The Robinson-Trautman Type III Prolongation Structure Contains $\mathbf{K_2}$

J. D. Finley, III

Department of Physics and Astronomy
University of New Mexico
Albuquerque, N.M. 87131

Internet: finley@tagore.phys.unm.edu
FAX: (505)-277-1520

The minimal prolongation structure for the Robinson-Trautman equations of Petrov type III is shown to always include the infinite-dimensional, contragredient algebra, K_2, which is of infinite growth. Knowledge of faithful representations of this algebra would allow the determination of Bäcklund transformations to evolve new solutions.

to be published in Communications in Mathematical Physics
Running Title: Robinson-Trautman Type III and K_2
MCS Numbers: 17B65, 83C20

The general class of Robinson-Trautman solutions [1] to the vacuum Einstein field equations have been important examples of exact solutions for many years, albeit they seem to have various difficulties with respect to their interpretation [2]. They are solutions characterized by having a repeated principal null direction, which is of course geodesic and shearfree, and is required to be diverging but not twisting. The standard reference [3] gives the general form of the metric which any Einstein space must have if it permits such a repeated principal null direction, and notes that all possible algebraically-special Petrov types are allowed. In the case of Petrov type III, the field equations are [3] first reduced to

$$K = \Delta \log P \equiv 2P^2 \partial_\zeta \partial_{\overline{\zeta}} \log P = -3[f(\zeta, u) + \overline{f}(\overline{\zeta}, u)] \quad , \tag{1}$$

where K is the Gaussian curvature of the 2-surface spanned by ζ and $\overline{\zeta}$. This equation determines the general RT-solution of Petrov type III. However, since u is nowhere explicitly mentioned within the partial differential equation (pde), it is well-known [3] that one could always simply ignore that dependence, perform a coordinate transformation sending $f(\zeta) \to \zeta$, leaving the curvature completely invariant, and reducing our equation to the rather simple-appearing equation

$$\begin{aligned} K \quad &= \quad 2P^2 \partial_\zeta \partial_{\overline{\zeta}} \log P = -\tfrac{3}{2}(\zeta + \overline{\zeta}) \quad , \\ &\text{or} \qquad u_{xy} = \tfrac{1}{2}(x+y)e^{-2u} \quad , \qquad \text{where} \quad \log P \equiv u \; , \end{aligned} \tag{2}$$

the subscripts denote partial derivatives, and the symbols $\{x, y\}$ have been introduced instead of $\{\zeta, \overline{\zeta}\}$, both to simplify the typography and to normalize the equation so that the coefficient has a value which will prove convenient.

As already pointed out, all Petrov type III solutions of the vacuum field equations with diverging, non-twisting null directions are determined by the general solution of Eq. (2). Nonetheless, only one rather trivial solution is available for study, namely $P = (\zeta + \overline{\zeta})^{3/2}$, even though all its Lie symmetries have been found [4]. This unfortunate situation has caused us to apply the general methods of Estabrook and Wahlquist [5] to this equation, for determination of (pseudo)-potentials, in the hope of generating new solutions. The EW procedure is a particular approach to the determination of non-local symmetries of a pde [6]. It has been used successfully [7-10] in many contexts, although it must be admitted that applications toward finding RT solutions of Petrov type II were unsuccessful [11-12].

The situation for Petrov type III seems to be much better. We have in fact obtained a complete, but so far still abstract, description of the space of allowed pseudopotentials. The unexpected consequence of this search was that the smallest space that allows a solution of the problem must be a carrier space of a realization, via vector fields, of an algebra of infinite growth [13,14], usually referred to as K_2 [14,15]. This algebra has so far resisted any attempts to find explicit realizations. The name was created by Kac [14] in his early article separating those algebras now called Kac-Moody algebras away from classes of much larger algebras. Kac used K_2 as a "simple" example of a contragredient algebra not in the Kac-Moody class.

We begin our investigation by describing the solution space for the equation as a surface, Y, in the jet space, $J^{(2)}$, that treats dependent functions and their first and second derivatives as independent quantities until a specific solution is obtained. The Estabrook-Wahlquist procedure guides one in searches for $\mathbf{F}$ and $\mathbf{G}$, vector fields over a space, W, of pseudopotentials, $\{w^A \mid A = 1, \ldots, N\}$, that we wish to adjoin to the original jet space. These vector fields provide prolongations of the (usual) total derivative operators on the jet space, *i.e.*, $\{D_x, D_y\}$, to the combined space of variables, $J \oplus W$, which then must satisfy the zero-curvature equations when restricted to Y:

$$\left[\tilde{D}_x + \mathbf{F}\,,\, \tilde{D}_y + \mathbf{G}\right] = 0 \quad , \tag{3}$$

where the tilde indicates that the operators have been restricted to the subspace, Y, of solutions. This is a slight generalization of the usual notion of the zero-curvature equations of Lax or of Zakharov and Shabat [16], since $\mathbf{F}$ and $\mathbf{G}$ are simply elements of an **abstract** Lie algebra, of vector fields, with neither the coordinates, nor even N, yet determined.

Following the approach of Cartan [17], the EW procedure for a pde in 2 independent variables may be described as follows [18]. We first choose a (closed) ideal, $\mathcal{K}$, generated by a set of 2-forms, $\{\alpha^r\}$, that describes the original pde. We then adjoin the variables w^A to the system by appending (to the original ideal) contact forms, ω^A, for each of these new variables, and insisting that the ideal remain closed:

$$\begin{aligned} \omega^A &= -\,dw^A + F^A dx + G^A dy, \\ dF^A \wedge dx + dG^A \wedge dy &= f^A_{\ r}\,\alpha^r + \eta^A{}_B \wedge \omega^B \quad , \end{aligned} \qquad A = 1, \ldots, N \quad , \tag{4}$$

where the functions F^A and G^A are the coefficients of the vector fields, $\mathbf{F}$ and $\mathbf{G}$, that define the zero-curvature representation of the problem. These brief sentences describe the "essence" of the EW procedure, which embodies two notions, the first being the choice of a sufficiently-small ideal that calculations can be carried out successfully, while the other is that the new potentials are all allowed, from the beginning, to depend on each other, thereby rendering the process of discovering them nonlinear, and justifying the name, "pseudopotentials," for these additional variables. The newly-introduced functions ${f^A}_r$ and 1-forms ${\eta^A}_B$ constitute "Lagrange multipliers" for the system, their existence being the explicit characterization of closure of the prolonged ideal. Because of this the final choices of $\mathbf{F}$ and $\mathbf{G}$ must maintain **non-zero** the multipliers ${f^A}_r$, since they retain the information needed by the procedure to "remember" the original ideal, and therefore the given pde.

Comparison of the coefficients of the various independent 2-forms on both sides of Eq. (4) determines those jet-variables on which F^A and G^B do **not** depend, **and** expresses the Lagrange multipliers, in terms of derivatives of the F^A and G^B. The only remaining requirements of closure, in Eq. (4), are that the coefficient of $dx \wedge dy$ should vanish. This particular coefficient is the expression of the commutator in Eq. (3), obtained by this method.

