Non-Abelian Infinite Algebra of Generalized Symmetries for the SDiff(2)Toda Equation

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We determine the (non-Abelian) algebra of generalized symmetries for the SDiff(2)Toda equation, a pde for a single function of 3 independent variables, the solutions of which determine self-dual, vacuum solutions of the Einstein field equations. This algebra is a realization of two copies of the abstract algebra SDiff(2), along with an additional pair of elements that have derivation-like properties on both of the copies. It contains as a subalgebra the doubly-infinite, Abelian algebra, equivalent to the infinite hierarchy of higher flows found by Takasaki and Takebe. An infinite prolongation of the jet bundle for the original pde, to include all the variables allowed in their hierarchy, is required for the presentation of this generalization. Because these symmetries have non-zero commutators, they generate a recursion relation, allowing the generation and description of the entire algebra.

1. The SDiff(2)Toda Equation, and Its Generalized Symmetries

This equation has been of interest in general relativity in various contexts, as well as some other fields of theoretical physics, for over twenty years. One derivation was given by one of us and Charles Boyer[1] in 1982, showing that it determines all self-dual, vacuum solutions of the Einstein field equations which admit a rotational Killing vector. (The description of that metric is given in Appendix A.) The equation is a partial differential equation (pde) for a single function of 3 independent variables, which may be written in the form

$$\Omega_{,xy} + \left(e^{\Omega}\right)_{,ss} = 0 \ , \tag{1.1}$$

where partial derivatives are indicated by a subscript which begins with a comma. Extensive study during that time has uncovered various classes of solutions; however, almost all of those describe metrics which possess additional Killing vectors as well. In particular, when the one rotational Killing vector is part of an entire $\boldsymbol{SU}(2)$ of symmetries for the metric, sometimes referred to as a Bianchi IX metric, this pde is reduced to a system of ordinary differential equations. This system has been shown to be resolved via the Painlevé VI, and Painlevé III, functions[2]. Other details of the history of the search for solutions may be seen at this reference[3]. Nonetheless, very few solutions of general type are known, even though there has been a resurgence of interest in this problem in recent years[4], along with a few new solutions described quite recently[5]. In particular the complete set of generalized symmetries has not been known before; it is hoped that this characterization of them will facilitate the search for additional classes of general solutions.

The complete algebra of generalized symmetries that we find may be described as the semi-direct sum of the (unique) non-Abelian, 2-dimensional Lie algebra with the direct sum of two copies of $\boldsymbol{SDiff}(2)$, i.e., $\boldsymbol{S}_2 \ni \{\boldsymbol{SDiff}(2)\oplus\boldsymbol{SDiff}(2)\}$. One of the copies of $\boldsymbol{SDiff}(2)$ is built over s-potentials of quantities made from x-derivatives, while the other is built over similar s-potentials of quantities made from y-derivatives, so that those two independent variables play identical but independent roles. We can describe those subalgebras via two arbitrary constants, for the solvable algebra, and two countable sequences of arbitrary functions of 1 variable, one for each of the copies of $\boldsymbol{SDiff}(2)$. By expanding those functions in series about the origin, we may span those copies by two doubly-infinite sets, $\{X^n_p \mid p = 1, 2, \dots \ ; \ n = 0, 1, 2, \dots\}$ and $\{Y^m_q \mid q = 1, 2, \dots \ ; \ m = 0, 1, 2, \dots\}$. If we then also span the solvable algebra by the set $\{S_1, S_0\}$, we have a (vector-space) basis for the entire algebra, and may define the details of the construction by giving the appropriate Lie products of this entire set:

$$
\begin{aligned}
&\{X^n_p, X^m_q\} = (qn - pm) X^{n+m-1}_{p+q-1}\ , \qquad \{X^n_p, Y^m_q\} = 0\ , \quad \{Y^n_p, Y^m_q\} = (qn - pm) Y^{n+m-1}_{p+q-1}\ ,\\
&\{X^n_p, S_0\} = (p-1)\, X^n_p\ , \quad \{X^n_p, S_1\} = n(p-1)\, X^{n-1}_{p-1}\ , \\
&\{Y^n_p, S_0\} = (p-1)\, Y^n_p\ , \quad \{Y^n_p, S_1\} = n(p-1)\, Y^{n-1}_{p-1}\ , \qquad \{S_0, S_1\} = S_1\ .
\end{aligned}
\tag{1.2}
$$

As this is an algebra of generators for symmetries, the (vector-space) basis for the algebra could be described in terms of tangent vectors on (the appropriate infinite prolongation of) the manifold used to describe the pde, or, as is more usual, in terms of their characteristics[6,7], which are functions defined over that manifold. In that presentation, the Lie product for the algebra elements is given in terms of the associated Poisson-type brackets for the characteristic functions.

The Lie symmetries, i.e., those involving only the first level on the jet bundle, J^1, for this equation are well-known[8], and constitute the (infinite-dimensional) subalgebra spanned by $\{X^n_1,\ Y^m_1,\ S_0,\ S_1 \mid n,m = 0,1,2,\ \ldots\ \}$. Another important subalgebra is Abelian and is spanned by $\{X^0_p,\ Y^0_q \mid p,q = 0,1,2,\ \ldots\ \}$. It is this algebra that generates the compatible hierarchies of higher-order pde's that are associated with this equation via the work of Takasaki and Takebe[9]. That those entire infinite sets of pde's are compatible is what we would now expect, given that the associated subalgebras of generalized symmetries are Abelian and therefore generate *commuting flows* on the jet bundle.

While Eq. (1.1) has been given quite a few names over the last twenty years, the name we use was first used by Mikhail Saveliev[10] and also Takasaki and Takebe[9], emphasizing the fact that some definition of "the symmetry algebra" for this equation ought to be $\boldsymbol{SDiff}(2)$. Saveliev's description[10] was built on his construction of continuum Lie algebras[11], which gave them a formal, infinite series as an expression for the "general solution," built over this algebra. Unfortunately, his result seems to be too formal and not practically useful for describing solutions so as to be able to use them in applications, but do see the more detailed descriptions of Bakas[10]. Takasaki's approach was considerably more practical, and indeed created the infinite hierarchy of commuting flows over the (restricted) infinite jet bundle, built over this pde at the lowest level[9]. That hierarchy provided a convenient structure which allowed a (functional) realization of $\boldsymbol{SDiff}(2)$, which they describe. It is this Abelian structure, mentioned above, along with the investigations of the generalized symmetries of the (2-dimensional) Toda lattice pde's made by Kajiwara and Satsuma[12], (built on the earlier work on the KdV-type hierarchy for those lattice equations of Takasaki and Ueno[13]) that led us to investigate the generalized symmetries of this equation. In the sections below we explain in detail how we define our jet bundles, and what is necessary to arrive at these conclusions. We trust that this larger explication of the generalized symmetries of the equation will eventually be helpful in a better understanding of the solution manifold for the problem.

2. The Infinite Jet Bundle and the Earlier Additional Potentials

A k-th order pde may be realized as a subvariety, Y, of a finite jet bundle, $J^{(k)}(M,N)$, where M is the space of independent variables and N the space of dependent variables in the original pde. That subvariety is most easily described, in local coordinates, by resolving the pde for some appropriate derivative and using that equation to locally describe a surface in the jet bundle. At such a level it is straightforward to look for the Lie symmetries as the generators of flows in the jet bundle that remain on this surface, so that they map the solution manifold into itself. They are just vector fields over $J^{(1)}(M,N)$, prolonged to this k-th jet. However, the search for generalized symmetries is most easily performed on the infinite prolongation of that pde, prolonging Y to $Y_\infty \subset J^{(\infty)}$, a proper subset of the complete infinite jet over those variables, where arbitrarily many derivatives are allowed, as described for instance by Vinogradov[6,7,14].

We use the obvious choices $\{x, y, s, \Omega\}$ for coordinates on J^0 and then introduce for each integer $k \geq 1$, a notation $\Omega_{(\sigma)}$, where (σ) is an unordered list of length k, of the symbols for the independent variables, x, y, and s. For a given k the set of all of these constitutes a set of coordinates for $J^{(k)}/J^{(k-1)}$; for instance at second order these coordinates are $\{\Omega_{xx}, \Omega_{xy}, \Omega_{xs}, \Omega_{yy}\, \Omega_{ys}, \Omega_{ss}\}$, where we do not use a comma in the subscript to simply denote variables in the various jet bundles. This allows us to write out the total derivatives on the entire (infinite) jet:

$$D_{x_i} = \partial_{x_i} + \sum_{k=|\sigma|=0}^{\infty} \Omega_{(\sigma)x^i}\, \partial_{\Omega_{(\sigma)}} \ , \quad x_i = x, y, s \ , \tag{2.1}$$

where $\Omega_{(\sigma)x^i}$ is of order $k+1$ when $|\sigma| = k$. We must then restrict our consideration to the variety defined by solutions of the pde. On this variety we use the pde to make Ω_{xy} a function of the other coordinates, and then use its derivatives to remove all other coordinates which contain one or more x and also one or more y. When this is done, we will denote these functions by the use of "overtildes" above the symbol that might otherwise have simply labelled a coordinate, on the un-restricted bundle. We refer to these functions as "co-coordinates"; the infinite set of them define Y_∞ as a subvariety of $J^{(\infty)}(M,N)$. Some examples would be $\widetilde{\Omega_{xy}} = -(\Omega_{ss} + \Omega_s^2)e^\Omega$ or $\widetilde{\Omega_{xxy}} = D_x\, \widetilde{\Omega_{xy}} = -(\Omega_{xss} + 2\Omega_s\Omega_{xs} + \Omega_x\Omega_{ss} + \Omega_x\Omega_s^2)e^\Omega$. Therefore, at level k, the coordinates on this restricted bundle now correspond to just those k-tuples either with ℓ x's and $(k-\ell)$ s's or ℓ y's and $(k-\ell)$ s's, where ℓ varies from 0 to k:

$$\text{on } J^k/J^{k-1}: \quad \Omega_{xx\ldots x}, \ldots, \Omega_{x\ldots xs\ldots s}, \ldots, \Omega_{ss\ldots s}, \ldots, \Omega_{y\ldots ys\ldots s}, \ldots, \Omega_{yy\ldots y}\ , \quad , \tag{2.2}$$

The total derivatives pull back to this variety, with the restricted total derivatives (denoted by an overbar) including only derivatives with respect to these local coordinates:

$$\begin{aligned}
\overline{D}_x =& \partial_x + \Omega_x\partial_\Omega + \Omega_{xx}\partial_{\Omega_x} + \widetilde{\Omega_{xy}}\partial_{\Omega_y} + \Omega_{xs}\partial_{\Omega_s} + \Omega_{xxx}\partial_{\Omega_{xx}} + \widetilde{\Omega_{xyy}}\partial_{\Omega_{yy}} \\
&+ \Omega_{xxs}\partial_{\Omega_{xs}} + \widetilde{\Omega_{xys}}\partial_{\Omega_{ys}} + \Omega_{xss}\partial_{\Omega_{ss}} + \Omega_{xxxx}\partial_{\Omega_{xxx}} + \widetilde{\Omega_{xyyy}}\partial_{\Omega_{yyy}} \\
&+ \Omega_{xxxs}\partial_{\Omega_{xxs}} + \Omega_{xxss}\partial_{\Omega_{xss}} + \Omega_{xsss}\partial_{\Omega_{sss}} + \widetilde{\Omega_{xyys}}\partial_{\Omega_{yys}} + \widetilde{\Omega_{xyss}}\partial_{\Omega_{yss}} \\
&+ \ \ldots \ = \ \partial_x + \sum_{(\sigma)}^{\infty} \Omega_{(\sigma)x}\partial_{\Omega_{(\sigma)}} \ , \\
\overline{D}_y =& \partial_y + \Omega_y\partial_\Omega + \widetilde{\Omega_{yx}}\partial_{\Omega_x} + \Omega_{yy}\partial_{\Omega_y} + \Omega_{ys}\partial_{\Omega_s} + \widetilde{\Omega_{yxx}}\partial_{\Omega_{xx}} + \Omega_{yyy}\partial_{\Omega_{yy}} \\
&+ \widetilde{\Omega_{xys}}\partial_{\Omega_{xs}} + \Omega_{yys}\partial_{\Omega_{ys}} + \Omega_{yss}\partial_{\Omega_{ss}} \ + \ \ldots \ = \ \partial_y + \sum_{(\sigma)}^{\infty} \Omega_{(\sigma)y}\partial_{\Omega_{(\sigma)}} \ , \\
\overline{D}_s =& \partial_s + \Omega_s\partial_\Omega + \Omega_{xs}\partial_{\Omega_x} + \Omega_{ys}\partial_{\Omega_y} + \Omega_{ss}\partial_{\Omega_s} + \Omega_{xxs}\partial_{\Omega_{xx}} + \Omega_{yys}\partial_{\Omega_{yy}} \\
&+ \Omega_{xss}\partial_{\Omega_{xs}} + \Omega_{yss}\partial_{\Omega_{ys}} + \Omega_{sss}\partial_{\Omega_{ss}} \ + \ \ldots \ = \ \partial_s + \sum_{(\sigma)}^{\infty} \Omega_{(\sigma)s}\partial_{\Omega_{(\sigma)}} \ ,
\end{aligned} \tag{2.3}$$

where the sum over all possible values of (σ) here only includes those symbols that label coordinates *on the restricted variety,* as shown in the examples above. As well, in these sums when we consider the coefficients above of the form $\Omega_{(\sigma)x}$ or $\Omega_{(\sigma)y}$, we must remember that if the subscript does not include both an x and a y then it is simply a jet coordinate, while if it does include both an x and a y, then the object appearing there is a co-coordinate, which must be replaced by its explicit value in terms of the coordinates, as defined by some total derivative of the original pde.

