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Transitions in coherent oscillations between two trapped Bose-Einstein condensates

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By solving numerically the tunneling quantum dynamics between two trapped Bose-Einstein condensates (BEC's), we find two distinct time scales. On the short time scale, we recover semiclassical predictions such as ''macroscopic quantum self-trapping'' and the ''amplitude transition,'' first studied in the polaron context, and the ''π states,'' previously discovered in the context of BEC's. On a much longer time scale, quantum dynamics shows that self-trapping is destroyed in contrast to semiclassical behavior. However, this time scale increases exponentially with the total number of condensate atoms, indicating that self-trapping and ''π states'' would be relevant and experimentally observable in the tunneling of BEC's. [S1050-2947(99)51409-1]

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Bose-Einstein condensation in weakly interacting alkali-metal gases was detected initially by several groups [1]. The precise manipulation of these Bose-Einstein condensates (BEC's) [2,3] has raised the possibility of tailoring these new BEC systems to a degree not possible with superfluid systems such as ^{4}He, ^{3}He, as well as with superconductors. The experimental demonstration of spatial coherence through the observation of interference fringes in two overlapping condensates [4] and the measurement of the relative phase of two condensates in different hyperfine spin states [5] naturally raise the question of measurement and exploitation of temporal phase coherence by means of a Josephson junction between two condensates. Aspects of the question have already been theoretically addressed in the context of BEC in the limit of noninteracting atoms [6] and for small-amplitude Josephson oscillations [7,8]. The semiclassical mean-field dynamics using the Gross-Pitaevskii equation has been studied and interesting phenomena such as macroscopic quantum self-trapping discussed by Milburn *et al.* [9] and Smerzi *et al.* [10,11], and the existence of π states and oscillations (dynamical states, wherein the time-averaged quantum phase difference across the junction equals π), have been predicted by Smerzi *et al.* [10,11] in a weakly coupled double BEC forming a boson Josephson junction. Similar studies have been conducted to investigate driven two-component BEC [12]. Finite-temperature effects describing damping have also been studied [8,13]. Quantum corrections have been included to describe collapse and revival sequences [9,14], phase decoherence and dephasing of Josephson oscillations [15–17], phase squeezing [18], and phase diffusion and renormalization of oscillation frequencies [14]. The approaches to include quantum corrections that gave rise to collapses and revivals and departures from semiclassical dynamics [9,14] have been different and complementary.

The Gross-Pitaevskii equation describing the mean-field dynamics of a BEC is formally identical to the nonlinear Schrödinger equation that has appeared earlier in other fields. Therefore, many of the results, methods, and insight can be fruitfully applied from those other fields to the study of BEC dynamics. Several such useful tools and results exist in the context of interacting quasiparticle-boson systems (polaron dynamics), where the discrete nonlinear Schrödinger equation (DNLSE) has served as a powerful instrument for describing self-trapping and several related phenomena. These include a self-trapping transition [19], a static transition [20], and an additional amplitude transition [21] in the dynamics. While recent investigations have cast doubt on the validity of the DNLSE for conserved quasiparticles interacting with a boson field [22,23], the objections do not appear to apply to the Gross-Pitaevskii equation, which emerges from the microscopic dynamics in a substantially different way. It is therefore of interest to return to the results previously obtained [20] from the DNLSE in the polaron context, and, with their help, to ask about the relationship of the full quantum dynamics of the BEC system to its semiclassical counterpart as represented in the Gross-Pitaevskii equation. By focusing on two specific phenomena, we will show below that results obtained earlier through the DNLSE in the context of coupled quasiparticle-boson systems [19–21], and through a semiclassical analysis in the context of BEC [10,11], are indeed observed in the fully quantum version of the BEC tunneling model on short times, whereas, for long times, self-trapping is destroyed, as is always expected in a translationally invariant quantum system [24]. We will also show the dependence of this long time scale on the total number of condensate atoms, including an onset of a true self-trapping in the limit that this number becomes infinite.