For the RT equation, we choose our ideal, $\mathcal{K}$, as that ideal within $\Lambda^2(J^{(1)})$ generated by

$$(du - p\,dx) \wedge dy \quad , \quad (du - q\,dy) \wedge dx \quad , \quad dp \wedge dx - dq \wedge dy + (x+y)e^{-2u}\,dx \wedge dy \quad . \tag{5}$$

While this is in fact not the smallest choice, its symmetry makes the problem rather easier. (We will show in Appendix I that making other choices does not change our (minimal) result, concerning K_2.) Comparing coefficients gives us the non-dependencies, the Lagrange multipliers, and the commutator equation for this particular ideal:

$$\begin{aligned} &\mathbf{F}_q = 0 = \mathbf{G}_p \quad ; \qquad \lambda_1^A = G_u^A \quad , \quad \lambda_2^A = F_u^A \quad , \quad \lambda_3^A = Z^A \quad , \\ &[\mathbf{F} + \partial_x, \mathbf{G} + \partial_y] = -p\,\mathbf{G}_u + q\,\mathbf{F}_u + \tfrac{1}{2}(x+y)e^{-2u}(\mathbf{F}_p - \mathbf{G}_q) \quad . \end{aligned} \tag{6}$$

Comparing coefficients in Eq. (6), the stated dependencies allow us to infer the existence of vertical vector fields $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{Z}$ such that

$$\mathbf{F} = p\,\mathbf{Z} + \mathbf{B} \quad , \quad \mathbf{G} = -q\,\mathbf{Z} + \mathbf{C} \quad , \quad \mathbf{Z}_u = 0 \quad , \tag{7}$$

Re-inserting these forms into Eq. (6), it becomes a polynomial in p and q, so that the vanishing of all of the separate coefficients gives the following equations:

$$[\mathbf{Z}\,,\,\mathbf{C}] = -\mathbf{C}_u + \mathbf{Z}_y \quad , \quad [\mathbf{Z}\,,\,\mathbf{B}] = +\mathbf{B}_u + \mathbf{Z}_x \quad , \tag{8a}$$

$$[\mathbf{B}\,,\,\mathbf{C}] = \mathbf{B}_y - \mathbf{C}_x + (x+y)e^{-2u}\mathbf{Z} \quad . \tag{8b}$$

To proceed further with the integration of these equations, we must make *some* assumption concerning the dependence on the independent variables. In most studies of pde's via the EW prolongation procedure, it is common to assume no dependence on the independent variables [19], although the Ernst equation has indeed been an exception [10] to this. However, having explicit dependence on those variables, it should be clear that this equation will require some dependence of $\mathbf{F}$ and $\mathbf{G}$ on $\{x, y\}$. Originally, we argued that the most reasonable approach would be to assume that $\mathbf{F}_x = 0 = \mathbf{G}_y$, since those derivatives would not appear in the final expression anyway. This approach can in fact be completed, and will be discussed in Appendix II; however, at least in this instance, we will show that it is gauge equivalent to the opposite approach, namely that $\mathbf{F}_y = 0 = \mathbf{G}_x$, which is the rather simpler road we shall now follow. Since $\mathbf{Z}$ appears in the expressions for both of $\mathbf{F}$ and $\mathbf{G}$, this requires that $\mathbf{Z}_x = 0 = \mathbf{Z}_y$, and reduces Eqs. (8) to

$$[\mathbf{Z}\,,\,\mathbf{C}] = -\mathbf{C}_u \quad , \quad [\mathbf{Z}\,,\,\mathbf{B}] = +\mathbf{B}_u \quad , \quad [\mathbf{B}\,,\,\mathbf{C}] = (x+y)e^{-2u}\,\mathbf{Z} \quad , \tag{9}$$

where $\mathbf{B} = \mathbf{B}(x)$ and $\mathbf{C} = \mathbf{C}(y)$.

The first two of Eqs. (9) are simply flow equations for a vector field [18], which are immediately integrated to give

$$\mathbf{B}(x,u) = e^{+u(\mathrm{ad}\,\mathbf{Z})}\,\mathbf{R}(x) \qquad , \qquad \mathbf{C}(y,u) = e^{-u(\mathrm{ad}\,\mathbf{Z})}\,\mathbf{S}(y) \quad , \tag{10}$$

where we have indicated explicitly the assumed x- and y-dependence. Inserted into the last of Eqs. (9), these forms give us the "last" requirement,

$$[e^{+u(\mathrm{ad}\,\mathbf{Z})}\,\mathbf{R}(x)\,,\,e^{-u(\mathrm{ad}\,\mathbf{Z})}\,\mathbf{S}(y)] \quad = \quad (x+y)e^{-2u}\,\mathbf{Z} \quad . \tag{11}$$

We interpret this condition as the agreement of two power series in u; the coefficients of $u^k/k!$ are given by

$$\sum_{m=0}^{k}\left(\frac{-1}{2}\right)^k\binom{k}{m}[\mathbf{R}_{(k-m)}\,,\,\mathbf{S}_{(m)}] = (x+y)\,\mathbf{Z} \quad , \forall k = 0,1,2,\ldots \quad , \tag{12}$$

where the subscripts in parentheses indicate repeated commutators with $\mathbf{Z}$:

$$\mathbf{R}_{(m)} \equiv (-1)^m(\mathrm{ad}\,\mathbf{Z})^m\,\mathbf{R}(x) \quad , \quad \mathbf{S}_{(m)} \equiv (\mathrm{ad}\,\mathbf{Z})^m\,\mathbf{S}(y) \quad , \forall m = 0,1,2,\ldots \quad . \tag{13}$$

The 0-th order term of Eq. (11) implies that $[\mathbf{Z}\,,[\mathbf{R}(x)\,,\mathbf{S}(y)]] = 0$. Inserting this fact into the Jacobi identity shows that the commutators $[\mathbf{R}_{(k-m)}\,,\mathbf{S}_{(m)}]$ are actually independent of the value of m, allowing us to sum the series in Eqs. (12):

$$[\mathbf{R}_{(i)}\,,\mathbf{S}_{(j)}] = (x+y)\,\mathbf{Z} \quad , \forall i,j = 0,1,2,\ldots \quad . \tag{14}$$

While there is no obvious requirement that the various $\mathbf{R}_{(i)}$, for example, be parallel, the fact that the right-hand side of the equation depends on neither i nor j does surely suggest such a thought. In Appendix II, we describe the more general case, while here we take as an additional assumption that all $\mathbf{R}_{(i)}$ are parallel, and that all $\mathbf{S}_{(j)}$ are parallel. The coefficients of proportionality are determined uniquely, causing the infinite sums for $\mathbf{B}$ and $\mathbf{C}$ to both become proportional to e^{-u}. This allows all u-dependence to be factored out of Eq. (11), reducing it to the simpler requirement:

$$[\mathbf{Z},\, \mathbf{S}] = \mathbf{S} \quad , \quad [\mathbf{Z},\, \mathbf{R}] = -\mathbf{R} \quad , \quad \Longrightarrow \quad [\mathbf{R}(\mathbf{x}),\, \mathbf{S}(\mathbf{y})] = (x+y)\, \mathbf{Z} \quad . \tag{15}$$