On the infinite jet, characteristics, φ, for (generalized) symmetries are searched for as functions on the prolongation up to order ℓ, i.e., on $Y_\ell \subset J^{(\ell)}$, for any finite integer $\ell \geq 1$. They must be solutions of the following equation, which Vinogradov refers to as the universal linearization equation[6]:

$$\left\{\overline{D}_x\overline{D}_y + e^{\Omega}\left[\overline{D}_s\overline{D}_s + 2\,\Omega_s\,\overline{D}_s + (\Omega_{ss} + \Omega_s^2)\right]\right\}\varphi = 0\ . \tag{2.4}$$

Since the pde itself is of second order, this equation will contain coordinates on the jet that involve derivatives of Ω no higher than 2 above the highest order, ℓ, on which φ depends. In fact the highest orders cancel completely, so that it is actually to be resolved on $Y_{(\ell+1)}$. We begin by first asking only for the Lie symmetries, i.e., those built on J^1. The general solution to that problem is then given by the following:

$$\varphi = A(x)\,\Omega_x + B(y)\,\Omega_y + (\alpha s + \beta)\Omega_s + A_{,x}(x) + B_{,y}(y) - 2\alpha\ , \tag{2.5}$$

which may therefore be parametrized by two arbitrary functions of 1 variable, and two additional constants. For comparison with later results, it will be convenient to create a basis for this set of (Lie) characteristics in the following way:

$$\begin{gathered}
GX_1[A] \equiv A(x)\,\Omega_x + A_{,x}(x) \equiv \sum_{n=-\infty}^{+\infty} A_n X_1^n\ ,\ \text{where}\ \ A(x) \equiv \sum_{n=-\infty}^{+\infty} A_n x^n\ ,\ X_1^n \equiv GX_1[x^n]\ ,\\
GY_1[B] = B(y)\,\Omega_y + B_{,y}(y) \equiv \sum_{n=-\infty}^{+\infty} B_n Y_1^n\ ,\ \text{where}\ \ B(y) \equiv \sum_{n=-\infty}^{+\infty} B_n y^n\ ,\ Y_1^n \equiv GY_1[y^n]\ ,\\
S_0 \equiv s\Omega_s - 2\ , \qquad S_1 \equiv \Omega_s\ , \qquad GS(\alpha,\beta) \equiv \alpha\, S_0 + \beta\, S_1\ .
\end{gathered} \tag{2.6}$$

which have the following commutators:

$$\begin{gathered}
\{GX_1[A_1], GX_1[A_2]\} = GX_1[A_1A_2' - A_2A_1']\ , \quad \{GY_1[B_1], GY_1[B_2]\} = GY_1[B_1B_2' - B_2B_1']\ ,\\
\text{or, equivalently,} \quad \{X_1^n, X_1^m\} = (m-n)\,X_1^{n+m-1}\ , \quad \{Y_1^n, Y_1^m\} = (m-n)\,Y_1^{n+m-1}\ ,\\
\text{and}\\
\{GX_1[A], GY_1[A]\} = 0 = \{GX_1[A], S_j\} = \{GY_1[B], S_j\}\ ,\ \ \{S_0, S_1\} = S_1\ ,\\
\text{or, equivalently,} \quad \{X_1^n, Y_1^m\} = 0\ ; \qquad \{X_1^n, S_j\} = 0 = \{Y_1^m, S_j\} = 0\ ,
\end{gathered} \tag{2.7}$$

where j takes on the values 0 and 1, and the prime indicates the derivative with respect to that functions's argument. Each of the arbitrary functions can be seen to generate a copy of the Virasoro algebra (without center), namely $\boldsymbol{SDiff}(1)$. It will also be useful later to have simpler names for those symmetries when the arbitrary function is chosen constant, and then normalized to 1, i.e., for $A(x) = 1$ and also $B(y) = 1$:

$$X_1 \equiv X_1^0 = GX_1[1] = \Omega_x\ , \qquad Y_1 \equiv Y_1^0 = GY_1[1] = \Omega_y\ . \tag{2.7a}$$

The set of all generalized symmetries forms a Lie algebra. When those symmetries are expressed as vector fields over $J^{(\infty)}$, the (skew-symmetric) Lie product for the symmetries is simply the usual

Lie bracket, or commutator bracket, for the vector fields. However, since we are describing our symmetries in terms of their characteristics, the commutators must be determined in terms of some Poisson-bracket style of calculation for functions. Therefore, let ϕ and ψ be two arbitrary characteristics, with $\vec{v}_\phi$ the vector field associated to ϕ and $\vec{v}_\psi$ the field associated with ψ. Furthermore, let the commutator of these two vector fields be given by $\vec{v}_\omega$, associated with a characteristic ω. Then we have the following general theorem[6]:

$$[\vec{v}_\phi, \vec{v}_\psi] = \vec{v}_\omega \quad \Leftrightarrow \quad \omega = \{\phi, \psi\} \equiv \mathfrak{Z}_\phi(\psi) - \mathfrak{Z}_\psi(\phi) \;, \tag{2.8}$$

where the operator $\mathfrak{Z}$ maps an arbitrary function on the jet bundle, say α, into a linear, (first-order) differential operator acting on (other) functions on that infinite jet. This operator is a sum of derivatives with respect to each of the coordinates on the jet, excluding the independent variables, with a coefficient that depends on the differential concomitants of α. Those coefficients are defined in the following way: We associate with each of the coordinates on the (restricted) infinite jet a product of total derivative operators which would act on the basic coordinate Ω to create that particular coordinate; i.e., if the coordinate in question is $\Omega_{(\sigma)}$, then we denote that product of total derivative operators by $\overline{D}_{(\sigma)}$. An example would be $\overline{D}_x \overline{D}_s \Omega = \Omega_{xs}$; note that $(\sigma) = 0$ corresponds to just the identity operator. The corresponding coefficient is then the result of letting that product of derivative operators act on α:

$$\begin{aligned} \mathfrak{Z}_\alpha\, \beta &\equiv \sum_{\sigma=0}^{(\infty)} [\overline{D}_{(\sigma)}(\alpha)]\, \partial_{\Omega_{(\sigma)}} \beta \\ &= \Big\{ \alpha\, \partial_\Omega + [\overline{D}_x(\alpha)]\, \partial_{\Omega_x} + [\overline{D}_x^2(\alpha)]\, \partial_{\Omega_{xx}} + \;\ldots\; + [\overline{D}_y(\alpha)]\, \partial_{\Omega_y} + [\overline{D}_y^2(\alpha)]\, \partial_{\Omega_{yy}} + \;\ldots \\ &\qquad + [\overline{D}_s(\alpha)]\, \partial_{\Omega_s} + [\overline{D}_s \overline{D}_x(\alpha)]\, \partial_{\Omega_{sx}} + \ldots + [\overline{D}_s \overline{D}_y(\alpha)]\, \partial_{\Omega_{sy}} + \ldots \Big\} \beta \;, \end{aligned} \tag{2.9}$$

where α and β are two arbitrary functions on the (restricted) jet. Therefore, for two of our characteristics, as described above in Eqs. (2.7), we would have

$$\{X_1^n, X_1^m\} \equiv \mathfrak{Z}_{X_1^n}\, X_1^m - \mathfrak{Z}_{X_1^m}\, X_1^n \;. \tag{2.10}$$

As was already noted, the infinite algebra of Lie symmetries has been known for some time[8]. On the other hand, we were quite surprised when we attempted to solve this equation by allowing φ to depend on coordinates on higher-level jet bundles. When this search was carefully made, we found that there were none! This was particularly troublesome since we were certainly aware of the (doubly-infinite) hierarchy of commuting flows discovered by Takasaki and Takebe[9], which should certainly be related to the desired generalized symmetries. After considerable thought, we decided that the problem might well be analogous to the behavior of the generalized symmetries for the KdV equation, as explained by Krasil'shchik[15]. In that case, there is a very well-known, infinite hierarchy of commuting flows, which is one-to-one related with an infinite, Abelian algebra of generalized symmetries. Even though this algebra is Abelian, there is a recursion operator for these symmetries, originally found by Olver[14]. Krasil'shchik and Vinogradov[15] showed that one may generalize that Abelian algebra to a larger, no-longer-Abelian algebra by prolonging the original infinite jet with an additional set of fibers, referred to by them as *coverings* of $J^{(\infty)}$. Their prolongation is defined by the introduction of a potential of the original dependent variable, i.e., a first integral of that variable, and its higher derivatives, as coordinates on these fibers. This allowed them to use the non-zero commutators in this enlarged algebra to derive the (already-known) form of Olver's recursion operator.

Indeed our pde has two very obvious potentializations, based on integrals with respect to s, that are well known in the literature:

$$
\begin{aligned}
\Theta_{,xy} &= -e^{\Theta_{,ss}} \ , \\
\Phi_{,xy} &= -\left(e^{\Phi_{,s}}\right)_{,s} = -\Phi_{,ss}\, e^{\Phi_{,s}} \ , \quad \Phi = \Theta_{,s} \ , \\
\Omega_{,xy} &= -\left(e^{\Omega}\right)_{,ss} = -\left[\Omega_{,ss} + (\Omega_{,s})^2\right] e^{\Omega} \ , \quad \Omega = \Phi_{,s} = \Theta_{,ss} \ .
\end{aligned}
\tag{2.11}
$$

To include these potentials in our bundle, we must prolong the jet bundle with still additional (infinite-dimensional) fibers, which may be defined as having coordinates, for the first potential, $\{\Phi,\, \Phi_x,\, \Phi_{xx},\ \ldots,\ \Phi_y,\, \Phi_{yy},\ \ldots\}$, and then also $\{\Theta,\, \Theta_x,\, \Theta_{xx},\ \ldots,\ \Theta_y,\, \Theta_{yy},\ \ldots\}$, for the second potential. Having done this, we must also prolong the total derivatives accordingly. This gives the following result, where, again, the sum is over the coordinates already given above as appropriate, and we denote the prolongation of the original total derivatives with a caret over the symbol:

$$
\widehat{\overline{D}}_x = \overline{D}_x + \sum_{(\sigma)=0}^{(\infty)} \Phi_{(\sigma)x} \partial_{\Phi_{(\sigma)}} + \sum_{(\sigma)=0}^{(\infty)} \Theta_{(\sigma)x} \partial_{\Theta_{(\sigma)}} \ , \quad \widehat{\overline{D}}_y = \overline{D}_y + \sum_{(\sigma)=0}^{(\infty)} \Phi_{(\sigma)y} \partial_{\Phi_{(\sigma)}} + \sum_{(\sigma)=0}^{(\infty)} \Theta_{(\sigma)y} \partial_{\Theta_{(\sigma)}} \ ,
$$

$$
\widehat{\overline{D}}_s = \overline{D}_s + \sum_{(\sigma)=0}^{(\infty)} \Omega_{(\sigma)} \partial_{\Phi_{(\sigma)}} + \sum_{(\sigma)=0}^{(\infty)} \Phi_{(\sigma)} \partial_{\Theta_{(\sigma)}} \ . \tag{2.12}
$$

As before, those "derivatives" of Φ, or Θ, that correspond to mixed x- and y-derivatives are to be determined by the pde's given above, in Eqs. (2.11), while those that correspond to s-derivatives are of course already determined in terms of derivatives of Ω.