Our model describes tunneling between two condensates via a weak Josephson link created by focusing blue-detuned far-off-resonant light into the center of the trap containing a BEC, generating a repulsive optical dipole force [4]. We note here that our model could also describe tunneling of atoms between two internal atomic states with the help of a laser-induced Raman transition [5]. The second quantized Hamiltonian can be written as

$$H=-2\mathcal{K}(b_1^\dagger b_2+b_2^\dagger b_1)+U(b_1^\dagger b_1^\dagger b_1 b_1+b_2^\dagger b_2^\dagger b_2 b_2), \tag{1}$$

where the operators $b_i, b_i^\dagger$ are boson operators satisfying the

usual commutation rule $[b_i, b_j^\dagger] = \delta_{ij}$. The tunneling matrix element is described by $\mathcal{K}$ and the nonlinear mean-field interaction is described by U. As is standard and as has been considered by earlier authors [8–15], the parameters κ and U are considered to be constant and independent of the difference in the number of atoms. This is an approximation that is valid even for reasonably large fractional population difference between the two traps. The error involved in this approximation has been estimated to be small [8,11]. The procedure to obtain the above two-mode BEC Hamiltonian from the underlying many-body Hamiltonian describing a trapped atomic BEC [25] is well-known and has been outlined, for example, in Refs. [9,16,18]. Earlier use of Hamiltonian (1) has occurred in the work of Scott, Eilbeck, and co-workers [26] on energy transfer and trapping in coupled anharmonic oscillators.

The analysis of Eq. (1) is simplified by the use of angular-momentum operators $\hat{J}_x = \frac{1}{2}(b_1^\dagger b_2 + b_2^\dagger b_1)$, $\hat{J}_y = \frac{i}{2}(b_1^\dagger b_2 - b_2^\dagger b_1)$, $\hat{J}_z = \frac{1}{2}(b_2^\dagger b_2 - b_1^\dagger b_1)$, and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ in terms of which Hamiltonian (1) may be rewritten as

$$H = -4\mathcal{K}\hat{J}_x + 2U\hat{J}_z^2, \tag{2}$$

barring constant energy shifts that depend on the total number of atoms. If we denote the total number of atoms in the two traps [conserved by Hamiltonian (1)] by N_T, define the normalized operators $\hat{S}_i = \hat{J}_i/(N_T/2)$, and define $\Lambda = UN_T/2\mathcal{K}$, which measures the ratio of the mean-field energy to the tunneling energy, we obtain the operator equations of motion corresponding to Hamiltonian (2):

$$\dot{\hat{S}}_z = -\hat{S}_y, \tag{3a}$$

$$\dot{\hat{S}}_y = \hat{S}_z + \frac{\Lambda}{2}(\hat{S}_z\hat{S}_x + \hat{S}_x\hat{S}_z), \tag{3b}$$

$$\dot{\hat{S}}_x = -\frac{\Lambda}{2}(\hat{S}_y\hat{S}_z + \hat{S}_z\hat{S}_y). \tag{3c}$$

In Eqs. (3), the time t has been rescaled to dimensionless time $\mathcal{K}t$. The semiclassical approximation can be derived by factorizing the operator products in Eqs. (3b) and (3c). This yields the tunneling equations for two coupled condensates [9–11], the discrete self-trapping equation [27], and the discrete nonlinear Schrödinger equation for a polaronic dimer [19],

$$\dot{z} = -\sqrt{1-z^2}\sin\phi, \tag{4}$$

$$\dot{\phi} = \Lambda z + \frac{z}{\sqrt{1-z^2}}\cos\phi, \tag{5}$$

where $\langle S_z\rangle \equiv z, \langle S_y\rangle \equiv \sqrt{1-z^2}\sin\phi, \langle S_x\rangle \equiv \sqrt{1-z^2}\cos\phi$.