The explicit existence of x and y in the original pde generated the need for x- and y-dependence of our prolongation vector fields. Since, however, they are linear in those variables, and display themselves explicitly that way in the last of the equations in Eqs. (15), it seems sufficient to consider the yet-further special case when $\mathbf{R}(x)$ and $\mathbf{S}(y)$ are just first-order polynomials in their respective jet variables:

$$\mathbf{R}(x) \equiv -\mathbf{f}_1 - x\mathbf{f}_2 \qquad , \qquad \mathbf{S}(y) \equiv +\mathbf{e}_2 + y\mathbf{e}_1 \quad . \tag{16}$$

Inserting these forms into Eqs. (15) gives the complete presentation of the prolongations of the total derivatives as explicit functions of the original jet variables:

$$\mathbf{F} = p\mathbf{Z} - e^{-u}(\mathbf{f}_1 + x\mathbf{f}_2) \quad , \quad \mathbf{G} = -q\mathbf{Z} + e^{-u}(\mathbf{e}_2 + y\mathbf{e}_1) \quad . \tag{17}$$

The final requirements on the system are simply statements of some of the commutators of the vector fields on the fibers (of pseudopotentials) themselves:

$$\begin{aligned} [\mathbf{Z}, \mathbf{e}_i] = \mathbf{e}_i \quad , \quad [\mathbf{Z}, \mathbf{f}_i] = -\mathbf{f}_i \quad , \; i = 1,2 \quad , \\ [\mathbf{e}_2, \mathbf{f}_1] = 0 = [\mathbf{e}_1, \mathbf{f}_2] \quad , \quad [\mathbf{e}_1, \mathbf{f}_1] = \mathbf{Z} = [\mathbf{e}_2, \mathbf{f}_2] \quad . \end{aligned} \tag{18}$$

Referring to Eqs. (6), we see that the three Lagrange multipliers are now proportional to $\{\mathbf{R}(x), \mathbf{S}(y), \mathbf{Z}\}$. Our next task is to determine a realization of the algebra defined by these 5 generators, which maintains these three quantities linearly independent. This algebra, defined by the 5 generators above, is still not completely displayed since the quantities $[\mathbf{e}_1, \mathbf{e}_2]$ and $[\mathbf{f}_1, \mathbf{f}_2]$ are not given, and are therefore to be considered arbitrary modulo the requirements of the Jacobi identity. As examples of these sorts of requirements, it is straight-forward to show that

$$\begin{aligned} [\mathbf{Z}, \{(\mathrm{ad}\,\mathbf{e}_1)^n \mathbf{e}_2\}] = (n+1)\{(\mathrm{ad}\,\mathbf{e}_1)^n \mathbf{e}_2\} \,, \\ [\mathbf{e}_2, \{(\mathrm{ad}\,\mathbf{f}_2)^{m+1} \mathbf{f}_1\}] = -\tfrac{1}{2}(m+1)(m+2)\{(\mathrm{ad}\,\mathbf{f}_2)^m \mathbf{f}_1\} \,. \end{aligned} \tag{19}$$

A presentation of any Lie algebra as a direct sum of subspaces, with the following requirement on the Lie bracket operation, is referred to as an (integer)-graded Lie algebra:

$$\mathcal{G} = \mathop{\oplus}_{i=-\infty}^{\infty} \mathcal{G}_i \ , \qquad [\mathcal{G}_i \, , \, \mathcal{G}_j] \subseteq \mathcal{G}_{i+j} \quad . \tag{20}$$

If d_i is the dimension of $\mathcal{G}_i$, as a vector space, then

$$r \equiv \overline{\lim_{i\to\infty}} \left\{ \log\left(\sum_{j=-i}^{i} d_j\right) \Big/ \log(i) \right\} \tag{21}$$

is called [13,14] the *growth* of the full Lie algebra, $\mathcal{G}$. Finite-dimensional Lie algebras have growth 0, while those usually referred to as Kac-Moody algebras have finite growth. We refer to $\hat{\mathcal{G}} \equiv \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ as *the local part* of $\mathcal{G}$, and supplement our definition of a graded algebra by insisting that it should be (algebraically) generated by commutators of its local part, so that the grading is then well-defined. (This is of course what one would expect.) For our algebra, we take the local part as follows:

$$\mathcal{G}_0 \equiv \{\mathbf{Z}\} \quad , \quad \mathcal{G}_{-1} \equiv \{\mathbf{f}_i \mid i = 1, 2\} \quad , \quad \mathcal{G}_1 \equiv \{\mathbf{e}_i \mid i = 1, 2\} \quad . \tag{22}$$

The first equality in Eqs. (19) then tells us that the dimension of $\mathcal{G}_i$ is growing rapidly **unless** the objects $\{(\mathrm{ad}\,\mathbf{e}_1)^n \mathbf{e}_2\}$ were to vanish from some value of n onward, which is indeed what occurs in a Kac-Moody algebra. The second equality in Eqs. (19) tells us that if those quantities were to vanish, there would be a downward cascade causing that particular entire part of the structure to vanish, leaving us with zero values for our Lagrange multipliers, which is of course unacceptable. We may therefore conclude that this algebra does indeed grow quite fast.

In fact this algebra may be completely identified. It is the simplest contragredient algebra of infinite growth, referred to as K_2. In Chapter II of Ref. 14, Kac defines general contragredient algebras associated with a given matrix A with integer elements. They are integer-graded algebras with certain requirements on the Lie brackets of the basis elements of the local part. Let $\{f_i\}$, $\{h_i\}$, and $\{e_i\}$, be basis vectors for $\mathcal{G}_{-1}$, $\mathcal{G}_0$, and $\mathcal{G}_1$, respectively. We first require that their commutators satisfy the following:

$$[\mathbf{e}_i, \mathbf{f}_j] = \delta_{ij}\mathbf{h}_i \quad , \quad [\mathbf{h}_i, \mathbf{h}_j] = 0 \quad , \quad [\mathbf{h}_i, \mathbf{e}_j] = A_{ij}\mathbf{e}_j \quad , \quad [\mathbf{h}_i, \mathbf{f}_j] = -A_{ij}\mathbf{f}_j \tag{23}$$

The contragredient Lie algebra is then the minimal graded Lie algebra, with local part $\hat{\mathcal{G}}$. (Beginning with any algebra generated by this local part, finding the largest homogeneous ideal that contains no elements of $\mathcal{G}_0$ (except 0) and then factorizing the algebra over this ideal will create the minimal one.) In the special case that the matrix A has its diagonal elements positive (usually normalized to $+2$), its off-diagonal elements non-positive, and all $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$, for $i \neq j$, then it is called a generalized Cartan matrix.

For our problem, we may now consider a contragredient algebra with matrix A such that

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad . \tag{24}$$

Our 3 sets of basis vectors are $\{h_1, h_2\}$, $\{e_1, e_2\}$, and $\{f_1, f_2\}$, so that $[h_i, e_j] = e_j$, etc., for $i, j = 1, 2$. Therefore, we see that $(h_1 - h_2)$ is central, so that we may factor our algebra by it. The resulting algebra is K_2, except that Kac normalizes his vectors so that all the elements of A have the value 2 instead of 1. K_2 is now easily seen to be isomorphic to the prolongation algebra we have determined for the RT equation of type III, with $Z \rightarrow h_1 \pmod{h_1 - h_2}$.