The first new potential, Φ, already allows two new solutions to the equation for generalized symmetries (for our original pde), one of which involves Φ_x and Φ_{xx}, and an arbitrary function of x, and the other involves the same sorts of objects, involving the independent coordinate y:

$$
\begin{aligned}
GX_2[A] &= 2A(x)\, X_2 + A'(x)(s\Omega_x + 2\Phi_x) + A''(x)\, s \ , \\
GY_2[B] &= 2B(y)\, Y_2 + B'(y)(s\Omega_y + 2\Phi_y) + B''(y)\, s \ , \\
X_2 &\equiv \Phi_{xx} + \Phi_x \Omega_x \ , \quad Y_2 \equiv \Phi_{yy} + \Phi_y \Omega_y \ .
\end{aligned}
\tag{2.13}
$$

As these characteristics involve simple, explicit polynomials in s, as well as their dependence on either x or y, their commutation relations will be more complicated, and more interesting. However, in order to compute those commutation relations we must also prolong the $\mathfrak{Z}$ operator to this (larger) prolonged bundle. This prolongation is slightly more complicated than before, because the coefficient of an arbitrary derivative in the linearization operator, $\mathfrak{Z}$, involves that particular action of total derivative operators that generates the new coordinate from Ω. Since $\Omega = \Phi_s$, so that Φ is the first s-integral of Ω, the term in the operator $\mathfrak{Z}_\alpha$ that involves ∂_Φ will need as coefficient the first s-integral of α. For arbitrary functions α, this obviously cannot be done in any closed form; however, we only need this operator to act on characteristics of symmetries. We will see that all of our characteristics may in fact be written as perfect s-derivatives of other quantities, and even perfect second s-derivatives, of yet other quantities defined on the (sufficiently-prolonged) jet bundle. The existence of these second s-derivatives will be needed when, eventually, our characteristics involve Θ, which is defined via $\Theta_{ss} = \Omega$. For the moment we will simply introduce the operators ${\overline{D}_s}^{-1}$ and ${\overline{D}_s}^{-2}$ for this purpose, although we will eventually become more systematic about it. We also understand that such an "integration" is not unique, and the form is even dependent on one's choice of coordinates. This lack of uniqueness has generated some amount of discussion in the literature. Our approach is similar to that given by Guthrie[16], who desires that all such "integrations" should in fact be re-described so that equations containing an inverse (of a differential) operator are replaced by a system of first order differential equations to be resolved. We begin with some such equations here, but will use such an approach even more systematically in the next section. We use the following differential equations to define the potentializations in that sense:

$$
\begin{aligned}
GX_1[A] &= A\Omega_x + A' = \overline{D}_s\Big\{A\Phi_x + A's\Big\} = \overline{D}_s^2\Big\{A\Theta_x + \tfrac{1}{2}A's^2\Big\}\ ,\\
GY_1[B] &= B\Omega_y + B' = \overline{D}_s\Big\{B\Phi_y + B's\Big\} = \overline{D}_s^2\Big\{B\Theta_y + \tfrac{1}{2}B's^2\Big\}\ ,\\
GS(\alpha,\beta) &= (\alpha s+\beta)\Omega_s - 2\alpha = \overline{D}_s\Big\{(\alpha s+\beta)\Omega - \alpha\Phi - 2\alpha s\Big\}\\
&\qquad = \overline{D}_s^2\Big\{(\alpha s+\beta)\Phi - \alpha\Theta - \alpha s^2\Big\}\ .\\
GX_2[A] &= \overline{D}_s\Big\{2A(\Theta_{xx} + \tfrac{1}{2}\Phi_x^2) + A'(s\Phi_x + 2\Theta_x) + \tfrac{1}{2}A''s^2\Big\}\ ,\\
GY_2[B] &= \overline{D}_s\Big\{2B(\Theta_{yy} + \tfrac{1}{2}\Phi_y^2) + B'(s\Phi_y + 2\Theta_y) + \tfrac{1}{2}B''s^2\Big\}\ .
\end{aligned}
\tag{2.14}
$$

This allows us to write down the appropriate prolongations for the form of the linearization operator appropriate at this stage, where the "caret" indicates that this is a prolongation of the original operator:

$$
\widehat{\mathfrak{Z}}_\alpha = \mathfrak{Z}_\alpha + \sum_{\sigma=0}^{(\infty)}\{\widehat{\overline{D}}_{(\sigma)}\overline{D}_s^{\,-1}(\alpha)\}\partial_{\Phi_{(\sigma)}} + \sum_{\sigma=0}^{(\infty)}\{\widehat{\overline{D}}_{(\sigma)}\overline{D}_s^{\,-2}(\alpha)\}\partial_{\Theta_{(\sigma)}}\ . \tag{2.15}
$$

With this prolongation, the calculation of the commutators with our earlier Lie characteristics is straight-forward:

$$
\begin{aligned}
\{GX_2[A], GX_1[R]\} &= GX_2[RA' - 2AR'] &&\Leftrightarrow &&\{X_2^a, X_1^b\} = (a-2b)\,X_2^{a+b-1}\ ,\\
\{GX_2[A], GS(\alpha,\beta)\} &= \alpha GX_2[A] + \beta GX_1[A'] &&\Leftrightarrow &&\{X_2^a, GS(\alpha,\beta)\} = \alpha X_2^a + a\beta X_1^{a-1}\ ,\\
\{GY_2[B], GS(\alpha,\beta)\} &= \alpha GY_2[B] + \beta GY_1[B'] &&\Leftrightarrow &&\{Y_2^a, GS(\alpha,\beta)\} = \alpha Y_2^a + a\beta Y_1^{a-1}\ ,\\
\{GY_2[B], GY_1[S]\} &= GY_2[SB' - 2BS'] &&\Leftrightarrow &&\{Y_2^a, Y_1^b\} = (a-2b)\,Y_2^{a+b-1}\ ,\\
\{GX_2[A], GY_1[B]\} &= 0 = \{GX_1[A], GY_2[B]\} &&\Leftrightarrow &&\{X_2^n, Y_1^m\} = 0 = \{X_1^n, Y_2^m\}\ ,
\end{aligned}
\tag{2.16}
$$

where as before, at Eqs. (2.7), we must use arbitrary functions to parametrize our set of characteristics by defining a basis in the following way:

$$
X_2^n \equiv GX_2[x^n]\ ,\quad Y_2^m \equiv GY_2[y^m]\ ,\quad \text{and}\quad X_2 \equiv X_2^0 = \tfrac{1}{2}\,GX_2[1]\ ,\quad Y_2 \equiv \tfrac{1}{2}\,Y_2^0 = GY_2[1]\ , \tag{2.17}
$$

The extra factor of one half in the definition of the symbols X_2 and Y_2 differs from the similar definition, for X_1 and Y_1, in Eqs. (2.7a). We will say more about this as we find more characteristics.

At this point we would like to determine the commutator of two different versions of this newer characteristic. As the commutator of two characteristics is always again a (linear combination of) characteristics, the fact that this commutator turns out to be non-zero provides a desirable object, namely a "recursion operator," that will generate higher-order characteristics from the lower ones, just as was the case with the Olver recursion operator, or the Krasil'shchik version of it, for the KdV equation. The calculation of this commutator will require the use of the second potential, Θ, such that $\overline{D}_s^2\Theta = \Omega$, and the further prolongations involving it, and will give us two new characteristics:

$$\begin{aligned}
\{GX_2[A], GX_2[R]\} &= GX_3[2RA' - 2AR'] &\quad\Leftrightarrow\quad& \{X_2^a, X_2^b\} = 2(a-b)\, X_3^{a+b-1}\ ,\\
\{GY_2[B], GY_2[S]\} &= GY_3[2SB' - 2BS'] &\quad\Leftrightarrow\quad& \{Y_2^a, Y_2^b\} = 2(a-b)\, Y_3^{a+b-1}\ ,\\
\{GX_2[A], GY_2[B]\} &= 0 &\quad\Leftrightarrow\quad& \{X_2^a, Y_2^b\} = 0\ ,
\end{aligned} \tag{2.18}$$

where the quantities $GX_3[A(x)]$ and $GY_3[B(y)]$ are our new symmetry characteristics, one for the "x-direction," and one for the "y-direction." These have the following forms:

$$\begin{aligned}
GX_3[A] &= 3A(x)\, X_3 + 2A'(x)[sX_2 + 2\Theta_{xx} + \tfrac{3}{2}\Phi_x^2 + \tfrac{1}{2}\Theta_x\Omega_x]\\
&\qquad + A''(x)(\tfrac{1}{2}s^2\Omega_x + 2s\Phi_x + \Theta_x) + \tfrac{1}{2}A'''(x)s^2\ ,\\
X_3 &\equiv \Theta_{xxx} + 2\Phi_x\Phi_{xx} + \Omega_x(\Theta_{xx} + \Phi_x^2)\ ;\\
GY_3[B] &= 3B(y)\, Y_3 + 2B'(y)[sY_2 + 2\Theta_{yy} + \tfrac{3}{2}\Phi_y^2 + \tfrac{1}{2}\Theta_y\Omega_y]\\
&\qquad + B''(y)(\tfrac{1}{2}s^2\Omega_y + 2s\Phi_y + \Theta_y) + \tfrac{1}{2}B'''(y)s^2\ ,\\
Y_3 &\equiv \Theta_{yyy} + 2\Phi_y\Phi_{yy} + \Omega_y(\Theta_{yy} + \Phi_y^2)\ ,
\end{aligned} \tag{2.19}$$

where we have now defined the Abelian elements of this set by $X_3 \equiv \frac{1}{3}\, X_3^0$ and the same for Y_3.

At this point we note that the recursive nature of our commutators, with these higher-order characteristics, comes about because we are allowed to use arbitrary functions, instead of simply constants. The restriction of these characteristics when the arbitrary functions are chosen to be just constants, and therefore normalized to have value 1, are the quantities we have been describing as $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$. They are "Abelian characteristics" in the sense that they commute one with another, i.e., their span constitutes an Abelian (sub-)algebra of the entire algebra of characteristics. In fact, they are exactly that subalgebra that creates the compatible flows discovered by Takasaki and Takebe[9]. In general the presentation of these Abelian restrictions will be simplified by the use of a factor of $1/n$, as will be described more generically below.

We also already have enough structure to compute commutators for this new characteristic with the Lie symmetries:

$$\begin{aligned}
\{GX_3[A], GX_1[R]\} &= GX_3[RA' - 3AR'] &\quad\Leftrightarrow\quad& \{X_3^a, X_1^b\} = (a-3b)\, X_3^{a+b-1}\ .\\
\{GY_3[B], GY_1[S]\} &= GY_3[SB' - 3BS'] &\quad\Leftrightarrow\quad& \{Y_3^a, Y_1^b\} = (a-3b)\, Y_3^{a+b-1}\ ,\\
&\text{and also}\quad \left\{\begin{aligned}&\{GX_3[A], GS(\alpha,\beta)\} = 2\alpha\, GX_3[A] + \beta\, GX_2[A']\ ,\\ &\{GY_3[B], GS(\alpha,\beta)\} = 2\alpha\, GY_3[B] + \beta\, GY_2[B']\ .\end{aligned}\right.
\end{aligned} \tag{2.20}$$

However, the next plausible commutator cannot yet be computed because we do not have a structure that allows to determine the s-integrals of $GX_3[A]$, nor even the second s-integral of $GX_2[A]$. Since we have generated one pair of new characteristics—depending on one pair of arbitrary functions of one variable—for each new potential introduced into the bundle, it seems plausible to now introduce yet more new potentials. On the other hand, while the earlier potentials were obvious as simple integrals, the next ones are certainly no longer obvious. There are of course similar questions that occur in the study of the KP equation, for example, where the standard (Japanese school[17]) approach involves an infinite hierarchy of dependent variables, all satisfying more- and more-involved equations as one climbs upward in the hierarchy. Therefore, we used as a guide the hierarchical approach to this equation taken by Takasaki and Takebe[9], as already mentioned. We introduce (re-normalized versions of) their (infinite sequences of) quantities v_k and $\hat{v}_j$. This set of potentials is defined via (two) first-order pde's that define the solutions as first s-integrals of differential polynomials in the preceding potentials, and has many convenient aspects for the problem. In the next section a general approach will be given for an infinite hierarchy of such potentials.