We present two results. One is motivated by the *amplitude transition* discovered by Tsironis and Kenkre [21] and the related π states and oscillations shown in Refs. [10,11]. The other governs the dependence on the atom number of the self-trapping transition [9–11] which is similar to the polaron transition in the DNLSE/DSTE analysis in Refs. [19,27].

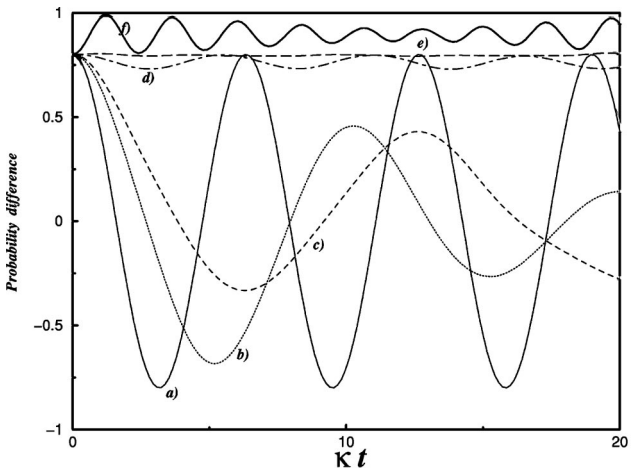

FIG. 1. Time evolution of $\langle S_z\rangle(t) = z(t)$ as a function of dimensionless time $\mathcal{K}t$. Λ increases as one goes from (a) through (f). The details of the numerics are given in text.

Results for the amplitude transition and π states. To see the amplitude transition and π states, we numerically solve the set of equations [(3a),(3b),(3c)] for short times with the help of initial states that are expected to closely mimic the semiclassical counterparts. These states are the atomic coherent states [28] defined as follows. A general state of the system can be written as a superposition of number-difference states

$$|\Psi\rangle = \sum_{N_2=0}^{N} c_{N_2}|N_2, N_1\rangle, \tag{6}$$

where N_1, N_2 are the number of atoms in traps 1 and 2, whose sum we will denote by N_T. This conservation is connected with the fact that the total angular-momentum operator $\hat{J}^2$ commutes with the Hamiltonians (1) and (2). This helps us define the total angular-momentum quantum number $j = (N_1 + N_2)/2$ and the z-projection quantum number $m = (N_2 - N_1)/2$. In terms of these number-difference states, an atomic coherent state is defined [28] as

$$|\theta, \varphi\rangle = \sum_{N_2=0}^{N_T} \left(\frac{N_T!}{N_2!N_1!}\right)^{1/2} \sin^{N_2}\left(\frac{\theta}{2}\right) \times \cos^{N_1}\left(\frac{\theta}{2}\right) e^{-iN_2\varphi}|N_2, N_1\rangle \tag{7}$$

and the expectation values of the angular-momentum operators are given by

$$\langle\hat{S}_z\rangle = -\cos\theta, \quad \langle\hat{S}_x\rangle = \sin\theta\cos\varphi, \quad \langle\hat{S}_y\rangle = \sin\theta\sin\varphi. \tag{8}$$

Thus the state is specified uniquely once the expectation values of the operators $\hat{S}_i$ are specified. We use the above prescription to specify the initial states for the numerical calculation shown in Fig. 1. Further, in order to make contact with the semiclassical equations (4) and (5), we operationally define the probability difference $z(t) \equiv \langle S_z\rangle(t)$ and the phase difference $\phi(t) \equiv \cos^{-1}(\langle S_x\rangle(t)/\sqrt{1-\langle S_z\rangle(t)^2})$. The figure shows the time evolution of the probability difference, de-