Having determined the smallest prolongation algebra, the next step in the process of finding new solutions is to write down explicit vector- field (or matrix) realizations of this algebra, use the variables in the carrier space as pseudopotentials, pick out a Bäcklund transformation, take the one existing solution, and begin to generate new ones, as has been done many times before with many other interesting pde's. The difficulty, in this case, is that **no** realizations of K_2 have yet been discovered. Since this is indeed the minimal prolongation algebra, we see that there is considerable correspondence between the two problems. It seems reasonable to suppose that finding new solutions is equivalent to evolving realizations of this algebra. Therefore, the main purpose of this report is to encourage its listeners to try to achieve at least some non-trivial realization of K_2.

Acknowledgments: Considerable appreciation needs to be expressed to Alice Fialowski who better acquainted me with contragredient algebras, to Mark Hickman, who suggested that this problem should be amenable to prolongation techniques, to John K. McIver, who listened to very many descriptions of the difficulties of the problem, to Mikhail Saveliev, who noticed the gauge transformations used in reducing the problem to a size that could be dealt with, and to Michael Wester, who created the computer code for Macsyma that allowed me to know that reductions of the algebras were (or were not) compatible with the Jacobi identity.

Appendix I: Other Choices of Generating Ideal

We have found it convenient to use a symmetric choice for the generators of our subideal of the complete, restricted contact module. Other particular choices of ideal may generate distinct maximal algebras [20,21]; nonetheless we now show that other plausible choices do not actually change the minimal algebra involved in the prolongation process for this pde. The sine-Gordon equation is very similar to our equation, simply not involving explicitly the independent variables. Our symmetric ideal is actually modelled on that used by Shadwick [22] for the sine-Gordon equation. On the other hand, many other authors, including in particular Hoenselaers [8], have used an ideal for the sine-Gordon equation that is asymmetric, **and** contains fewer generators. These two actually constitute all the reasonable choices one can make [23].

Following Hoenselaers' model [8], an alternative ideal would have the following generators:

$$(du - p\,dx)\wedge dy \quad , \quad dp\wedge dx + \tfrac{1}{2}(x+y)e^{-2u}\,dx\wedge dy \quad . \qquad (A1.1)$$

Since this does have both fewer generators and fewer variables, not using q, than the one we described in Eqs. (6), one could indeed hope for "nicer" results. Following the same procedure as before, the analogue of Eqs. (7) is quickly found to be

$$\begin{gathered}\mathbf{F}=\mathbf{F}(x,y,p)\ ,\ \mathbf{G}=\mathbf{G}(x,y,u)\quad ;\qquad \lambda_1=\mathbf{G}_u \quad , \quad \lambda_2=\mathbf{F}_p \quad ,\\ [\mathbf{F}+\partial_x,\mathbf{G}+\partial_y] = -p\,\mathbf{G}_u + \tfrac{1}{2}(x+y)e^{-2u}\,\mathbf{F}_p \quad .\end{gathered} \qquad (A1.2)$$

Introducing the new quantity $\mathbf{P}\equiv\mathbf{G}+\frac{1}{2}\mathbf{G}_u$, reduces the commutator equation in Eqs. (A1.2) to the simpler form $[\mathbf{F}+\partial_x\,,\mathbf{P}+\partial_y] = -p\,\mathbf{P}_u$. The general solution can be worked out in an analogous fashion to that shown in Ref. 21; however, by first taking two successive derivatives with respect to p, resulting in $[\mathbf{F}_{pp}\,,\mathbf{P}]=0$, we can pick out the smallest interesting piece of it, again in a manner analogous to Ref. 8, by simply setting both the objects in this last commutator, separately, to zero, which gives us

$$\mathbf{F}=p\mathbf{A}+\mathbf{B} \quad , \qquad \mathbf{G}=e^{-2u}\mathbf{C} \quad , \qquad (A1.3)$$

where $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ all depend on both x and y, but with no necessity for dependence more complicated than linear. Inserting these forms back into the original commutator equation produces the following results, with the u-dependence already completely satisfied:

$$\mathbf{A}_y = 0 = \mathbf{B}_y\ ; \quad [\mathbf{A}\,,\mathbf{C}] = 2\mathbf{C}\ ; \quad \mathbf{C}_x + [\mathbf{B}\,,\mathbf{C}] = \tfrac{1}{2}\,(x+y)\,\mathbf{A} \quad . \qquad (A1.4)$$

Expanding each of these vector fields as first-order polynomials, in the form

$$\mathbf{A} = 2\mathbf{A}_0 + 2x\mathbf{A}_1 \ , \ \mathbf{B} = \mathbf{B}_0 + x\mathbf{B}_1 \ , \ \mathbf{C} = \mathbf{C}_0 + x\mathbf{C}_1 + y\mathbf{C}_2 \quad , \tag{A1.5}$$

we easily calculate the commutators required, and find that linear independence of our Lagrange multipliers insists that the set $\{\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0, \mathbf{C}_2\}$ must remain linearly independent, and of course non-zero. There are two plausible special cases of interest here: Case 1 sets $\mathbf{A}_1 = 0 = \mathbf{B}_1$, while case 2 sets $\mathbf{A}_1 = 0 = \mathbf{C}_1$. For case 1, we have 5 generators, with all but 4 of the 10 commutator products already determined. We present the commutators in the form of a table, and do not bother to indicate the lower-triangular portion since it is of course skew-symmetric:

	$\mathbf{A}_0$	$\mathbf{B}_0$	$\mathbf{C}_0$	$\mathbf{C}_1$	$\mathbf{C}_2$
$\mathbf{A}_0$	0		$\mathbf{C}_0$	$\mathbf{C}_1$	$\mathbf{C}_2$
$\mathbf{B}_0$		0	$-\mathbf{C}_1$	$\mathbf{A}_0$	$\mathbf{A}_0$
$\mathbf{C}_0$			0		
$\mathbf{C}_1$				0	
$\mathbf{C}_2$					0

(A1.6)

The 4 omitted entries in the upper-triangular portion must still be determined. Taking $\mathbf{A}_0$ as an element in the Cartan subalgebra, $\mathcal{G}_0$, we see that all of $\mathbf{C}_i$ constitute positive roots, but there are no immediately-determined negative roots. A plausible "cure" for this is to identify the undetermined commutator, $[\mathbf{A}_0\,,\mathbf{B}_0]$ as a negative root; i.e., to assume that $[\mathbf{A}_0\,,[\mathbf{A}_0\,,\mathbf{B}_0]] = -[\mathbf{A}_0\,,\mathbf{B}_0]$, consistent with the Jacobi identity. Another identification that reduces the number of unknown commutators in a manner consistent with the Jacobi identity is to identify $\mathbf{C}_1 = \mathbf{C}_2$; this has the obvious justification that it causes $\mathbf{C}$ to depend only on $x + y$, just as does the pde itself. At this point, the entire algebra—with the notable exception of $\mathbf{B}_0$—can be identified with (our version of) K_2 again, using the contragredience matrix A given by Eq. (24):

$$\mathbf{A}_0 \to \mathbf{h}\,,\ \mathbf{C}_0 \to \mathbf{e}_1\,,\ \mathbf{C}_1 = \mathbf{C}_2 \to \mathbf{e}_2\,,\ [\mathbf{A}_0\,,\mathbf{B}_0] \to \mathbf{f}_2\,,\ [\mathbf{B}_0\,,[\mathbf{A}_0\,,\mathbf{B}_0]] \to \mathbf{f}_1\,. \tag{A1.7}$$

Since $\mathbf{B}_0$ is a subalgebra, we may then identify the algebra at this point with the semi-direct sum of K_2 and $\{\mathbf{B}_0\}$, which is satisfactory for our current purposes. (See the related result in Appendix II.)