3. Prolongations for An Infinite Hierarchy of Potentials

The previous two integrals, of our original dependent variable, $\Omega = \Omega(x, y, s)$, were very natural in the current context. The next ones are somewhat more complicated since they involve integrands nonlinear in the previous variables. Those first two potentials, in the previous section, allowed us to determine two new characteristics each, but required prolongation to new fibers which required the jet coordinates for all their x- and y-derivatives, although not of course their s-derivatives. The newer potentials we will introduce now will come in pairs, as is required to maintain the symmetry between the x- and y-directions, since each one will only allow a single new characteristic. However, because of this they will only require new fiber coordinates in all derivatives with respect to a single one of the variables x or y, with the derivatives with respect to the other variables being given by the pair of defining (first-order) pde's. Therefore the total number of new fiber dimensions introduced will be the same as before, for each new characteristic.

With an aim toward a better explication of a fairly complicated process, we will initially introduce just the first pair of newer potentials, and go through the process they engender to generate their associated (pair of) new characteristics. Then we will retreat and set down a general formulation that allows us to define the entire new infinite set of pairs. Therefore, we now introduce a new pair of potentials, q_2 and w_2. Each of them is defined as the solution of a pair of first-order pde's, which are compatible because of the original pde:

$$
\begin{aligned}
&q_2 \;\text{ defined by } \begin{cases} \widehat{\widehat{D}}_s\, q_2 = \Theta_{xx} + \frac{1}{2}\Phi_x^2 \;\equiv\; \eta_2 \,, \quad \text{with } \; X_2 = \widehat{\widehat{D}}_s\, \eta_2 \,, \\ \widehat{\widehat{D}}_y\, q_2 = -\,\Phi_x\, e^{\Omega} \;\equiv\; -\rho_2\, e^{\Omega} \,, \end{cases} \\
&w_2 \;\text{ defined by } \begin{cases} \widehat{\widehat{D}}_s\, w_2 = \Theta_{yy} + \frac{1}{2}\Phi_y^2 \;\equiv\; \zeta_2 \,, \quad \text{with } \; Y_2 = \widehat{\widehat{D}}_s\, \zeta_2 \,, \\ \widehat{\widehat{D}}_x\, w_2 = -\,\Phi_y\, e^{\Omega} \;\equiv\; -\sigma_2\, e^{\Omega} \,. \end{cases}
\end{aligned}
\tag{3.1}
$$

This pair of potentials allows us to re-consider the second characteristic as a second s-derivative, namely $X_2 = {\widehat{\widehat{D}}_s}^{\,2}\, q_2$. In fact it also provides enough structure to write down the third characteristic family as a first s-integral. The general forms for the second- and third-level characteristics are given in Eqs. (2.13) and Eqs. (2.19), respectively. With these additional potentials, we may now re-write them as s-derivatives of more primitive structures, which we do below, allowing, in each, for an arbitrary function $A = A(x)$:

$$
\begin{aligned}
GX_2[A] &= \widehat{\widehat{D}}_s{}^2 \Big\{ A q_2 + \tfrac{1}{2} A' s \Theta_x + \tfrac{1}{12} A'' s^3 \Big\} \,, \\
GX_3[A] &= \widehat{\widehat{D}}_s \Big\{ A(q_{2x} + \Theta_{xx}\Phi_x + \tfrac{1}{3}\Phi_x^3) + \tfrac{2}{3} A' [s(\Theta_{xx} + \tfrac{1}{2}\Phi_x^2) + q_2 + \tfrac{1}{2}\Theta_x \Phi_x] \\
&\qquad + \tfrac{1}{6} A'' s (s\Phi_x + 2\Theta_x) + \tfrac{1}{18} A''' s^3 \Big\} \,.
\end{aligned} \tag{3.2}
$$

We do not bother to write down the associated formulations for $GY_2[B]$ and $GY_3[B]$, as they are completely identical modulo changing x's to y's, and also q_2 to w_2. On the other hand, it is of course important that the total derivatives have been prolonged to accommodate the new fibres which may have coordinates chosen as $\{q_2, q_{2x}, q_{2xx}, \ldots\}$ and also $\{w_2, w_{2y}, w_{2yy}, \ldots\}$. As well the associated linearization operator, $\mathfrak{Z}$, must be prolonged to this next level as well. However, we assume that those prolongations have been performed at this point, but defer the explicit explanation of how it is done until we describe the complete structure, beginning with the paragraph that contains Eqs. (3.4). On the other hand, we do now, again, in this jet bundle with a prolongation to 4 sets of additional fibers, have sufficient structure to calculate yet one more characteristic. That this calculation gives a non-zero result, again shows the value of $GX_2[R(x)]$ as a generating function for new symmetry characteristics:

$$
\{GX_3[A], GX_2[R]\} \equiv GX_4[2RA' - 3AR'] \Leftrightarrow \; \{X_3^a, X_2^b\} = (2a - 3b)\, X_4^{a+b-1} \,,
$$

$$
\begin{aligned}
GX_4[A] \equiv{}& 4A(x)\, X_4 + 3A'(x)[sX_3 + 2\eta_3 + \tfrac{2}{3}(\Theta_x X_2 + 2\Phi_x \eta_2 + \Omega_x q_2)] + \tfrac{1}{6} A'''' s^3 \\
&+ A''[s^2 W_2 + 4s\eta_2 + 2q_2 + s(\Theta_x \Omega_x + \Phi_x^2) + 2\Theta_x \Phi_x] + A''' s (\tfrac{1}{6} s^2 \Omega_x + s\Phi_x + \Theta_x) \,,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
X_4 \equiv \tfrac{1}{4} X_4^0 &= q_{2xx} + 2\Phi_x \Theta_{xxx} + 2\Phi_{xx}\Theta_{xx} + 3\Phi_x^2 \Phi_{xx} + \Omega_x (q_{2x} + 2\Phi_x \Theta_{xx} + \Phi_x^3) \,, \\
\eta_3 &\equiv q_{2x} + \Theta_{xx}\Phi_x + \tfrac{1}{3}\Phi_x^3 = \widehat{\widehat{D}}_s{}^{-1}(X_3) \,, \qquad \eta_2 \equiv \Theta_{xx} + \tfrac{1}{2}\Phi_x^2 \,.
\end{aligned}
$$

To calculate additional commutators, and characteristics, we must define yet another pair of potentials, q_3 and w_3, and perform appropriate prolongations. It is therefore, instead, time to go ahead and describe the details of the entire sequence of (pairs of) potentials that we want to introduce, which will allow us to introduce the entire sequence of (pairs of) characteristics for our equation. Therefore, we define a doubly-infinite sequence of (nonlinear) potentials, $\{q_j, w_j \mid j = 0, 1, 2, \ldots\}$, which will allow the description of a doubly-infinite sequence of Abelian characteristics, $\{X_j, Y_j \mid j = 0, 1, 2, \ldots\}$, for symmetries. Following the mode of description used for q_2, and w_2 in Eqs. (3.1), we define, for instance, q_j by giving the system of (compatible) pde's that define its s- and y-derivatives in terms of lower-order quantities. We do this via a pair of intermediary functions, $\{\eta_j \mid j = 0, 1, \ldots\}$ and $\{\rho_j \mid j = 0, 1, \ldots\}$, which are "mid-way" between an Abelian symmetry characteristic and its associated potential:

$$
\left.
\begin{aligned}
\widehat{\widehat{D}}_s\, q_j &= \eta_j \,, \\
\widehat{\widehat{D}}_y\, q_j &= -\rho_j\, e^{\Omega}
\end{aligned}
\right\}
\quad \Longrightarrow \quad X_j = \widehat{\widehat{D}}_s\, \eta_j = \widehat{\widehat{D}}_s{}^2\, q_j \,. \tag{3.4}
$$

As already noted after Eqs. (2.11), the inclusion of new potentials into our jet bundle requires not only the prolongation of the original bundle to include these quantities themselves but also their further prolongation to the infinite jet. More precisely, this is a prolongation of the space, N, of dependent variables, or, equivalently $J^{(1)}(M,N)/J^{(0)}(M)$. When this larger space is extended, now, to its infinite jet the new coordinates on the additional fibers could be taken, for instance, as $\{q_j, q_{j(\sigma)} \mid j = 0,1,2,\ldots\ ; |(\sigma)| = 1,2,3,\ldots\}$, with all possible combinations of x, y, and s included in the list denoted by (σ), with $|(\sigma)|$ indicating its length. On the other hand, remembering the role of the q_j's as potentials, here we only want the restricted variety in that prolongation, i.e., the prolongation of our earlier variety, Y_∞. Referring to that prolonged variety by $\widehat{Y}_\infty$, it is a surface defined by all the pde's in the new version of the system. Therefore, as the sets $\{q_{js}\}$ and $\{q_{jy}\}$, for all non-zero values of j, are defined by Eqs. (3.4) above, we see that the fibers in this prolonged variety need only have $\{q_j, q_{jx}, q_{jxx}, q_{jxxx}, \ldots\}$ as additional coordinates, with all other new coordinates being reduced to the status of co-coordinates by those pde's in Eqs. (3.4). Of course, when we do also consider the case for the alternate (infinite) set of potentials, w_k, those must also be included. We will describe them soon, but will first explain in detail these (differential) polynomials, η_j and ρ_k.

These new functions, η_j and ρ_j, are (weighted) polynomials over the set of quantities $\{\widehat{\widehat{D}}_x\, q_m \equiv q_{mx} \mid m = 0,\ldots, j-1\}$, only, involving no higher (or lower) coordinates on the infinitely prolonged variety $\widehat{Y}_{(\infty)}$. In terms of these coordinates, the η_j may be written out explicitly, in terms of a sum over all the (additive) partitions of their (integer) index:

$$
\begin{aligned}
\eta_k = & \sum_{a\in P(k)} \frac{(|a|-1)!}{\{a\}!} \prod_{j=1}^{k} \left(\widehat{\widehat{D}}_x q_{j-1}\right)^{a_j} = \sum_{a\in P(k)} \begin{pmatrix} |a|-1 \\ a_1\, a_2\, \ldots\, a_k \end{pmatrix} \prod_{j=1}^{k} \left(\widehat{\widehat{D}}_x q_{j-1}\right)^{a_j} \\
& \eta_0 = \Omega\ , \\
& \eta_1 = q_{0x}\ , \\
& \eta_2 = q_{1x} + \tfrac{1}{2}(q_{0x})^2\ , \\
& \eta_3 = q_{2x} + q_{1x}q_{0x} + \tfrac{1}{3}(q_{0x})^3\ , \\
& \eta_4 = q_{3x} + q_{2x}q_{0x} + \tfrac{1}{2}(q_{1x})^2 + q_{1x}(q_{0x})^2 + \tfrac{1}{4}(q_{0x})^4\ , \\
& \eta_5 = q_{4x} + q_{3x}q_{0x} + q_{2x}q_{1x} + q_{2x}(q_{0x})^2 + (q_{1x})^2 q_{0x} + q_{1x}(q_{0x})^3 + \tfrac{1}{5}(q_{0x})^5\ , \\
& \qquad \ldots \qquad .
\end{aligned}
\tag{3.5}
$$

The differential polynomials ρ_k are closely related to the η_{k-1}, being a sum of the same terms, but with different coefficients:

$$
\rho_k = \sum_{a\in P(k-1)} \frac{|a|!}{\{a\}!} \prod_{j=1}^{k-1} (\widehat{\widehat{D}}_x q_{j-1})^{a_j} = \left\{ \sum_{m=0}^{k-2} q_{mx} \frac{\partial}{\partial q_{mx}} \right\} \eta_{k-1}\ , \qquad k \geq 2\ ,
$$

$$
\begin{aligned}
& \rho_1 = 1\ , \\
& \rho_2 = q_{0x}\ , \\
& \rho_3 = q_{1x} + (q_{0x})^2\ , \\
& \rho_4 = q_{2x} + 2q_{1x}q_{0x} + (q_{0x})^3\ , \\
& \rho_5 = q_{3x} + 2q_{2x}q_{0x} + (q_{1x})^2 + 3q_{1x}(q_{0x})^2 + (q_{0x})^4\ , \\
& \rho_6 = q_{4x} + 2q_{3x}q_{0x} + 2q_{2x}q_{1x} + 3q_{2x}(q_{0x})^2 + 3(q_{1x})^2 q_{0x} + 4q_{1x}(q_{0x})^3 + (q_{0x})^5\ , \\
& \qquad \ldots \qquad .
\end{aligned}
\tag{3.6}
$$

The other sequence of new potentials, $\{w_j \mid j = 0, 1, 2, \dots\}$, are related to similar functions ζ_j and σ_j, polynomials in the $\{w_{my} \equiv \widehat{\widehat{D}}_y w_m \mid m = 0, 1, \dots, j-1\}$, coordinates on the prolonged $J^{(1)}$, in exactly the same way as before for the η_j and ρ_j, except that one must change all x's to y's, and also all q_j's to w_j's:

$$\left.\begin{aligned} \widehat{\widehat{D}}_s\, w_j &= \zeta_j \ , \\ \widehat{\widehat{D}}_x\, w_j &= \ -\, \sigma_j\, e^{\Omega} \end{aligned}\right\} \qquad \Longrightarrow \qquad Y_j = \widehat{\widehat{D}}_s\, \zeta_j = \widehat{\widehat{D}}_s{}^2\, w_j \ . \tag{3.7}$$

Because these pde's define the sets $\{w_{j,x}, w_{j,s} \mid j = 0, 1, 2, \dots\}$ as co-coordinates, we must only include as (new) coordinates for our prolonged variety, $\widehat{Y}_{(\infty)}$, the set $\{w_j, w_{jy}, w_{jyy}, w_{jyyy}, \dots\}$ for each value of j.