fined as $z(t)$, as a function of dimensionless time $\mathcal{K}t$. We have used 35 atoms for our numerical calculation and the initial condition chosen is $\langle S_z\rangle(0)=0.8,\langle S_x\rangle(0)=-0.6$. (This implies that $\langle\phi\rangle(0)=\pi$.) We emphasize here that we get the same results qualitatively when we increase the number of atoms by two orders of magnitude and use a different numerical procedure than the one used throughout this paper. The value of nonlinearity defined as Λ increases as one goes from (a) through (f). In passing from Fig. 1(a) to Fig. 1(d), the frequency of oscillations first decreases and then increases. This is exactly like the semiclassical prediction [29] and the point at which the frequency starts to increase upon increase of Λ marks self-trapping [11,19]. Note also that, in Fig. 1(d), the oscillations are with a nonzero value of $\langle S_z\rangle(t)$ signifying self-trapping. What is even more remarkable is that in Fig. 1(e), the probability difference evolution is practically stationary. This is analogous to the self-trapped stationary state in the polaron context discussed in Refs. [20,21] and to the z-symmetry breaking state discussed in Ref. [11]. Upon increase of the mean-field interaction, the probability evolution actually increases from its initial value, signaling the ''amplitude transition'' precisely like its semiclassical counterpart. It is important to note that the critical values of Λ where the self-trapping and amplitude transitions take place in Fig. 1 are very close to the values predicted by the semiclassical analysis even though we have used only 35 particles. For the self-trapping transition, the semiclassical and fully quantum values of Λ are 1.25 and around 1.3, respectively, whereas for the amplitude transition, the semiclassical and fully quantum values are 1.67 and 1.75, respectively. We display in Fig. 2 a phase-plane plot with z against ϕ. Different contours define different values of Λ. The number of condensate atoms is the same as in Fig. 1. This phase-plane plot is remarkably similar to the semiclassical analog of Refs. [10,11]. The dynamics with the average phase difference across the junction locked to π, giving rise to π states and oscillations [11], is displayed even by the fully quantum model, at least for short times. The quantum dynamics is complicated by a complex structure with sequences of collapses and revivals, which will be analyzed elsewhere. These features in different dynamical regimes have been studied in Refs. [9,14].

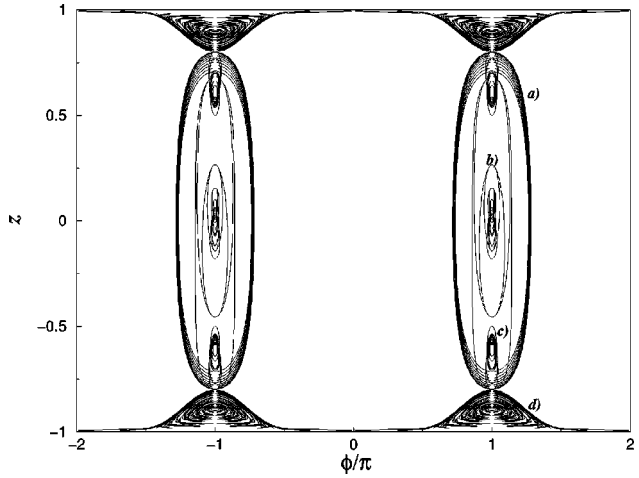

FIG. 2. Phase-plane plot of z vs ϕ. Λ increases as one goes from (a) through (d) and different contours signify different values of Λ.

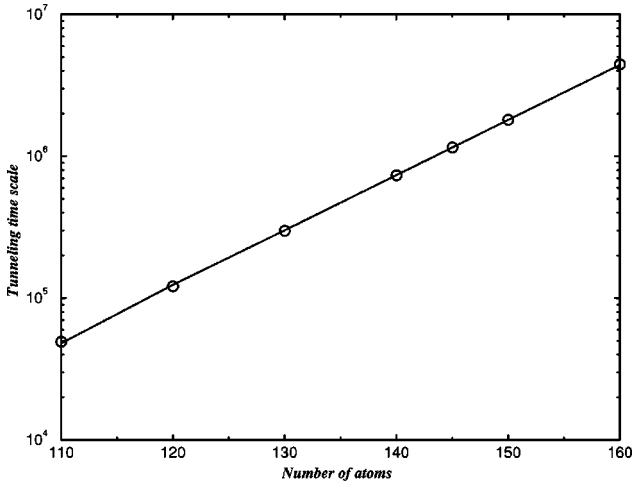

FIG. 3. Tunneling time scale (in units of $\mathcal{K}^{-1}$) plotted logarithmically vs the total number of condensate atoms, N_T.