Following case 2 equally far, $\mathbf{F}$ depends only on x (and p) while $\mathbf{G}$ depends only on y (and u), as was the case for the symmetric ideal already discussed. Again we have five generators with only six of the commutators already determined. The known commutators are

	$\mathbf{A}_0$	$\mathbf{B}_0$	$\mathbf{B}_1$	$\mathbf{C}_0$	$\mathbf{C}_2$
$\mathbf{A}_0$	0			$\mathbf{C}_0$	$\mathbf{C}_2$
$\mathbf{B}_0$		0		0	$\mathbf{A}_0$
$\mathbf{B}_1$			0	$\mathbf{A}_0$	0
$\mathbf{C}_0$				0	
$\mathbf{C}_2$					0

$(A1.8)$

The 4 omitted entries in the upper-triangular portion must still be determined. Taking $\mathbf{A}_0$ as an element in the Cartan subalgebra, $\mathcal{G}_0$, we see that the $\mathbf{C}_i$ constitute positive roots, but there are no immediately-determined negative roots. A plausible "cure" identifies the $\mathbf{B}_j$ as negative roots, thereby determining two of the previously-unknown commutators. This is consistent with the Jacobi identity and directly identifies the algebra as K_2, with matrix A given by Eq. (24):

$$\mathbf{A}_0 \to \mathbf{h}\,,\ \mathbf{B}_0 \to -\mathbf{f}_2\,,\ \mathbf{B}_1 \to -\mathbf{f}_1\,, \mathbf{C}_0 \to \mathbf{e}_1\,,\ \mathbf{C}_2 \to \mathbf{e}_2\,. \qquad (A1.9)$$

Therefore, this ideal also always leads to algebras of infinite growth, certainly containing K_2.

Appendix II: Other Choices of $\{x, y\}$-dependence

Following Eqs. (9), we considered further only the special case where $\mathbf{F}_y = 0 = \mathbf{G}_x$, resulting in Eqs. (10), which were much simpler. However, the alternative set of assumptions is also viable and justifiable, i.e., setting $\mathbf{F}_x = 0 = \mathbf{G}_y$. We argue that this is reasonable since these particular derivatives never appear, explicitly, within Eqs. (9). At this point the first two of Eqs. (9) each have the form of a vector-field-valued pde which is somewhat more complicated than simply the usual flow equations. However, since the derivative operator acting on either one of the (to-be-determined) vector fields gives exactly zero when operating on the other—due to the assumptions just made—we can actually still manage to integrate these equations, the solution to which we describe below in the following terms.

Lemma: Solution of the pde $\big[\mathbf{A},\, \mathbf{R}\big] = \mathbf{A}_x + \mathbf{R}_u$.

We suppose given two vertical vector fields, $\mathbf{A}$ and $\mathbf{R}$, elements of the Lie algebra of vector fields over the space W of our pseudopotentials. As these lie in the tangent bundle to fibers over $J^{(2)}$, they also depend on two **disjoint** sets of other variables, say $\mathbf{A} = \mathbf{A}(x, y)$ and $\mathbf{R} = \mathbf{R}(u, v)$, and are required to satisfy the pde

$$\big[\mathbf{A},\, \mathbf{R}\big] = \mathbf{A}_x + \mathbf{R}_u \quad , \tag{A2.1}$$

where as usual the subscripts indicate partial derivatives. The solution is determined by first differentiating the equation with respect to, say, x, which annuls the derivative of $\mathbf{R}$, providing a flow equation for $\mathbf{A}_{xx}$ along $\mathbf{R}$, which we integrate, taking proper care of the fact that while the general form of $\mathbf{R}$ depends on both u and v, $\mathbf{A}_{xx}$ depends on neither one. We then differentiate with respect to u, and follow an analogous procedure for $\mathbf{R}_{uu}$. The general solution is then obtained by substituting back into the original equation and making all "constants of integration" behave properly. That solution is determined by the sets of vector fields $\mathbf{A}_0(y) \equiv \sum_{m=0}^{\infty} \frac{y^m}{m!} \mathbf{A}_{0m}$ and $\mathbf{R}_0(v) = \sum_{k=0}^{\infty} \frac{v^k}{k!} \mathbf{R}_{0m}$, and the field $\mathbf{A}_{10}$ or $\mathbf{R}_{10}$, which are related symmetrically by $\mathbf{A}_{10} - \mathbf{R}_{10} = [\mathbf{R}_{00}\, , \mathbf{A}_{00}]$, such that

$$\begin{aligned}
\mathbf{A}(x, y) &= \mathbf{A}_0(y) + \sum_{m=0}^{\infty} \frac{(-x)^{m+1}}{(m+1)!} (\mathrm{ad}\, \mathbf{R}_{00})^m \mathbf{A}_1(y)\, , \text{ with } \mathbf{A}_1(y) \equiv \mathbf{R}_{10} + [\mathbf{R}_{00},\, \mathbf{A}_0(y)]\, , \\
\mathbf{R}(u, v) &= \mathbf{R}_0(v) + \sum_{k=0}^{\infty} \frac{(+u)^{k+1}}{(k+1)!} (\mathrm{ad}\, \mathbf{A}_{00})^k \mathbf{R}_1(v)\, , \text{ with } \mathbf{R}_1(v) \equiv \mathbf{A}_{10} + [\mathbf{A}_{00},\, \mathbf{R}_0(v)]\, ,
\end{aligned} \tag{A2.2}$$

along with a collection of requirements on the commutators of these vector fields, which are most easily expressed by setting $\mathbf{A}_{m+1}(y)$ as the $(m+1)$-st term in the expansion, in powers of x, of $\mathbf{A}(x,y)$, above, and $\mathbf{R}_{k+1}(v)$ as the $(k+1)$-st term in the expansion, in powers of u, of $\mathbf{R}(u,v)$ The entire collection of constraints is then easily stated as the quadruply countable set:

$$\left[\mathbf{A}_{m+1}(y)\,,\mathbf{R}_{k+1}(v)\right]=0\quad\forall\,k,m=0,1,2,\,\ldots\quad. \tag{A2.3}$$