The indices for the q_j and w_k, and, especially the potentials q_2 and w_2, were chosen quite deliberately since the definitions "backtrack" so that this sequence includes the simpler (linear) potentials, Φ and Θ already introduced. We use Φ as an initial point for both sequences, but then diverge from there, using instead Θ_x as q_1 and Θ_y as w_1:

$$\begin{gathered} \widehat{\widehat{D}}_s\, q_0 \equiv \Omega = \widehat{\widehat{D}}_s\, w_0 \quad \Longrightarrow \ q_0 = \Phi = w_0 \ , \\ \widehat{\widehat{D}}_s\, q_1 \equiv \widehat{\widehat{D}}_x\, q_0 \ , \widehat{\widehat{D}}_y\, q_1 \equiv -e^{\Omega} \ ; \quad \widehat{\widehat{D}}_s\, w_1 \equiv \widehat{\widehat{D}}_y\, w_0 \ , \widehat{\widehat{D}}_x\, w_1 \equiv -e^{\Omega} \quad \Longrightarrow \ q_1 = \Theta_x \ , \ w_1 = \Theta_y \ , \end{gathered} \tag{3.8}$$

and of course use the definitions given just above, Eqs. (3.1), for q_2 and w_2.

That all these pde's are compatible is just a consequence of the pde itself. Alternatively, one may say that they simply are a re-definition of the doubly-infinite hierarchy of commuting flows for this pde, given already by Takasaki and Takebe[9]. In particular, since all the equations in that hierarchy constitute distinct, commuting flows over the manifold, the various flow parameters along those curves may be taken as new, independent variables. These variables are just the doubly-infinite set of potentials which we have taken, instead, as additional variables to constitute prolongations of our original jet bundle. An additional fascinating and unexpected consequence of these definitions, and their compatibility with the original pde, is the fact that they satisfy a "linearization" of the original pde:

$$\widehat{\widehat{D}}_x \widehat{\widehat{D}}_y\, q_k + e^{\Omega}\, \widehat{\widehat{D}}_s{}^2\, q_k = 0 \ . \tag{3.9}$$

Of course the pde is not truly linear since the q_k's and Ω are tightly related via other pde's.

Appendix B has some details of what little part of the theory of additive partitions of integers that we need, this theory having been elaborated and studied in many ways. Here we simply note that the set of all additive partitions of an integer k we denote by the symbol $\mathcal{P}(k)$. If a is an element of this set, i.e., $a \in \mathcal{P}(k)$, then a is an ordered list of integers, $a_i \le k$, where a_i tells us how many times the integer i is repeated in that particular partition; obviously $1 \le i \le k$, and in any particular partition, many of the a_i's will be 0:

$$a \in \mathcal{P}(k) \iff a \equiv \{a_1, a_2, a_3, \ \dots \ , a_k\} \ , \ a_p \ge 0 \ , \qquad \text{such that } k = \sum_{p=1}^{k} p\, a_p \ . \tag{3.10a}$$

Two very useful functions on these lists, $|a|$ and $\{a\}!$, will be used often:

$$|a| \equiv \sum_{p=1}^{k} a_p \ , \qquad \text{and also} \quad \{a\}! \equiv \prod_{p=1}^{k} (a_p)! \ . \tag{3.10b}$$

Obviously $|a|$ satisfies the constraint that it must not be larger than k.

Having the explicit sequence of these polynomials, from which our potentials, q_j (and also w_k) are first integrals, it is straightforward to calculate the sequence of Abelian characteristics, X_j. They can be determined either from the polynomials η_j or from the ρ_j:

$$\begin{aligned} X_j = \widehat{\overline{D}}_s\eta_j \; &= \widehat{\overline{D}}_s{}^2 q_j = e^{-\Omega}\widehat{\overline{D}}_x\left(e^{+\Omega}\rho_j\right) = \widehat{\overline{D}}_x\rho_j + \Omega_x\,\rho_j \\ &= \; q_{j-2,xx} + 2q_{0x}q_{j-3,xx} + \;\ldots\; + \Omega_x(q_{j-2,x} + \;\ldots\; + q_{0x}^{j-1})\ , \end{aligned}$$

$$\begin{gathered} X_1 = \Omega_x\ , \qquad\qquad X_2 = q_{0xx} + \Omega_x q_{0x}\ , \\ X_3 \;=\; q_{1xx} + 2q_{0x}q_{0xx} + \Omega_x(q_{1x} + q_{0x}^2)\ , \\ X_4 = \; q_{2xx} + 2q_{0x}q_{1xx} + 2q_{0xx}q_{1x} + 3q_{0x}^2q_{0xx} + \Omega_x(q_{2x} + 2q_{0x}q_{1x} + q_{0x}^3)\ , \\ \ldots\qquad . \end{gathered} \tag{3.11}$$

(In the second line above we use commas to separate x from a complicated index value such as $j-3$, just to make the meaning clear.) Of course the Y_k's are made in the same way. As already noted these X_j's, and separately the Y_k's, form Abelian algebras of characteristics for generalized symmetries.

To determine the more general versions of these characteristics, that involve arbitrary functions (of one variable), we must first establish the complete prolongation of the total derivatives, and also of the linearization operator. The new, infinitely-prolonged total derivatives then have the form:

$$\begin{aligned} \widehat{\overline{D}}_x \; &= \overline{D}_x + \Phi_x\partial_\Phi + \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\left\{q_{k,(m+1)}\partial_{q_{k,(m)}} + \widetilde{w_{k,x(m)}}\partial_{w_{k,(m)}}\right\}\ , \\ \widehat{\overline{D}}_y \; &= \overline{D}_y + \Phi_y\partial_\Phi + \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\left\{\widetilde{q_{k,y(m)}}\partial_{q_{k,(m)}} + w_{k,(m+1)}\partial_{w_{k,(m)}}\right\}\ , \\ \widehat{\overline{D}}_s \; &= \overline{D}_s + \sum_{k=0}^{\infty}\left\{\sum_{m=0}^{\infty}\eta_{k,(m)}\partial_{q_{k,(m)}} + \sum_{n=0}^{\infty}\zeta_{k,(n)}\partial_{w_{k,(n)}}\right\}\ , \end{aligned} \tag{3.12}$$

where, for instance, the notation $q_{k,(m)}$ means the coordinate on the prolonged bundle that is equal to $(\widehat{\overline{D}}_x)^m\,q_k$. As before the (prolonged) co-coordinates, denoted with over-tildes, always correspond to derivatives of some q_k or w_j that involve both x and y-values of the independent variables. For instance, those with one y and m x's on a q_k, i.e., $\widetilde{q_{k,y(m)}}$, are determined by the action of $(\widehat{\overline{D}}_x)^m$ on $\widetilde{q_{k,y}}$, which is given by the differential polynomial ρ_k, above; likewise those with one x and m y's on a w_j, i.e., $\widetilde{w_{k,x(m)}}$, are determined by the action of $(\widehat{\overline{D}}_y)^m$ on $\widetilde{w_{j,x}}$, determined by the polynomial σ_j. [The use of this newer notation changes slightly the earlier form of the prolongation: since $q_0 = \Phi = w_0$, all the coordinates related to Φ, that appear in Eqs. (2.4) are all still contained in this newer version, and not counted twice; and, since $q_1 = \Theta_x$ and $w_1 = \Theta_y$, all terms that were in Eqs. (2.12) related to Θ are here also, with one exception. That exception is the quantity Θ itself, as opposed to any of its derivatives. It appears that Θ itself is never explicitly necessary in the prolongation structure; only its derivatives are ever used. On the other hand, do note the comments in the conclusions concerning the possible relationship between e^Θ and a τ-function for this problem.]

For an α defined over the complete, prolonged variety $\widehat{Y}_\infty$, the appropriate prolongation of the $\mathfrak{Z}_\alpha$ operator must now contain terms involving ∂_{q_j}, $\partial_{q_{jx}}$, …, ∂_{w_j}, $\partial_{w_{jy}}$, …, where j varies from 0 to infinity, with α-dependent coefficients. We label those coefficients for ∂_{q_j} and ∂_{w_j} as $Q_j(\alpha)$ and $W_j(\alpha)$, respectively, while the coefficients for some higher-level fiber coordinate, say $\partial_{q_{jxx}}$, would just be $(\widehat{\overline{D}}_x)^2\,Q_j(\alpha)$, etc. Since $q_0 = \Phi = \overline{D}_s^{\;-1}\Omega$ and $q_1 = \widehat{\overline{D}}_x\widehat{\overline{D}}_s^{\;-2}\,\Omega$ are linear we already understand how to construct prolongations corresponding to them: the appropriate coefficients for ∂_{q_0} and ∂_{q_1} would be $Q_0(\alpha) \equiv \widehat{\overline{D}}_s^{\;-1}\,\alpha$ and $Q_1(\alpha) \equiv \widehat{\overline{D}}_x\widehat{\overline{D}}_s^{\;-2}\,\alpha$, respectively. However, $q_2 = \widehat{\overline{D}}_s^{\;-1}\,\eta_2 = \widehat{\overline{D}}_s^{\;-1}(\Theta_{xx} + \frac{1}{2}\Phi_x^2) = \widehat{\overline{D}}_s^{\;-1}(q_{1x} + \frac{1}{2}q_{0x}^2)$ depends on Ω in a nonlinear way—as do all higher q_j's—so that the prolongation appropriate for them is not as immediately obvious. The coefficient for, say, ∂_{q_2}, namely $Q_2(\alpha)$, should in fact be the linear part of $q_2(\Omega+\epsilon\alpha)$, or, equivalently, the first functional derivative, with respect to α, of the expression $q_2(\Omega)$:

$$q_2(\Omega) = \widehat{\overline{D}}_s^{\;-1}\left\{\widehat{\overline{D}}_x^{2}\widehat{\overline{D}}_s^{\;-2}(\Omega) + \tfrac{1}{2}\{\widehat{\overline{D}}_x\widehat{\overline{D}}_s^{\;-1}\Omega\}^2\right\}\,, \tag{3.13}$$

$$q_2(\Omega+\epsilon\alpha) - q_2(\Omega) = \widehat{\overline{D}}_s^{\;-1}\left\{\epsilon\,\widehat{\overline{D}}_x^{2}\widehat{\overline{D}}_s^{\;-2}\alpha + \tfrac{1}{2}\{2q_{0x}\epsilon\widehat{\overline{D}}_x\widehat{\overline{D}}_s^{\;-1}\alpha\} + O(\epsilon^2)\right\} \;\equiv\; \epsilon\,Q_2(\alpha) + O(\epsilon^2)\,.$$

$$\Longrightarrow\; Q_2(\alpha) \equiv \widehat{\overline{D}}_s^{\;-1}\left\{\widehat{\overline{D}}_s^{\;-2}\widehat{\overline{D}}_x^{2}\alpha + q_{0x}\,\widehat{\overline{D}}_s^{\;-1}\widehat{\overline{D}}_x\alpha\right\} = \widehat{\overline{D}}_s^{\;-1}\left\{\widehat{\overline{D}}_xQ_1(\alpha) + q_{0x}\,\widehat{\overline{D}}_xQ_0(\alpha)\right\}\,. \tag{3.14}$$