Dependence of self-trapping on the number of atoms in the condensate. As explicitly shown earlier [24], self-trapping or z-symmetry breaking in a translationally invariant Hamiltonian of the type of Eq. (1) is an artifact of the factorization assumption inherent in the semiclassical dynamics and must be destroyed at long times, for a system of finite number, N_T, of condensate atoms. True self-trapping or z-symmetry breaking [11] can, however, occur as $N_T\rightarrow\infty$. In order to understand this dependence on N_T clearly, we first note that the ground state and first excited state of the system described by Eqs. (1) and (2), like all the other eigenstates of the system, are characterized by $\langle S_z\rangle=0$. Further, in the limit of large particle number, these two states are very nearly degenerate even for finite Λ. This means that if one were to prepare the state initially as a superposition of these two lowest states, the longest time scale in the dynamics will be the inverse of the energy spacing between these two states.

We plot in Fig. 3 this tunneling time scale (logarithmically), in units of $\mathcal{K}^{-1}$, as a function of the total condensate atom number N_T. As we increase N_T, we decrease U so that the dimensionless ratio $UN_T/2\mathcal{K}=\Lambda$ is kept constant [30]. In Fig. 3, $\Lambda=1.25$, which would put the semiclassical system in the self-trapped regime for the initial chosen value of $\langle S_z\rangle$. The open circles are the numerical calculations and the solid line is the fit to a straight line. It is clear that the tunneling time scale increases exponentially with the total number of condensate atoms. This means that the self-trapping, described by semiclassical equations of motion like the DNLSE [20] or the BEC tunneling equations [9–11], increases in validity remarkably quickly as one increases the number of condensate atoms because of the exponential dependence. This exponential dependence is similar to the well-known behavior of the polaronic tunneling time scale with respect to the quasiparticle-phonon coupling constant [23]. We also note that the semiclassical regime will be always obtained in the limit of small Λ compared to the total number of atoms [31], that the observed behavior is certainly related to the choice of the initial state, and that damping effects could play an important role in the dynamics [8].

To summarize, we have studied the fully quantum dynamics of a coupled two-mode BEC tunneling model in relation

to its nonlinear mean-field counterpart. We have shown that there exists a clear separation of two time scales. For short times, the fully quantum dynamics reproduces several features of the semiclassical dynamics. These include a dynamic self-trapping transition, an amplitude transition, the existence of self-trapped (*z-symmetry-broken*) stationary states, and π states. For longer times, the quantum dynamics shows destruction of self-trapping in contrast to the prediction of the nonlinear semiclassical equations. We show, however, that the time scale associated with the destruction of self-trapping increases exponentially with the total number of condensate atoms, suggesting that self-trapping could be robust for the experimentally relevant and macroscopically large number of condensate atoms.

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[1] M. H. Anderson, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995); K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, *ibid.* **75**, 1687 (1995).

[2] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science **275**, 637 (1997); D. S. Hall, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1543 (1998); D. M. Stamper-Kurn *et al.*, *ibid.* **80**, 2027 (1998).

[3] B. P. Anderson and M. A. Kasevich, Science **282**, 1686 (1998).

[4] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science **275**, 637 (1997).

[5] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1539 (1998); **81**, 1543 (1998).

[6] J. Javanainen, Phys. Rev. Lett. **57**, 3164 (1986).

[7] F. Dalfovo, L. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, 4213 (1996).