We now apply this lemma to the two appropriate equations, in Eqs. (9). The second of these equations involves $\mathbf{B}(u,y)$ and $\mathbf{Z}(x,y)$, so that the y-dependence overlaps, but is "irrelevant" for this particular pde. Choosing to use only $\mathbf{X}_{10}$, our lemma provides us with new vector fields, $\mathbf{R}(y)$, $\mathbf{X}_0(y)$, and $\mathbf{X}_1(y)$, such that

$$\begin{aligned}\mathbf{B}(u,y)&=e^{u(\mathrm{ad}\,\mathbf{X}_0)}\,\mathbf{R}(y)+\sum_{m=0}^{\infty}\frac{u^{m+1}}{(m+1)!}(\mathrm{ad}\,\mathbf{X}_0)^m\,\mathbf{X}_1(y)\quad,\\ \mathbf{Z}(x,y)&=\mathbf{X}_0(y)+\sum_{\ell=0}^{\infty}\frac{(-x)^{\ell+1}}{(\ell+1)!}((\mathrm{ad}\,\mathbf{R}))^{\ell}\,\mathbf{X}_1(y)\quad,\end{aligned} \tag{A2.4}$$

along with the commutator requirements that

$$\left[(\mathrm{ad}\;\mathbf{R})^{\ell+1}\mathbf{X}_0\,,\,(\mathrm{ad}\,\mathbf{X}_0)^m(\mathbf{X}_1+[\mathbf{X}_0\,,\,\mathbf{R}])\right]=0\quad,\quad\forall\,\ell,m=0,1,2,\ldots\quad. \tag{A2.5}$$

The same lemma applied to the first equation, involving $\mathbf{C}(u,x)$ and $\mathbf{Z}(y,x)$, gives us the existence of vector fields $\mathbf{S}(x)$, $\mathbf{Y}_0(x)$ and $\mathbf{Y}_1(x)$ such that

$$\begin{aligned}\mathbf{C}(u,x)&=e^{-u(\mathrm{ad}\,\mathbf{Y}_0)}\,\mathbf{S}(x)+\sum_{n=0}^{\infty}\frac{(-u)^{n+1}}{(n+1)!}(\mathrm{ad}\,\mathbf{Y}_0)^m\,\mathbf{Y}_1(x)\quad,\\ \mathbf{Z}(y,x)&=\mathbf{Y}_0(x)+\sum_{\ell=0}^{\infty}\frac{(-y)^{\ell+1}}{(\ell+1)!}(\mathrm{ad}\,\mathbf{S})^{\ell}\,\mathbf{Y}_1(x)\quad,\end{aligned} \tag{A2.6}$$

along with the commutator requirements that

$$\left[(\mathrm{ad}\,\mathbf{S}))^{\ell+1}\,\mathbf{Y}_0\,,\,(\mathrm{ad}\,\mathbf{Y}_0)^n\,(\mathbf{Y}_1+[\mathbf{Y}_0\,,\,\mathbf{C}_0])\right]=0\quad,\quad\forall\,\ell,n=0,1,2,\ldots\quad. \tag{A2.7}$$

The requirement that $\mathbf{Z}(x,y)$, as presented in Eqs. (A2.4) and (A2.6), should be the same is a very strong constraint on the underlying vector fields. A term-by-term comparison is, in principle, required. For instance, the lowest-order requirement is that $\mathbf{Y}(x)$ and $\mathbf{X}(y)$ should be related as follows:

$$\mathbf{Y}_0(x)\;=\mathbf{Z}_0+\sum_{\ell=0}^{\infty}\frac{(-x)^{\ell+1}}{(\ell+1)!}(\mathrm{ad}\,\mathbf{R}_{00})^{\ell}\,\mathbf{X}_{10}\,,\quad\mathbf{X}_0(y)\;=\mathbf{Z}_0+\sum_{\ell=0}^{\infty}\frac{(-y)^{\ell+1}}{(\ell+1)!}(\mathrm{ad}\,\mathbf{S}_{00})^{\ell}\,\mathbf{Y}_{10}\,, \tag{A2.8}$$

while we could also write out the requirements on $\mathbf{Y}_1(x)$, $\mathbf{X}_1(y)$, etc. These requirements would then have to be inserted into the shapes for $\mathbf{B}$ and $\mathbf{C}$. While we have indeed written out yet much more general series of equations for this problem, we nonetheless feel justified at this point to append to these equations some additional assumptions that simplify the problem enough to be presented with only a finite amount of formalism. Therefore, at this point, we consider only the case when $\mathbf{Z}$ is completely independent of both x and y, thus reducing all the expressions for $\mathbf{Z}$ above to a single term, which we will refer to as $\mathbf{Z}_0$, a vector field defined only over the fiber variables, w^A. As well, we again *assume* that it is reasonable to truncate the series for $\mathbf{R}(y)$ and $\mathbf{S}(x)$ to make them first-order polynomials:

$$\mathbf{R}(y) = \mathbf{R}_0 + y\mathbf{R}_1 \quad , \quad \mathbf{S}(x) = \mathbf{S}_0 + x\mathbf{S}_1 \quad , \tag{A2.9}$$

Insertion of Eqs. (A2.4, A2.6) and (A2.9) into the remaining portion of Eqs. (9) gives us a collection of requirements on the vector fields already named, i.e., $\{\mathbf{Z}_0, \mathbf{R}_0, \mathbf{R}_1, \mathbf{S}_0, \mathbf{S}_1\}$, as well as additional ones which involve repeated commutators with $\mathbf{Z}_0$ that have not yet been named. We first list all the requirements that follow when one evaluates at $u = 0$, therefore involving only those vector fields just named above:

	$\mathbf{R}_0$	$\mathbf{R}_1$	$\mathbf{S}_0$	$\mathbf{S}_1$	$\mathbf{Z}_0$
$\mathbf{R}_0$	0		$\mathbf{R}_1 - \mathbf{S}_1$	$\mathbf{Z}_0$	$+\mathbf{T}_0$
$\mathbf{R}_1$		0	$\mathbf{Z}_0$	0	$+\mathbf{V}_0$
$\mathbf{S}_0$			0		$-\mathbf{U}_0$
$\mathbf{S}_1$				0	$-\mathbf{W}_0$
$\mathbf{Z}_0$					0

(*A*2.10)

The quantities in the last row and column are new quantities, that will be needed at higher powers in u, which we now define generically, $\forall n = 0, 1, 2, \ldots$:

$$\begin{gathered} \mathbf{T}_n \equiv (-1)^n (\operatorname{ad}\mathbf{Z}_0)^n [\mathbf{Z}_0\,,\,\mathbf{R}_0]\,, \quad \mathbf{V}_n \equiv (-1)^n (\operatorname{ad}\mathbf{Z}_0)^n [\mathbf{Z}_0\,,\,\mathbf{R}_1] \\ \mathbf{U}_n \equiv (\operatorname{ad}\mathbf{Z}_0)^n [\mathbf{Z}_0\,,\,\mathbf{S}_0]\,, \qquad \mathbf{W}_n \equiv (\operatorname{ad}\mathbf{Z}_0)^n [\mathbf{Z}_0\,,\,\mathbf{S}_1] \qquad . \\ \mathbf{X}_{\ell p} \equiv \big[\mathbf{T}_\ell,\,\mathbf{V}_p\big]\,, \qquad \mathbf{Y}_{mn} \equiv \big[\mathbf{U}_m,\,\mathbf{W}_n\big] \end{gathered} \tag{A2.11}$$

The quantities in the last line are not determined by the requirements of the equation; however, we will show below that it does not permit them to vanish, so that we now give them names.