The entire set of new terms in $\mathfrak{Z}_\alpha$, for the additional potential, q_2, should be the one with coefficient Q_2 plus all those generated by its x-derivatives, i.e., the following (infinite) sequence: $Q_2(\alpha)\,\partial_{q_2} + [\widehat{\overline{D}}_xQ_2(\alpha)]\,\partial_{q_{2x}} + [\widehat{\overline{D}}_x^{\;2}\,Q_2(\alpha)]\,\partial_{q_{2xx}} + \;\ldots\;.$

We must then continue in this manner, to include the corresponding sequences for q_3, q_4, etc. Next we must also consider the various coefficients $W_j(\alpha)$ that must multiply ∂_{w_j}, for the other (infinite) sequence of potentials, w_j:

$$\begin{gathered}\frac{\partial}{\partial\epsilon}w_j(\Omega+\epsilon\alpha)\Big|_{\epsilon=0} \;\equiv\; W_j(\alpha) \;=\; \widehat{\overline{D}}_s^{\;-1}\left\{\sum_{k=0}^{j-1}\zeta_{j-k}\,\widehat{\overline{D}}_y\,W_k(\alpha)\right\}\,,\\ W_0(\alpha) = \widehat{\overline{D}}_s^{\;-1}(\alpha)\,,\quad W_1(\alpha) = \widehat{\overline{D}}_y\widehat{\overline{D}}_s^{\;-2}(\alpha)\,.\end{gathered} \tag{3.15}$$

This finally gives us sufficient structure to provide the necessary generalization of our linearization operator, which generalizes completely the earlier, provisional form given in Eq. (2.15):

$$\widehat{\mathfrak{Z}}_\alpha = \mathfrak{Z}_\alpha + \sum_{k=0}^{\infty}\left\{\sum_{m=0}^{\infty}[\widehat{\overline{D}}_x^{\;(m)}Q_k(\alpha)]\partial_{q_{k,(m)}} + \sum_{n=0}^{\infty}[\widehat{\overline{D}}_y^{\;(n)}W_k(\alpha)]\partial_{w_{k,(n)}}\right\}\,. \tag{3.16}$$

As the process of determining $Q_2(\alpha)$, as described in Eqs. (3.13-14), seems fairly complicated and would appear to become worse for Q_3, etc., we now show the existence of a recursive algorithm that allows us to calculate the $Q_k(\alpha)$ sequentially, always giving us the next one in terms of those with lower indices. However, to explain this, we must retreat slightly, and study in more detail the relationship between the η_j's and ρ_k's. Taking the definitions given earlier, it is straightforward to show the following relation between them, and then a recursion algorithm for these polynomials:

$$\frac{\partial\eta_j}{\partial q_{kx}} = \rho_{j-k} \quad\Longrightarrow\quad \rho_{j+1} = \sum_{m=0}^{j-1}q_{mx}\,\rho_{j-m}\,. \tag{3.17a}$$

Since the η_j depend only on these particular jet coordinates, $\{q_{kx} \mid k=0,\ldots,j-1\}$, we may also determine the following additional useful recursion relation:

$$\widehat{\widehat{D}}_x \eta_j = \left\{ \sum_{k=0}^{j-1} q_{kxx} \frac{\partial}{\partial q_{kx}} \right\} \eta_j = \sum_{k=0}^{j-1} q_{kxx} \rho_{j-k} \ . \tag{3.17b}$$

To invert these relations it is useful to re-write them, using the form of a (lower-triangular) matrix, ${P_j}^k$, with $j=1,\,2,\,\ldots$ while $k=0,\,1,\,2,\,\ldots$:

$$ {P_j}^k \equiv \frac{\partial \eta_j}{\partial q_{kx}} = \begin{cases} \rho_{j-k}\ , & j>k, \\ 0\ , & j \le k. \end{cases} \tag{3.18}$$

Taking, now, the quantities $\{\widehat{\widehat{D}}_x \eta_j \mid j=1,2,3,\ldots\}$ as the components of a column vector, and $\{q_{kxx} \mid k=0,1,2,\ldots\}$ as the components of another, we see that Eqs. (3.17b) may be taken in the form

$$\widehat{\widehat{D}}_x \eta_j = \sum_{k=0}^{j-1} {P_j}^k\, q_{kxx} \tag{3.19}$$

while Eqs. (3.17a) are essentially a statement defining the matrix Q, the inverse of the matrix P:

$$\begin{gathered} \text{for} \quad k=0,1,2,3,\ldots \qquad j=1,2,3,\ldots\ , \qquad {Q_k}^j = \begin{cases} -q_{k-j,x}\ , & k \ge j, \\ 1\ , & k=j-1,\ , \\ 0\ , & k<j-1. \end{cases} \\ {P_{j+1}}^k\, {Q_k}^{\ell+1} = \begin{cases} \rho_{j-\ell+1} - \{q_{0x}\rho_{j-\ell} - \ldots - q_{j-\ell-1,x}\rho_1\} = 0\ , & \ell<j\ , \\ 1\ , & \ell=j\ , \\ 0\ , & \ell>j\ . \end{cases} \end{gathered} \tag{3.20}$$

With this information about the inverse, we may now solve Eqs. (3.17b) for the column vector with components q_{kxx}:

$$q_{kxx} = \sum_{j=1}^{k+1} {Q_k}^j\, \widehat{\widehat{D}}_x \eta_j \ = \widehat{\widehat{D}}_x \eta_{k+1} - \sum_{m=0}^{k-1} q_{mx}\, \widehat{\widehat{D}}_x \eta_{k-m} \ . \tag{3.21}$$

Since the process of determining the linear part of $q_j(\Omega+\epsilon\alpha)$ is just a derivation, we simply follow the same procedure as was used to determine Eqs. (3.17b), replacing the derivation $\widehat{\widehat{D}}_x$ acting on η_j treated as a function of $\{x,y,s\}$, with the determination of this linear part, treating the η_j instead as $\widehat{\widehat{D}}_s\, q_j(\Omega)$. That process gives us the desired recursive algorithm to obtain $\widehat{\widehat{D}}_s\, Q_k(\alpha)$ in terms of the set $\{\widehat{\widehat{D}}_x Q_j \mid j=0,\ \ldots\ ,\ k-1\}$:

$$\widehat{\widehat{D}}_s\, Q_k(\alpha) = \left\{ \sum_{m=0}^{k-1} [\widehat{\widehat{D}}_x\, Q_m(\alpha)] \frac{\partial}{\partial q_{mx}} \right\} \eta_k = \sum_{m=0}^{k-1} \rho_{k-m}\, \widehat{\widehat{D}}_x\, Q_m(\alpha)\ . \tag{3.22}$$

The general equation for the $Q_k(\alpha)$'s involves an s-integration, which obviously cannot be performed exactly for any arbitrary function α. As usual, however, we only need to perform that integration when the argument is a characteristic for a (generalized) symmetry. Therefore, choosing that α as a characteristic, X_j, we can accomplish explicitly the s-integration. The forms given above will always involve some second derivatives of q_m, i.e., terms of the form q_{mxx}. We may use Eqs. (3.21) to replace these in terms of a series of quantities involving $\widehat{\overline{D}}_x\eta_n = \widehat{\overline{D}}_s\, q_{nx}$, which allows the desired integration. This must be done sequentially; therefore, we now write down the first two, which are very simple, then proceed onward to $Q_2(X_j)$ explicitly, and then consider the more general case. Those first two are just the following:

$$Q_0(X_j) = \overline{D}_s^{\,-1}(X_j) = \eta_j \ , \qquad Q_1(X_j) = \overline{D}_x\overline{D}_s^{\,-2}(X_j) = q_{jx} \ . \tag{3.23}$$

On the other hand, returning to Eq. (3.14) for the interesting case $\alpha = X_j$, we have

$$Q_2(X_j) = \overline{D}_s^{\,-1}\{q_{jxx} + q_{0x}\,\overline{D}_x\eta_j\} \ . \tag{3.24}$$

When $j = 1$ the integrand above is simply the form for $\widehat{\overline{D}}_x\eta_1$, so that $Q_2(X_1) = q_{1x}$. For larger values of j we may proceed as already described, by eliminating the displayed q_{jxx} via Eqs. (3.21), which gives us

$$\begin{aligned}
Q_2(X_1) \ &= q_{1x} \ , \\
Q_2(X_j) \ &= \overline{D}_s^{\,-1}\left\{\overline{D}_x\eta_{j+1} - \sum_{m=1}^{j-1} q_{mx}\overline{D}_x\eta_{j-m}\right\} \\
&= q_{j+1,x} - \overline{D}_s^{\,-1}\left\{\sum_{m=1}^{j-1} q_{mx}\overline{D}_x\eta_{j-m}\right\} = q_{j+1,x} - \tfrac{1}{2}\sum_{m=1}^{j-1} q_{mx}q_{j-m,x} \ , \quad j \geq 2 \ .
\end{aligned} \tag{3.25}$$

We can continue onward, then, to $Q_3(X_j)$:

$$\begin{aligned}
Q_3(X_j) &= \widehat{\overline{D}}_s^{\,-1}\left\{\rho_1\widehat{\overline{D}}_x Q_2(X_j) + \rho_2\widehat{\overline{D}}_x Q_1(X_j) + \rho_3\widehat{\overline{D}}_x Q_0(X_j)\right\} \ = \ \ldots \\
&= \ \widehat{\overline{D}}_s^{\,-1}\Bigg\{\widehat{\overline{D}}_x\eta_{j+2} - \sum_{k=0}^{j} q_{kx}\widehat{\overline{D}}_x\eta_{j+1-k} - \sum_{k=1}^{j-1} q_{kx}q_{j-k,xx} \\
&\qquad + q_{0x}\left[\widehat{\overline{D}}_x\eta_{j+1} - \sum_{m=0}^{j-1} q_{mx}\widehat{\overline{D}}_x\eta_{j-m}\right] + [q_{1x} + (q_{0x})^2]\widehat{\overline{D}}_x\eta_j\Bigg\} \ , \\
&= \ \ldots \ = \ q_{j+2,x} - \sum_{k=1}^{j-1} q_{kx}q_{j+1-k,x} \ + \ \tfrac{1}{3}\left\{\sum_{k=1}^{j-2}\sum_{m=1}^{j-k-1} q_{kx}q_{mx}q_{j-k-m,x}\right\} \ , \ j \geq 3 \ .
\end{aligned} \tag{3.26}$$

For smaller values of j, one can simply terminate the derivation earlier. On the other hand, it turns out that they are more easily described by simply noting that

$$Q_k(X_j) = Q_j(X_k) \ . \tag{3.27}$$

As these forms are becoming lengthy, we now simply note another couple of examples, and then describe the structure in a general way:

$$\begin{aligned}
Q_4(X_5) &= q_{8x} - q_{6x}q_{1x} - 2q_{5x}q_{2x} - 3q_{4x}q_{3x} + (q_{2x})^3 + 4q_{3x}q_{2x}q_{1x} + q_{4x}(q_{1x})^2 - q_{2x}(q_{1x})^3 \ , \\
Q_5(X_5) &= q_{9x} - q_{7x}q_{1x} - 2q_{6x}q_{2x} - 3q_{5x}q_{3x} + q_{5x}(q_{1x})^2 - 2(q_{4x})^2 + 4q_{4x}q_{2x}q_{1x} \\
&\qquad + 3(q_{3x})^2q_{1x} + 4q_{3x}(q_{2x})^2 - q_{3x}(q_{1x})^3 - 3(q_{2x}q_{1x})^2 + \tfrac{1}{5}(q_{1x})^5 \ .
\end{aligned} \tag{3.28}$$

To consider the general case, we first note that we may always use Eq. (3.27) to convert those objects with $j < k$ into those where $j \geq k$. We ascribe a "grade" of $m+1$ to the quantity q_{mx} and note that $Q_k(X_j)$ has grade $k+j$. For $j \geq k$, it is composed of a sum of all products of k or fewer q_{mx}'s, such that the grade of the entire product equals $k+j$. The single term with only one element in the product will, of course, always be $q_{k+j-1,x}$ and is positive. From there on the signs alternate so that the sign of a term with n elements in the product will have sign $(-1)^{n-1}$. The explicit coefficients vary depending on the number of repetitions of a single quantity in an individual product. However, any individual one may be calculated explicitly by the method described above.

