallel-transported along the geodesic with $\mathbf{u} = \mathbf{e}_{(4)}$. We can use the following proposition to single out those geometrically privileged geodesics along which the interpretation frames are indeed parallel-transported. The detailed proof can be found in [8] with no difficulty to be generalized for the non-holonomic basis (6) with the metric (21).

Proposition 2 ([3, 8])**.** *Given a geodesic with the tangent vector* $u^\mu = (u^1, u^2, u^3, u^4)$ *in the spacetime (21), the interpretation null basis (22) and the orthonormal frame (23) are parallel-transported along the geodesic if*

$$0 = \Gamma^1{}_{4\mu}u^\mu \tag{24}$$

and

$$\dot{\vartheta}_\parallel(\tau) = i\,\Gamma^2{}_{2\mu}u^\mu \tag{25}$$

where $\vartheta_\parallel$ *is the rotation angle for* $\mathbf{m} \to \mathbf{m}_\parallel = e^{i\vartheta_\parallel}\mathbf{m}$ *in order that* $D\mathbf{m}_\parallel/d\tau = 0$.

6. Privileged geodesics

Now we apply the above results to our twisting type N spacetimes with Λ. First recall that $\Gamma_{242} = \Gamma_{243} = \Gamma_{244} = 0$. Then the condition (24) and its complex conjugate are tantamount to

$$u^1 = u^2 = 0, \tag{26}$$

which, by (13), leads to

$$\dot{x} = \dot{J} = 0. \tag{27}$$

Along such *radial* geodesics with fixed x and J, the geodesic equations are quite simplified with the first two of (14) given by

$$0 = \frac{i\Lambda R}{24P(\partial_J L)^2}\,(e^{-ir}+1)(P' + 2\Lambda J - 2iC_1)\,\dot{u}^2 \tag{28}$$

and its complex conjugate. The rest of (14) are given by

$$0 = \ddot{u} + \left(\frac{\Lambda}{6\partial_J L}\tan\frac{r}{2}\right)\dot{u}^2, \tag{29}$$

$$0 = \ddot{r} + \tan\frac{r}{2}\left(\dot{r}^2 - \frac{\Lambda}{3\,\partial_J L}\,\dot{r}\dot{u} - \frac{\Lambda^2\cos r}{18(\partial_J L)^2}\,\dot{u}^2\right). \tag{30}$$

The conserved quantities (15), i.e. first integrals of (29), (30), read

$$0 = \dot{r} + \frac{\Lambda \cos r}{3\partial_J L}\,\dot{u} + 4C_2(\partial_J L)\cos^2\frac{r}{2}, \tag{31}$$

$$0 = \frac{(W+\bar{W})}{\partial_J L}\,\dot{u} + 4(C_1C_2u - 2C_2L + C_3)\cos^2\frac{r}{2}, \tag{32}$$

in addition to the invariant length (12)

$$0 = \frac{\epsilon}{2} - \left(\frac{\Lambda\cos r}{24(\partial_J L)^2\cos^2\frac{r}{2}}\,\dot{u} + C_2\right)\dot{u}. \tag{33}$$

Note that $\partial_J L$ and L above are *constant* for fixed x and J and that W is given by (3).

Now we proceed to solve the system ((28)–(30)). First note that the equation (28) gives rise to the following two possibilities.

Case 1. We have $\dot{u} = 0$. Then (33) requires $\epsilon = 0$. This is the case corresponding to null geodesics along principle null directions. The geodesic equation (30) immediately gives us

$$0 = \ddot{r} + \dot{r}^2\tan\frac{r}{2},$$

which has the general solution

$$r = 2\arctan(A\tau + B)$$

with A, B arbitrary real constants of integration.

Case 2. Assuming $\dot{u} \neq 0$ and a given solution $P_0(J)$ to (4), we have, for (28) to hold,

$$0 = \frac{\mathrm{d}P_0(J)}{\mathrm{d}J} + 2\Lambda J, \qquad C_1 = 0. \tag{34}$$

The first equation above, with its right-hand side being a function of J, shall fix the value of J which we call J_0. In fact, combining (34) with (4), we know that J_0 is also the point at which the second derivative $P_0''(J)$ reaches its maximum value $-10\Lambda/3$. Hence for a given $P_0(J)$, we only have limited choices of J for the privileged geodesics described in proposition 2. Nonetheless, the coordinate x can take on any arbitrary constant value which we denote as $x = x_0$. In what follows, such $P = P_0(J_0)$ and $x = x_0$ will always be assumed.