The rest of Eqs. (9) requires two additional sets of commutators involving these new quantities. The first set involves the mixed commutators

$$\begin{array}{cc}
 & \begin{array}{cccccc} \mathbf{T}_n & \mathbf{V}_n & \mathbf{X}_{np} & \mathbf{U}_n & \mathbf{W}_n & \mathbf{Y}_{np} \end{array} \\
\begin{array}{c} \mathbf{Z}_0 \\ \mathbf{R}_0 \\ \mathbf{R}_1 \\ \mathbf{S}_0 \\ \mathbf{S}_1 \end{array} & \left(\begin{array}{cccccc}
-\mathbf{T}_{n+1} & -\mathbf{V}_{n+1} & \mathbf{X}_{n+1,p}+\mathbf{X}_{n,p+1} & +\mathbf{U}_{n+1} & +\mathbf{W}_{n+1} & -\mathbf{Y}_{n+1,p}-\mathbf{Y}_{n,p+1} \\
 & & & -\mathbf{W}_n & \mathbf{Z}_0 & \\
 & & & \mathbf{Z}_0 & 0 & \mathbf{W}_{p+1} \\
-\mathbf{V}_n & \mathbf{Z}_0 & & & & \\
\mathbf{Z}_0 & 0 & -\mathbf{V}_{p+1} & & &
\end{array}\right)
\end{array}, \tag{A2.12}$$

while the second set describes the commutators between the higher-level ones, themselves:

$$\begin{array}{cc}
 & \begin{array}{cccccc} \mathbf{T}_n & \mathbf{V}_n & \mathbf{U}_n & \mathbf{W}_n & \mathbf{X}_{np} & \mathbf{Y}_{np} \end{array} \\
\begin{array}{c} \mathbf{T}_m \\ \mathbf{V}_m \\ \mathbf{U}_m \\ \mathbf{W}_m \\ \mathbf{X}_{mk} \\ \mathbf{Y}_{mk} \end{array} & \left(\begin{array}{cccccc}
 & \mathbf{X}_{mn} & 0 & -\mathbf{Z}_0 & & +\mathbf{U}_{n+1} \\
-X_{nm} & & -\mathbf{Z}_0 & 0 & & -\mathbf{W}_{p+1} \\
0 & +\mathbf{Z}_0 & & \mathbf{Y}_{mn} & +\mathbf{T}_{n+1} & \\
+\mathbf{Z}_0 & 0 & -\mathbf{Y}_{nm} & & -\mathbf{V}_{p+1} & \\
 & & -\mathbf{T}_{m+1} & +\mathbf{V}_{k+1} & & 2\mathbf{Z}_0 \\
-\mathbf{U}_{m+1} & +\mathbf{W}_{k+1} & & & -2\mathbf{Z}_0 &
\end{array}\right)
\end{array}, \tag{A2.13}$$

The structure above is of course still frightfully complicated; therefore one surely wonders how much of it is "necessary." The Lagrange multipliers supply the answer to at least part of that question. Referring back to Eqs. (7), within the current notation they are simply the three vector fields $\{\mathbf{Z}_0\,,\,\mathbf{T}_0{+}y\,\mathbf{V}_0\,,\,\mathbf{U}_0{+}x\,\mathbf{W}_0\}$. Considering that $\left[\mathbf{X}_{mk},\,\mathbf{Y}_{np}\right]=+2\,\mathbf{Z}_0$, independent of the values of the indices $\{m,k,n,p\}$, we immediately see that none of those undetermined double commutators $\mathbf{X}_{mk}$, nor $\mathbf{Y}_{np}$, may vanish. However, if any of the individual terms within our three vector fields were to vanish, then one or more of these double commutators would indeed have to vanish, therefore requiring us to maintain, at the least, all 5 of those vector fields non-zero and linearly independent.

As a first approach to studying this structure, we go to a very simplified homomorphic image which we name $\mathcal{RT}_0$. The mapping is generated by dropping the subscripts on the newly-created quantities:

$$\begin{array}{c} \mathbf{T}_n \longrightarrow \mathbf{T}_0 \equiv \mathbf{T}\,, \quad \mathbf{V}_n \longrightarrow \mathbf{V}_0 \equiv \mathbf{V} \\ \mathbf{U}_n \longrightarrow \mathbf{U}_0 \equiv \mathbf{U}\,, \quad \mathbf{W}_n \longrightarrow \mathbf{W}_0 \equiv \mathbf{W} \\ \mathbf{Z}_0 \longrightarrow \mathbf{Z}\,, \end{array} \quad , \quad \forall n = 0,1,2,\,\ldots\,, \tag{A2.14}$$

where $\mathbf{Z}_0 \to \mathbf{Z}$ is just to make the typography all appear more consistent. (This is also a useful place to point out that if the second-order terms in x or y had been kept, all their commutators would now be either zero or undetermined, thus motivating our having already dropped them.) We should perhaps also mention that a somewhat more complicated mapping does not work, namely one where $\mathbf{T}_n$ might have been mapped to $(a_T)^n\mathbf{T}$, for some constant a_T. In fact, such quantities a_i are completely determined by the requirement that this mapping actually be a homomorphism.)

Our algebra $\mathcal{RT}_0$ is generated by $\{\mathbf{Z}, \mathbf{R}_0, \mathbf{R}_1, \mathbf{T}, \mathbf{V}, \mathbf{S}_0, \mathbf{S}_1, \mathbf{U}, \mathbf{W}, \mathbf{X}, \mathbf{Y}\}$, and the new Lie product table is just obtained by ignoring the subscripts in the previous tables. Using those tables, we note the existence of a very interesting subalgebra contained within $\mathcal{RT}_0$, namely the one where we drop out the $\{\mathbf{R}_i\,, \mathbf{S}_j\}$. This subalgebra is generated by $\{\mathbf{Z}, \mathbf{T}, \mathbf{V}, \mathbf{U}, \mathbf{W}\}$, and we will refer to it as $\mathcal{RT}_{00}$. Remembering that it will also have $\mathbf{X}$ and $\mathbf{Y}$ as elements, the appropriate commutator table is

$$\mathcal{RT}_{00}\;: \quad \begin{array}{c} \\ \mathbf{Z} \\ \mathbf{U} \\ \mathbf{W} \\ \mathbf{T} \\ \mathbf{V} \\ \mathbf{X} \\ \mathbf{Y} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{Z} & \mathbf{U} & \mathbf{W} & \mathbf{T} & \mathbf{V} & \mathbf{X} & \mathbf{Y} \end{array} \\ \begin{pmatrix} 0 & \mathbf{U} & \mathbf{W} & -\mathbf{T} & -\mathbf{V} & -2\,\mathbf{X} & +2\,\mathbf{Y} \\ & 0 & \mathbf{Y} & 0 & \mathbf{Z} & \mathbf{T} & \\ & & 0 & \mathbf{Z} & 0 & -\mathbf{V} & \\ & & & 0 & \mathbf{X} & & \mathbf{U} \\ & & & & 0 & & -\mathbf{W} \\ & & & & & 0 & +2\,\mathbf{Z} \\ & & & & & & 0 \end{pmatrix} \end{array} . \qquad (A2.15)$$