There are also some other quantities for which we know that $Q_k(\alpha)$ should be explicitly defined. These are of course the other objects which we need to use to calculate commutators with these generators, i.e., the characteristics Y_j and the characteristics for the Lie symmetries in the s-direction, $GS(\alpha,\beta)$. This last set is very straightforward, and the calculation gives us

$$Q_k[GS(\alpha,\beta)] = (\alpha s + \beta)\,\eta_k - (k+1)\alpha\, q_k \; . \tag{3.29}$$

The other set is obviously a larger question, if only because there are very many more of them. We begin with the straightforward ones as before:

$$\begin{gathered} Q_0(Y_j) = \widehat{\widehat{D}}_s^{\,-1}(Y_j) = \zeta_j \equiv \widehat{\widehat{D}}_s\, w_j \; , \\ Q_1(Y_k) = w_{kx} = -\sigma_k\, e^{\Omega} \; , \quad Q_k(Y_1) = q_{ky} = -\rho_k\, e^{\Omega} \; , \end{gathered} \tag{3.30}$$

where the polynomials ρ_k are given in Eqs. (3.6), while we recall that the σ_k's are just the polynomials ρ_k with all x's interchanged with y's and all q_m's interchanged with w_m's. The alternation between a function of the w_k's and a function of the q_k's, for the two options in the previous equations, suggests that we will need polynomials in both of these sets of potentials for the more general case, namely $Q_k(Y_j)$. We therefore first define generalizations of the polynomials, ρ_k, σ_k, etc. that we have already been using. We take $\mathcal{P}^j_k$ and $\mathcal{Q}^j_k$ as graded polynomials over all integer partitions of $k-1$, in the variables $\{q_{kx}\}$ or the $\{w_{jy}\}$, respectively, but otherwise the same:

$$\begin{gathered} \mathcal{P}^j_1 = 1 = \mathcal{Q}^j_1 \; , \\ \text{for} \quad k \geq 2, \; \begin{cases} \mathcal{P}^j_k \equiv \displaystyle\sum_{a\in P(k-1)} \frac{(|a|+j-1)!}{(j-1)!\{a\}!} \prod_{n=1}^{k-1} \left(\widehat{\widehat{D}}_x q_{n-1}\right)^{a_n} \; , \\ \mathcal{Q}^j_k \equiv \displaystyle\sum_{a\in P(k-1)} \frac{(|a|+j-1)!}{(j-1)!\{a\}!} \prod_{n=1}^{k-1} \left(\widehat{\widehat{D}}_y w_{n-1}\right)^{a_n} . \end{cases} \end{gathered} \tag{3.31}$$

Comparison with Eqs. (3.6) shows that $\mathcal{P}^1_k = \rho_k$, and $\mathcal{Q}^1_k = \sigma_k$, i.e., these earlier ones were just the lowest-order members of these new sequences. It is then straightforward to work out the simple descriptions for this last set of coefficients we need:

$$Q_\ell(Y_k) = \sum_{m=1}^{\min(\ell,k)} \frac{(-1)^m}{m}\, \mathcal{P}^m_{\ell+1-m}\, \mathcal{Q}^m_{k+1-m}\, e^{m\Omega} \; , \tag{3.30}$$

with some other particular examples being given as

$$\begin{aligned} &\text{for} \quad k \geq 2, \quad \begin{cases} Q_2(Y_k) \; = \frac{1}{2}\mathcal{P}^2_1 \mathcal{Q}^2_{k-1}\, e^{2\Omega} - \mathcal{P}^1_1 \mathcal{Q}^1_k\, e^{\Omega} \; , \\ Q_k(Y_2) = \; = \frac{1}{2}\mathcal{P}^2_{k-1} \mathcal{Q}^2_1\, e^{2\Omega} - \mathcal{P}^1_k \mathcal{Q}^1_1\, e^{\Omega} \; , \end{cases} \\ &\text{for} \quad k \geq 3, \quad \begin{cases} Q_3(Y_k) = -\frac{1}{3}\mathcal{P}^3_1 \mathcal{Q}^3_{k-2}\, e^{3\Omega} + \frac{1}{2}\mathcal{P}^2_2 \mathcal{Q}^2_{k-1}\, e^{2\Omega} - \mathcal{P}^1_3 \mathcal{Q}^1_k\, e^{\Omega} \; , \\ Q_k(Y_3) = -\frac{1}{3}\mathcal{P}^3_{k-2} \mathcal{Q}^3_1\, e^{3\Omega} + \frac{1}{2}\mathcal{P}^2_{k-1} \mathcal{Q}^2_2\, e^{2\Omega} - \mathcal{P}^1_k \mathcal{Q}^1_3\, e^{\Omega} \; , \end{cases} \end{aligned} \tag{3.32}$$

To continue this, we must also follow an entirely analogous procedure to calculate the coefficients $W_j(\alpha)$, $\widehat{\widehat{D}}_y W_j(\alpha)$, respectively. However, they may be obtained easily from the ones already given, by the process of interchanging all x's with y's, all q_k's with w_k's, η_k's with ζ_k's, $\mathcal{P}^j_k$ with $\mathcal{Q}^j_k$, etc. The process of these interchanges takes $Q_k(X_j)$ into $W_k(Y_j)$, $Q_k(Y_j)$ into $W_k(X_j)$, and $Q_k[GS(\alpha,\beta)]$ into $W_k[GS(\alpha,\beta)]$. This process is quite straightforward, if perhaps tedious, and we do not write them out.

4. The Two Infinite Sets of Symmetry Characteristics

The prolongations described above now allow the calculation of any commutators of characteristics desired. In particular, we recall that the discussions after Eqs. (2.17) noted that commutators with the characteristic $GX_2[R]$, i.e. commutators of the form $\{GX_j[A], GX_2[R]\}$ had the property of a recursion operator for those characteristics we had already found at that point, as described in detail at Eqs. (2.18), for $j=2$, giving GX_3, and Eqs. (3.3), for $j=3$, giving GX_4. We now are able to calculate such commutators for arbitrary values of j, which allows us easily to see that this gives an infinite sequence of characteristics, each with its own arbitrary function, and s-dependent polynomials. The structure as a recursion operator is as expected:

$$\{GX_j[A], GX_2[R]\} = GX_{j+1}[2RA' - jAR'] \quad \Leftrightarrow \quad \{X^a_j, X^1_2\} = (2a-j)X^a_{j+1}\,. \tag{4.1}$$

It is simplest to display these as second (total) s-derivatives of the appropriate polynomial forms. When this is done the result, for $GX_k[A]$, is a polynomial beginning with a term containing A, then a term containing A', a term containing A'', etc., up to a term containing $A^{(k)}$, the k-th derivative of the function A. For $m<k$, the coefficient of the term containing $A^{(m)}$ is a polynomial in s, of order m, and the coefficients in this polynomial are made only of products of the q_j's themselves. The last term, which contains $A^{(k)}$, is simply $s^{k+1}A^{(k)}/(k+1)!$. We display the general result below, along with some explicit examples to give a better "feel" for their form, noting that forms for $GX_1[A]$ and $GX_2[A]$ have already been given:

$$GX_k[A] = \widehat{\widehat{D}}_s{}^2 \left\{ \frac{s^{k+1}}{(k+1)!}A^{(k)} + \sum_{m=0}^{k-1} A^{(m)} \sum_{a\in\mathcal{P}(k-m)} \frac{\prod_{n=1}^{|a|} i_n}{\{a\}!\,(m-|a|+1)!}\, s^{m+1-|a|} \prod_{j=1}^{k-m} (q_j)^{a_j} \right\}\,, \tag{4.2}$$

$$\begin{aligned}
GX_3[A] &= \widehat{\widehat{D}}_s{}^2 \left\{ 3Aq_3 + A'(2sq_2 + \tfrac{1}{2}q_1^2) + \tfrac{s^2}{2}A''q_1 + \tfrac{s^4}{24}A''' \right\}\,,\\
GX_4[A] &= \widehat{\widehat{D}}_s{}^2 \left\{ 4Aq_4 + A'(3sq_3 + 2q_1q_2) + A''s(sq_2 + \tfrac{1}{2}q_1^2) + \tfrac{s^3}{6}A'''q_1 + \tfrac{s^5}{120}A^{(iv)} \right\}\,,\\
GX_5[A] &= \widehat{\widehat{D}}_s{}^2 \Big\{ 5Aq_5 + A'[4sq_4 + 3q_1q_3 + 2(q_2)^2] + \tfrac{1}{2}A''(3s^2q_3 + 4sq_1q_2 + (q_1)^3/3)\\
&\qquad + \tfrac{1}{3}A'''s^2(sq_2 + 3(q_1)^2/4) + A^{(iv)}s^4q_1/24 + A^{(v)}s^6/720 \Big\}\,.
\end{aligned} \tag{4.3}$$

As usual, there is the completely analogous infinite sequence of (non-Abelian) characteristics, each with its own arbitrary function of one variable, $B(y)$, which we label as $\{GY_k[B] \mid k = 1, \dots\}$, with the same structure. We may therefore display explicitly the commutators of each set, with themselves and with each other:

$$\begin{aligned}\{GX_j[A], GX_k[R]\} &= GX_{j+k-1}[kRA' - jAR'] \,,\\ \{GX_j[A], GY_k[B]\} &= 0\,,\\ \{GY_j[B], GY_k[S]\} &= GY_{j+k-1}[kSB' - jBS']\;.\end{aligned}\tag{4.4}$$

We may also pull out the two basis sets, and display their commutators, which of course have the same content as the ones just above:

$$\begin{gathered}X_a^b \equiv GX_a(x^b)\;,\quad Y_a^b \equiv GY_a(x^b)\;,\\ \{X_j^a, X_k^b\} = (ak-bj)X_{j+k-1}^{a+b-1}\;,\quad \{X_j^a, Y_k^b\}\;=0\,,\quad \{Y_j^a, Y_k^b\} = (ak-bj)Y_{j+k-1}^{a+b-1}\;.\end{gathered}\tag{4.5}$$

On the other hand, the two original Lie symmetry characteristics, S_0 and S_1, do not commute with them, but do treat the two sets equally, where we use $GS(\alpha,\beta) = \alpha\, S_0 + \beta\, S_1$:

$$\begin{aligned}\{X_a^b, S_0\} &= (a-1)\,X_a^b\;, &\quad \{Y_a^b, S_0\} &= (a-1)\,Y_a^b\;,\\ \{X_a^b, S_1\} &= b\,X_{a-1}^{b-1}\;, &\quad \{Y_a^b, S_1\} &= b\,Y_{a-1}^{b-1}\;.\end{aligned}\tag{4.6}$$

We may also recall that the Abelian subalgebras of this large algebra, which are responsible for the commuting hierarchy of pde's built over the original $\boldsymbol{SDiff}(2)$ Toda equation is defined by

$$\begin{gathered}X_a \equiv \tfrac{1}{a}\,X_a^0\;,\quad Y_b \equiv \tfrac{1}{b}\,Y_b^0\;,\\ \{X_a, X_r\} = 0 = \{X_a, Y_b\} = \{Y_s, Y_b\}\;.\end{gathered}\tag{4.7}$$

5. Conclusions

Our search for these generalized symmetries of this equation began with a somewhat different quest. We were looking for a generalization of the Estabrook-Wahlquist method of finding non-local potentials, and associated Bäcklund transformations, that would be generic for pde's with three or more independent variables. The SDiff(2) Toda equation seemed like an ideal candidate as a beginning for this project, since the more usual Toda lattice equations had well-defined non-local (EW) prolongation structures and Bäcklund transformations. Limits of those (systems of) 2-dimensional equations lead to our current pde in a straightforward way; however, the associated limits of the prolongation structures[3,18] led to nothing interesting. We still have no new directions for that search.

Nonetheless, in some attempt to "buy" new solutions from old ones, for this pde, we decided to consider the generalized symmetries, beyond the usual Lie symmetries. This also led to a null result. That problem was resolved by finding that each generalized symmetry required the addition of an additional pair of first-order equations to the original system, defining the inclusion of a new potential to the jet bundle. This has then generated the entire structure of generalized symmetries described here. We have taken the original, commuting hierarchy of symmetries, found by Takasaki and Takebe, and broadened it extremely into our Lie algebra of generalized symmetries, which is definitely no longer Abelian. This allowed it to be described via a recursion operation, which generates the entire doubly-infinite algebra.