To solve the system ((29), (30)), one can first solve (29) for r and then combine the result with the first integral (31) so as to obtain a third-order ODE for $u(\tau)$ alone

$$0 = \dot{u}\,\dddot{u} - \ddot{u}^2 - \frac{C_2\Lambda}{3}\,\dot{u}^3 - \frac{\Lambda^2}{36C_L^2}\,\dot{u}^4, \qquad \Lambda \neq 0. \tag{35}$$

Thus for this equation, we obtain in the *real* domain the following two different ways[2] to represent its general solution both of which we find suitable for timelike geodesics with $\Lambda > 0$ (cf (41)):

$$\tanh\left(\frac{\Lambda}{12C_L}\,(u - E_1)\right) = \pm(\mathrm{e}^{A_1\tau+B_1} + D_1), \tag{36}$$

$$\tanh\left(\frac{\Lambda}{12C_L}\,(u - E_2)\right) = \sqrt{1 + D_2^2}\,\tanh\left(\frac{A_2\tau + B_2}{2}\right) + D_2, \tag{37}$$

[2] In the complex domain, the expressions (36) and (37) are in fact equivalent. To see this, one can separate a constant from E_1 and then use the addition theorem of tanh on the left-hand side of (36) and compare it with (37). However in the real domain, this transformation from (36) to (37), for their full ranges of real integration constants, cannot be achieved without generally going into the complex domain (e.g., note that $\operatorname{arctanh}(x)$ is complex for $x > 1$). For certain limited ranges of real integration constants, the solutions (36) and (37) may represent the same solutions.

with $C_L \equiv \partial_J L(J_0) \neq 0$ and real integration constants A_1, B_1, D_1, E_1 ($C_2 = A_1 D_1/2C_L$) and A_2, B_2, D_2, E_2 $\left(C_2 = -A_2/2C_L\sqrt{1 + D_2^2}\right)$. By the equation (29), these $u(\tau)$s yield respectively

$$r(\tau) = \pm 2 \arctan\left(-\frac{1}{2}\,\mathrm{e}^{-(A_1\tau + B_1)}(1 - D_1^2) - \frac{1}{2}\,\mathrm{e}^{A_1\tau + B_1}\right), \tag{38}$$

$$r(\tau) = -2 \arctan\left(\frac{D_2^2}{\sqrt{1 + D_2^2}}\sinh(A_2\tau + B_2) + D_2 \cosh(A_2\tau + B_2)\right). \tag{39}$$

One can check that the expressions (36), (38) and (37), (39) all constitute general solutions to (29), (30) in the real domain. In particular when $D_2 = 0$ in (37), (39), one has a special solution

$$u(\tau) = \frac{6C_L}{\Lambda}(A_2\tau + B_2) + E_2, \qquad r(\tau) = 0, \tag{40}$$

which is not included in the solution (36), (38). From (33), we obtain for both sets of solutions

$$\epsilon = -\frac{3A_{1,2}^2}{\Lambda}, \tag{41}$$

with $\epsilon = -1$ for timelike geodesics. In addition, the equation (32) yields trivially $C_3 = 2C_2 L(J_0)$.

7. Radial timelike geodesics and Killing horizon

We continue to study timelike geodesics given by (36), (38) and (37), (39). For simplicity, we only consider the more physically relevant situation with $\Lambda > 0$. Hence from (41), one has

$$A_{1,2} = \pm\sqrt{\frac{\Lambda}{3}}, \qquad \Lambda > 0 \tag{42}$$

for timelike geodesics. Note that the range of the hyperbolic function tanh on the left-hand sides of (36) and (37) is limited to the interval $[-1, 1]$, while the right-hand sides are not. Therefore the solution $u(\tau)$ with its τ restricted by the reality condition may generally reach infinity at some *finite* proper time. Nonetheless, whenever this happens at the critical value $\tau = \tau_c$ such that