Since none of these elements is allowed to vanish, this is the fundamental subalgebra within our general prolongation structure. It is isomorphic to the contragredient algebra of infinite growth, K_2, described in the main text, as given by

$$\mathbf{h} \longrightarrow \mathbf{Z}\,,\ \mathbf{e}_1 \longrightarrow \mathbf{U} \quad , \quad \mathbf{e}_2 \longrightarrow \mathbf{W} \quad , \quad \mathbf{f}_1 \longrightarrow \mathbf{V} \quad , \quad \mathbf{f}_2 \longrightarrow \mathbf{T} \quad . \qquad (A2.16)$$

The subalgebra $\mathcal{RT}_{00}$, isomorphic to K_2, is however **not** the entirety of $\mathcal{RT}_0$, which also contains the generators $\mathbf{R}_i$ and $\mathbf{S}_j$. It is straight-forward to show that simply mapping them to zero will not work; i.e., the requirements of the Jacobi identity will cause the structure to collapse sufficiently far that it "forgets" the original pde. In an attempt to better understand the function of these objects, we look for linear combinations of them which are "eigenvectors" of $\mathbf{Z}$, and find that the four new combinations $\mathbf{A} \equiv \mathbf{R}_0 + \mathbf{T}$, $\mathbf{B} \equiv \mathbf{S}_0 - \mathbf{U}$, $\mathbf{M} \equiv \mathbf{R_1} + \mathbf{V}$, and $\mathbf{N} \equiv \mathbf{S}_1 - \mathbf{W}$ all commute with $\mathbf{Z}$. The requirements of the Jacobi identity and **also** the requirements of the Lagrange multipliers do not prevent us from setting $\mathbf{M}$ and $\mathbf{N}$ to 0, which we therefore do. On the other hand, taking the remaining two, $\mathbf{A}$, and $\mathbf{B}$, as generators in lieu of the original $\mathbf{R_0}$ and $\mathbf{S_0}$, respectively, we find that $\mathbf{A} + \mathbf{B}$ is a central element, while neither $\mathbf{A}$ nor $\mathbf{B}$ occur in the commutator ideal. We may therefore rescue the "rest" of the algebra by ignoring that central element, and viewing this algebra as the *semi-direct product* of K_2 and the algebra consisting of the single element $\mathbf{A}$. We maintain the mapping as given in Eq. (A2.16) and append to it the commutation relations with $\mathbf{A}$, as follows:

$$[\mathbf{A}, \mathbf{h}] = 0\,,\ [\mathbf{A}, \mathbf{e}_1] = 0\,,\ [\mathbf{A}, \mathbf{e}_2] = -\mathbf{e}_1\,,\ [\mathbf{A}, \mathbf{f}_1] = +\mathbf{f}_2\,,\ [\mathbf{A}, \mathbf{f}_2] = 0 \quad , \tag{A2.17}$$

and can present the prolongations of the total derivatives in the form

$$\mathbf{F} = p\mathbf{h} + \mathbf{A} - e^{-u}(\mathbf{f}_1 + y\,\mathbf{f}_2)\,, \quad \mathbf{G} = -q\mathbf{h} - \mathbf{A} + e^{-u}(\mathbf{e}_2 + x\mathbf{x}_1) \quad , \tag{A2.18}$$

which should be compared, for instance, with the result in the main text, at Eqs. (17).

The form of $\mathbf{F}$ and $\mathbf{G}$ given in Eqs. (A2.18) is clearly different from that in Eqs. (17); One would argue that this is not surprising since they correspond to two distinct sets of assumptions concerning the dependence of the prolongation quantities on the independent variables. However, it turns out that the two sets are in fact equivalent under a gauge transformation. From the viewpoint of Eq. (3), the quantities $\mathbf{F}$ and $\mathbf{G}$ are prolongations of the total derivatives on J^∞ to the (larger) covering space $J^\infty \otimes W$. Moreover, the so-prolonged derivatives only commute when one restricts the calculation to the subvariety defined by the pde being studied. Therefore it is reasonable to treat the quantities $F^A\,dx + G^A\,dt$ as (the coefficients of a Lie-algebra-valued) connection 1-form on the covering space. Therefore, it should transform in the usual way for connections. The transformations we want to consider correspond to flows of the covering space generated by particular tangent vector fields, so that we are simply moving along a congruence of curves. Moreover, since we are restricting our attention to vertical vector fields, different values of a parameter along the curves just correspond to different choices for values of the fiber coordinates over the same base point. (See Refs. 21 or 24 for considerably more discussion concerning this idea.) The structure of our theory should be independent of distinctions such as this; therefore, we refer to transformations of this type, which simply map different explicit presentations of the underlying geometry into one another, as **gauge** transformations. Given a vertical vector field, $\mathbf{R}$, defined over some (local) portion of our manifold, the flow of that vector field is a (local) mapping of the manifold into itself, that can be presented via a congruence of curves, described by $\Phi_t \equiv e^{t\mathbf{R}} : U \subseteq M \to M$. Under the induced mapping of the tangent bundle, the transformation law for an arbitrary connection 1-form, Γ, would be

$$\Gamma' \equiv \Gamma_t = e^{t(\mathrm{ad}\,\mathbf{R})}\Gamma - d(t\mathbf{R}) \quad . \tag{A2.19}$$

For our particular transformation, we will choose the vector field $\mathbf{R}$, above to be our "extra" algebra element, $\mathbf{A}$, and the flow parameter, t, as $x-y$. Performing the transformation on the $\mathbf{F}$ and $\mathbf{G}$ given in Eqs. (A2.18), we find that the so-transformed quantities, $\mathbf{F}'$ and $\mathbf{G}'$ are in fact identical to the ones given in Eqs. (17):

$$\begin{aligned} \mathbf{F}' \equiv e^{(x-y)(\mathrm{ad}\,\mathbf{A})}\mathbf{F} - D_x\{(x-y)\mathbf{A}\} &= +p\,\mathbf{h} - e^{-u}(\mathbf{f}_1 + x\mathbf{f}_2) \quad , \\ \mathbf{G}' \equiv e^{(x-y)(\mathrm{ad}\,\mathbf{A})}\mathbf{G} - D_y\{(x-y)\mathbf{A}\} &= -q\,\mathbf{h} + e^{-u}(\mathbf{e}_2 + y\mathbf{e}_1) \quad . \end{aligned} \tag{A2.20}$$

Since $\mathbf{A}$ is a vertical vector field, this transformation simplify re-defines the origin in our fiber spaces in a manner that depends explicitly upon the value of the independent variable $x-y$. This transformation has two immediate effects. It was chosen to remove the vector field $\mathbf{A}$ from the presentation for $\mathbf{F}$ and $\mathbf{G}$. As well, it has switched the x- and y-dependence of $\mathbf{F}$ and $\mathbf{G}$. Of course an arbitrarily chosen dependence of the connection on $\mathbf{A}$ would **not** have allowed one to remove it. That this was possible shows that the explicit dependence of $\mathbf{F}$ and $\mathbf{G}$ on the independent variables was in fact gauge-dependent.

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