An important and interesting question is just how one may use this new structure to create new (families of) solutions to the original pde. We trust that this larger explication of the generalized symmetries of the equation will eventually be helpful in a better understanding of the solution manifold for the problem. There are several possible routes to an answer to this question. A very interesting one involves the work of Hernández, Winternitz, et al[19], which provides correspondences between continuous symmetries and Bäcklund transformations for the Toda lattice equations. Whether such an idea can be moved over to this limiting equation we do not yet know, but the idea is a promising one. Another direction has to do with the τ-function for the hierarchy of Takasaki and Takebe. In other work, on the KP equation, the appropriate τ-function, considered as depending on all the (infinitely-many) independent variables of the hierarchy problem, has been used as a source to generate (almost) all solutions of the original nonlinear equation. Takasaki and Takebe characterize the τ-function for this particular problem, and it appears to us that the function we have called e^{Θ} satisfies all those criteria. Therefore further study of it may well show that it also has the virtue of being able to tell us how to find the desired general solutions. However, research on that question is just beginning.

Appendix A

We begin with the standard Plebański[20] formulation for an $\mathfrak{h}$-space, i.e., a 4-dimensional, complex manifold with a self-dual curvature tensor that satisfies the Einstein vacuum field equations. Such a space is determined by a single function of 4 variables, $\Omega = \Omega(p, \bar{p}, q, \bar{q})$, which must satisfy one constraining pde, and then determines the metric via its second derivatives, as follows:

$$\begin{aligned} \Omega_{,p\bar{p}}\Omega_{,q\bar{q}} - \Omega_{,p\bar{q}}\Omega_{,q\bar{p}} = 1 \ , \\ \mathbf{g} = 2(\Omega_{,p\bar{p}}\, dp\, d\bar{p} + \Omega_{,p\bar{q}}\, dp\, d\bar{q} + \Omega_{,q\bar{p}}\, dq\, d\bar{p} + \Omega_{,q\bar{q}}\, dq\, d\bar{q}) \ . \end{aligned} \tag{A1}$$

Restricting attention to those complex spaces that allow real metrics of Euclidean signature, there are only two possible “sorts” of Killing vectors, “translations” and “rotations.” Noting that the covariant derivative of any Killing tensor must be skew-symmetric, by virtue of Killing’s equations, we may make this division more technical by dividing the class of Killing vectors based on this skew-symmetric tensor’s anti-self-dual part, which Einstein’s equations require to be constant. The “translational” Killing vectors are those for which this anti-self-dual part vanishes, while it does not vanish for the “rotational” ones. The self-dual case—where the anti-self-dual part vanishes—has been completely resolved[21]. (In this case the constraining equation for Ω reduces simply to the 3-dimensional Laplace equation.)

We continue by insisting that the space under study admit a rotational Killing vector, $\widetilde{\xi}$, and then re-defining the variables so that they are adapted to it:

$$\widetilde{\xi} = i(p\partial_p - \bar{p}\partial_{\bar{p}}) \equiv \partial_\phi \ , \qquad \widetilde{\xi}(\Omega) = 0 \, , \qquad p \equiv \sqrt{r}\, e^{i\phi} \ , \ \bar{p} \equiv \sqrt{r}\, e^{-i\phi} \ , \tag{A2}$$

which changes the constraining equation as follows, construing Ω to now depend on the variables $\{r, q, \bar{q}\}$, but not ϕ:

$$(r\Omega_{,r})_{,r}\Omega_{,q\bar{q}} - r\,\Omega_{,qr}\Omega_{,\bar{q}r} = 1 \ . \tag{A3}$$

It is however often more convenient to rewrite the constraining equation, and the metric, in terms of a new set of coordinates, obtained from the original ones via a Legendre transform based on variables r and $s \equiv r\Omega_{,r}$. Taking $\{s, q, \bar{q}\}$, along with ϕ, as the new coordinates, and $v \equiv \ln r$ as the function of these coordinates that will generate the metric, we find the following new presentation, which shows the agreement with the SDiff(2) Toda equation, where we must simply identify this new function v with the function Ω as given in Eq. (1.1):

$$
\begin{gathered}
\mathbf{g} = V\,\gamma + V^{-1}(d\phi + \underset{\approx}{\omega})^2\ , \\
V \equiv \tfrac{1}{2} v_{,s}\ , \quad \gamma \equiv ds^2 + 4\, e^v\, dq \wedge d\bar{q}\ , \qquad \underset{\approx}{\omega} \equiv \tfrac{i}{2}\{v_{,q} dq - v_{,\bar{q}} d\bar{q}\} \\
v_{,q\bar{q}} + (e^v)_{,ss} = 0\ , \qquad \text{and} \qquad \underset{\gamma}{*}\,(d\underset{\approx}{\omega}) = -iV^2 d(2s - V^{-1})\ .
\end{gathered}
\tag{A4}
$$

Another distinct use for this equation is the desire to have a manifold which is scalar flat and Kähler. LeBrun[22] showed that the solutions of a pair of pde's was necessary to answer this question. One of those is the SDiff(2)Toda equation, and the other one the linearization of that equation, for a second dependent function. This has been an important impetus for some of the work on the problem of $\boldsymbol{SU}(2)$-invariant metrics[2].

Appendix B

We give here simply a somewhat more detailed description of the set of additive partitions of integers, which have been described and studied in many ways. For a given integer, k, any particular (additive, integer) partition is simply a list of positive integers with sum equal to the given integer, k. We label any one such partition by a, and may describe it in more detail as the sequence $[i_1, i_2, \ldots, i_{|a|}]$, with all the i_j's being non-zero, and where, by convention, we order the entries so that $i_j \geq i_{j+1}$, and $|a|$ is the number of (non-zero) entries in a. It may well turn out that some of these quantities are the same, in which case we may use a_m to count the number of times the integer $m \leq k$ appears in that sequence. The set of all such integer partitions for a given k is denoted by $\mathcal{P}(k)$, and we will denote its number of elements, i.e., the number of distinct partitions of k, by $p(k)$An example for $k = 5$ is given by the following:

$$\mathcal{P}(5) = \Big[[5], [4,1], [3,2], [3,1,1], [2,2,1], [2,1,1,1], [1,1,1,1,1]\Big]\ . \tag{B1a}$$

For larger k at least, a "shorter" alternative is to use "powers" for those integers that are repeated in a particular partition, with the previous example being shown below in this mode:

$$\mathcal{P}(5) = \Big[[5], [4,1], [3,2], [3,1^2], [2^2,1], [2,1^3], [1^5]\Big]\ . \tag{B1b}$$

In these examples, we have also introduced an ordering of the partitions relative to one another so that those with larger entries appear first, i.e., to the left.

On the other hand it is more useful at the moment to describe any particular partition of k, i.e., some $a \in \mathcal{P}(k)$, by giving an ordered list of *non-negative integers*, a_i, where a_i tells how many times the integer i is repeated in that particular partition. We note that obviously we must have $1 \leq i \leq k$. This corresponds to the list of all the powers that appear in the second presentation of the partitions of 5, above, except that we carefully consider **all** integers between 1 and k to be present, so that some integers have power 0:

$$a \in \mathcal{P}(k) \iff a \equiv \{a_1, a_2, a_3,\ \ldots\ , a_k\}\ ,\ a_p \geq 0\ , \qquad \text{such that } k = \sum_{p=1}^{k} p\, a_p\ . \tag{B2}$$

In this form our example above takes the form

$$\mathcal{P}(5) = \Big[\{0,0,0,0,1\},\{1,0,0,1,0\},\{0,1,1,0,0\},\{2,0,1,0,0\},\\ \{1,2,0,0,0\},\{3,1,0,0,0\},\{5,0,0,0,0\}\Big]\,. \tag{B3}$$

It is this form of description of the partitions that is used in the definitions of the various sets of polynomials given in the main text, such as the η_k in Eqs. (3.5).

As already noted in Eqs. (3.11), there are various useful functions that describe individual members of a the set of all partitions of k. Two of these that we need are $|a|$ and $\{a\}!$:

$$|a| \equiv \sum_{p=1}^{k} a_p \ \leq k\ , \qquad \text{and also} \qquad \{a\}! \equiv \prod_{p=1}^{k} (a_p)!\ . \tag{B4}$$

Continuing with our example above, for the partitions of 5, these mappings have the following values there:

$$\text{for}\ \ a \in P(5), \qquad |a| \ \Longrightarrow\ [1,2,2,3,3,4,5]\ , \quad \{a\}! \ \Longrightarrow\ [1,1,1,2,2,6,120]\ . \tag{B5}$$

The simple explanation as to why these coefficients enter into our calculation is that the coordinates on the jet bundle may be graded, i.e., assigned a weight so that the various pde's have a consistent weight. A reasonable way to describe that begins with the consideration of a formal infinite series, $\mathcal{L}$, in powers of some grading parameter, l:

$$\mathcal{L}^n = \left\{1 + \sum_{i=0}^{\infty} u_i l^{-i-1}\right\}^n \equiv \sum_{m=0}^{\infty} C^n_m \lambda^{-m}\ . \tag{B6}$$

The early values of the coefficients C^n_m are easily seen to satisfy the following simple relations:

$$C^n_0 = 1\,,\ C^n_1 = nu_0\,,\ C^n_2 = nu_1 + \binom{n}{2} u_0^2\,,\ C^n_3 = nu_2 + 2\binom{n}{2} u_1 u_0 + \binom{n}{3} u_0^3\ . \tag{B7}$$

However, we would like a more general description of them. Because of the association of the index on u_i with the power of λ one may ascribe a "weight" to the u_i's: give the weight $j+1$ to the factor u_j, which causes the coefficient C^n_m to be a sum of terms, with distinct coefficients, each of which has the same overall weight, namely m. Therefore those u_j's that contribute to a given coefficient C^n_m have weights described by the different (positive, integer) partitions of m; these form a set, which we label as $P(m)$. This tells us to display the C^n_m as a sum over all those terms, each with an appropriate coefficient, which is a pure (combinatorial) number:

$$\begin{aligned} C^n_m &= \sum_{a\in P(m)} C(n;m\mid a)\, u_0^{a_1} u_1^{a_2} \ldots u_{m-1}^{a_m} \\ &= \sum_{a\in P(m)} C(n;m\mid a) \prod_{j=1}^{m} (u_{j-1})^{a_j}\ , \qquad a = \{a_1,,a_2,\ldots,a_m\}\ , \end{aligned} \tag{B8}$$

$$C(n;m\mid a)\ \equiv\ \frac{n(n-1)\ldots[n-|a|+1]}{a_1!\,a_2!\,\ldots\,a_m!} = \frac{n!}{[n-|a|]!\,\{a\}!} = \binom{n}{|a|}\left(\frac{(|a|)!}{\{a\}!}\right)\ . \tag{B9}$$

The coefficients $C(n;m \mid a)$ exist for each integer value of n, and for every partition of m, i.e., for $a \in P(m)$. It works, in particular, for negative as well as positive values of n, provided we simply make the usual, standard substitutions for the binomial coefficients. For instance when we set $n = -p$, for negative values of n, we have

$$\binom{n}{r} r! = n(n-1)\ldots[n-r+1] \to (-1)^r p\,(p+1)\ldots[p+r-1] = (-1)^r \binom{p+r-1}{r} r!\,.$$

Therefore, in the case that $n \equiv -p$ is negative, we may determine the desired coefficients as follows:

$$\begin{aligned} C^n_m &= \sum_{a\in P(m)} E(p\,;m \mid a)(-1)^{|a|}\, u_0^{a_1} u_1^{a_2} \ldots u_{m-1}^{a_m}\,, \qquad a = \{a_1,, a_2, \ldots, a_m\}\,; \\ E(p\,;m \mid a) &= \frac{[p+|a|-1]!}{(p-1)!\,a_1!\,a_2!\,\ldots\,a_m!} = \binom{p+|a|-1}{|a|}\left(\frac{(|a|)!}{\{a\}!}\right)\,, \qquad (B10) \\ &\text{note that } E(1;m \mid a) = (-1)^{|a|}C(-1;m \mid a) \text{ just simplifies to } \left(\frac{[r(a)]!}{\{a\}!}\right). \end{aligned}$$

It is exactly these coefficients $E(p\,;m \mid a)$ that appear in the definitions of the polynomials $\mathcal{P}^p_k$, in Eqs. (3.31).

Good general references for the theory of partitions, and proofs of the properties of the coefficients $C(n;m \mid a)$, are found in the books by Comtet[23] and by Riordan[24].

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