[8] I. Zapata, F. Sols, and A. Leggett, Phys. Rev. A **57**, R28 (1998).

[9] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A **55**, 4318 (1997).

[10] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).

[11] S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A **59**, 620 (1999).

[12] J. Williams, R. Walser, J. Cooper, E. Cornell, and M. Holland, Phys. Rev. A **59**, R31 (1999).

[13] I. Marino, S. Raghavan, S. Fantoni, S. R. Shenoy, and A. Smerzi, Phys. Rev. A **60**, 487 (1999).

[14] A. Smerzi and S. Raghavan (unpublished).

[15] F. Sols, Physica B **194**, 1389 (1994).

[16] P. Villain, M. Lewenstein, R. Dum, Y. Castin, L. You, A. Imamoglu, and T. A. B. Kennedy, J. Mod. Opt. **44**, 1775 (1997).

[17] P. Villain and M. Lewenstein, Phys. Rev. A **59**, 2250 (1999).

[18] M. J. Steel and M. J. Collett, Phys. Rev. A **57**, 2920 (1998).

[19] V. M. Kenkre and D. K. Campbell, Phys. Rev. B **34**, 4959 (1986).

[20] For a review, see V. M. Kenkre in *Singular Behavior and Nonlinear Dynamics*, edited by St. Pnevmatikos, T. Bountis, and Sp. Pnevmatikos (World Scientific, Singapore, 1989); V. M. Kenkre, Physica D **68**, 153 (1993).

[21] G. P. Tsironis and V. M. Kenkre, Phys. Lett. A **127**, 209 (1988).

[22] D. W. Brown, K. Lindenberg, and B. J. West, Phys. Rev. A **33**, 4104 (1986); W. C. Kerr and P. S. Lomdahl, Phys. Rev. B **35**, 3629 (1987); D. W. Brown, K. Lindenberg, and B. J. West, *ibid.* **35**, 6169 (1987); L. Bonci, P. Grigolini, and D. Vitali, Phys. Rev. A **42**, 4452 (1990); L. Bonci, P. Grigolini, R. Roncaglia, and D. Vitali, *ibid.* **47**, 3538 (1993); D. Vitali, P. Allegrini, and P. Grigolini, Chem. Phys. **180**, 297 (1994).

[23] M. I. Salkola, A. R. Bishop, V. M. Kenkre, and S. Raghavan, Phys. Rev. B **52** R3824 (1995); V. M. Kenkre, S. Raghavan, A. R. Bishop, and M. I. Salkola, *ibid.* **53**, 5407 (1996); S. Raghavan, V. M. Kenkre, A. R. Bishop, and M. I. Salkola, *ibid.* **53** 8457 (1996).

[24] V. M. Kenkre, M. F. Jørgensen, and P. L. Christiansen, Physica D **90**, 280 (1996); see R. Amritkar and V. M. Kenkre, Phys. Rev. E (to be published) for an erratum related to the previous paper.

[25] *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke, and S. Stringari (Cambridge University Press, Cambridge, England, 1995).

[26] A. C. Scott and J. C. Eilbeck, Phys. Lett. A **119**, 60 (1986); L. Bernstein, J. C. Eilbeck, and A. C. Scott, Nonlinearity **3**, 293 (1990).

[27] J. C. Eilbeck, A. C. Scott, and P. S. Lomdahl, Chem. Phys. Lett. **113**, 29 (1985); J. C. Eilbeck, P. S. Lomdahl, and A. C. Scott, Physica D **16**, 318 (1985).

[28] J. M. Radcliffe, J. Phys. A **4**, 313 (1971); F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).

[29] V. M. Kenkre and G. P. Tsironis, Phys. Rev. B **35**, 1473 (1987).

[30] We note here that this limit is reminiscent of the ''massive oscillator'' limit discussed in Ref. [23].

[31] M. Tinkham, *Introduction to Superconductivity*, 2nd ed. (McGraw-Hill, New York, 1996).