$$\mathrm{e}^{A_1\tau_c + B_1} + D_1 = \pm 1, \qquad D_1^2 \neq 1,$$

or

$$\sqrt{1 + D_2^2}\tanh\left(\frac{A_2\tau_c + B_2}{2}\right) + D_2 = \pm 1, \qquad D_2 \neq 0,$$

it always corresponds to

$$r(\tau_c) = 2\arctan(\mp 1) = \mp\frac{\pi}{2}, \tag{43}$$

which are in fact very special hypersurfaces in the spacetime. We can see their significance from the metric (1) with constant x and J

$$\widetilde{\mathbf{g}} = -\frac{1}{2C_L\cos^2\frac{r}{2}}\,\mathrm{d}u\left(\mathrm{d}r + \frac{\Lambda}{6C_L}\cos r\,\mathrm{d}u\right), \qquad -\pi < r < \pi, \tag{44}$$

which indicates that the Killing vector ∂_u is timelike for $-\pi/2 < r < \pi/2$ but spacelike for $-\pi < r < -\pi/2$ and $\pi/2 < r < \pi$, and *null* at $r = \pm\pi/2$. Thus that $u(\tau)$ diverges at $r = \pm\pi/2$ under the metric (44) is very similar to that of the Schwarzschild solution in the

Eddington–Finkelstein coordinates near $r = 2m$, the fact of which suggests a *Killing horizon*[3] at $r = \pm\pi/2$. Inside the region $r = \pm\pi/2$, the spacetime is *stationary* with u being a time coordinate (e.g., (40)). Particularly in the weak field limit $\Psi_4 \to 0$ with finite $\Lambda > 0$ [1], one can expect that $r = \pm\pi/2$ approaches the cosmological horizon of the de Sitter universe.

8. Wave amplitudes

Now we calculate the wave amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$ in (19). The null bases $\{\mathbf{e}_\alpha\}$ and $\{\mathbf{e}_{\hat{\alpha}}\}$ (cf (6) and (18)) are related by the Lorentz transformation

$$\begin{aligned} \mathbf{e}_4 &= A\,\mathbf{e}_{\hat{4}}, \\ \mathbf{e}_1 &= \mathrm{e}^{\mathrm{i}\theta}\mathbf{e}_{\hat{2}} + \bar{B}\mathbf{e}_{\hat{4}}, \\ -\mathbf{e}_3 &= A^{-1}(\mathbf{e}_{\hat{3}} + B\mathrm{e}^{\mathrm{i}\theta}\mathbf{e}_{\hat{2}} + \bar{B}\mathrm{e}^{-\mathrm{i}\theta}\mathbf{e}_{\hat{1}} + B\bar{B}\mathbf{e}_{\hat{4}}), \end{aligned} \tag{45}$$

with $A = -\sqrt{2}u^3$, $B = -\sqrt{2}u_1$ and $\theta = \pi$. The Weyl scalar Ψ_4 transforms as [9]

$$\hat{\Psi}_4 = A^2\bar{\Psi}_4 = \frac{4\Lambda^2 P_0(J_0)}{9}\,(u^3)^2\,\mathrm{e}^{\mathrm{i}r/2}\cos^3\frac{r}{2} \propto \Lambda, \tag{46}$$

where $u^3 = -R(\dot{u} + 2L\dot{x})/(P\partial_J L)$ from (13) (also, use the re-scaled function $g(w)$ from (5) to see the proportionality to Λ). Hence we know

$$\begin{aligned} \mathcal{A}_+ &= \frac{\Lambda^2}{18C_L^2}\,\dot{u}^2\cos^2\frac{r}{2}, \\ \mathcal{A}_\times &= \frac{\Lambda^2}{18C_L^2}\,\dot{u}^2\sin\frac{r}{2}\,\cos\frac{r}{2}. \end{aligned} \tag{47}$$

Substituting either (36), (38) or (37), (39) for $u(\tau)$ and $r(\tau)$ above with (42), we find that as long as the limit $u(+\infty)$ exists (e.g., when $A_1 < 0$ in (36)) and $r(+\infty) = \pm\pi$, the amplitudes behave like

$$\mathcal{A}_+ \sim \Lambda\exp\left(-4\sqrt{\frac{\Lambda}{3}}\,\tau\right), \qquad \mathcal{A}_\times \sim \Lambda\exp\left(-3\sqrt{\frac{\Lambda}{3}}\,\tau\right) \tag{48}$$

as the proper time $\tau \to +\infty$. This means that the gravitational waves are decaying exponentially fast with the spacetime locally approaching the de-Sitter universe. Hence the cosmic no-hair conjecture [10] is demonstrated with these radial geodesics. Furthermore, due to the proportionality to Λ, the polarization modes $\mathcal{A}_+$ and $\mathcal{A}_\times$ cannot be generally separated from the isotropic background represented by $\Lambda/3$ in (19), when it comes to consider their local effects.

9. Concluding remarks

Like their non-twisting counterparts [3], the twisting spacetimes described by (1)–(4) bear a similar local interpretation as exact transverse gravitational waves in the de-Sitter universe. However, the *essential* nonzero requirement of the cosmological constant Λ in our radiative solutions (wave amplitudes proportional to Λ; for $\Lambda = 0$, spacetimes becoming flat), to our best knowledge, has not been seen in any other radiative exact solution (e.g., Kundt class and Robinson–Trautman class). This distinctive feature of these spacetimes suggests that the gravitational waves they represent may not be considered as being generated by usual astronomical sources which always radiate regardless of the *background* Λ being zero or not.

[3] Due to cross terms like $\mathrm{d}\zeta\mathrm{d}u$ in the metric with the current coordinate system, it is not at all clear how to show $r = \pm\pi/2$ is an actual horizon for non-radial causal geodesics. Hence we will not dwell on this issue.

Moreover, from the linear theory of gravitational waves with cosmological constant, one can indeed identify extra freedom in the higher-order terms of approximate solutions caused by the Λ parameter [11], which is consistent with our observation from the exact theory. In fact, according to [11], this extra freedom in the wave solutions has been described as being a 'coupling to matter sources with a strength proportional to the cosmological constant itself', though the actual physics behind this ad hoc coupling is quite unknown. Altogether, both theories suggest a quite different role that the cosmological constant (or perhaps, dark energy) may play in the process of gravitational radiations, other than simply the 'inactive' de-Sitter background as it is generally thought of. Since in the inflationary epoch, the effect of Λ on the cosmology was much greater than at present, we expect that these radiative solutions might be of more relevance to primordial gravitational waves from the Big Bang.

References

[1] Zhang X and Finley D 2012 Lower order ODEs to determine new twisting type *N* Einstein spaces via CR geometry *Class. Quantum Grav.* **29** 065010

[2] Hill C D and Nurowki P 2008 Twisting rays imply conformally periodic universes *Class. Quantum Grav.* **25** 035014

[3] Bičák J and Podolský J 1999 Gravitational waves in vacuum spacetimes with cosmological constant: II. Deviation of geodesics and interpretation of nontwisting type N solutions *J. Math. Phys.* **40** 4506–17

[4] Podolský J 1998 Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe *Class. Quantum Grav.* **15** 719–33

[5] Podolský J and Belnán M 2004 Geodesic motion in Kundt spacetimes and the character of the envelope singularity *Class. Quantum Grav.* **21** 2811–29

[6] Zhang X 2012 CR geometry and twisting type *N* vacuum solutions *Dissertation* University of New Mexico, Albuquerque

[7] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco, CA: Freeman)

[8] Podolský J 1993 On exact radiative space-times with cosmological constant *Dissertation* Charles University, Prague

[9] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge: Cambridge University Press)

[10] Maeda K 1989 Inflation and cosmic no hair conjecture *Proceedings of the 5th Marcel Grossman Conference on General Relativity* ed D G Blair, M J Buckingham and R Ruffini (Singapore: World Scientific) pp 145–55

[11] Bernabeu J, Espriu D and Puigdomènech 2011 Gravitational waves in the presence of a cosmological constant *Phys. Rev.* D **84** 063523