

Dynamic localization and related resonance phenomena

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Abstract. Resonance phenomena related to Bessel function roots appearing in several diverse areas of physics, namely, electron transport in solids, atom excitation under the influence of light, spin evolution under applied magnetic fields, device design in Josephson junction arrays, motion in optical lattices, dynamics of Bose condensates, nuclear magnetic resonance as a probe of confined spaces and laser damage in materials, are discussed in the context of a unified picture.

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1. Introduction

The practice of physics is sometimes described as having as its prime element the art of simplification: weeding out the irrelevant and focusing on the essentials. Resonance phenomena are of special interest to the physicist because they manifest the essence of this situation: attention is naturally drawn to characteristic values of parameters for which qualitatively new features emerge in observations. This article describes a family of such resonance phenomena unified by the appearance of striking similarities among certain rather unrelated systems. The phenomena may be loosely described by the term dynamic localization. The variety of unrelated areas exhibiting these phenomena covers electron transport in solids, atom excitation under the influence of light, spin evolution under orthogonal magnetic fields, device design in Josephson junction arrays, optical lattices, Bose condensates, nuclear magnetic resonance and laser damage in materials. A recent ingenious experimental observation made in one of these areas (optical lattices) by Raizen and collaborators [1] has resulted in a clean verification of a quantum transport prediction made a decade ago [2–4] in an entirely different area (electron transport in crystals), and has led to a general resurgence of theoretical interest in dynamic localization. However, investigations of the resonance phenomena in these various unrelated areas have been usually made with little interaction between the communities of scientists who have undertaken them. We have two purposes in this paper: to provide an overview of the resonance phenomena from the limited but useful vantage point of research techniques familiar to us, and to establish or elucidate links that would allow one to transfer insights and techniques from one area to another, and thereby enhance

the possibility of deeper understanding or efficient control in each area.

The first of the areas mentioned has to do with electrons, whose quantum motion in solids is dramatically influenced by the application of a time-dependent electric field: electron motion can be *hindered* by the field leading to localization under specific and controllable conditions [2, 5–9]. In the second, interference effects during light absorption can affect the transfer of excitation so profoundly that plateaus can appear [10–12] in the time dependence, followed by sudden spurts of transfer even though the applied field is sinusoidal, the localization within an atomic state during the plateau and the suddenness of transfer during the spurts both being interesting effects. In the field of magnetic macromolecules in giant-spin materials and cold atomic systems, orthogonal applied magnetic fields can lead to almost identical resonance phenomena and to the possibility of interesting quantum control [13–16]. Device design in Josephson junctions [17], and application of an ac modulation via optical lattice techniques [1] to the gravitational field acting on Bose condensates [18], provide further instances of related resonances. In the distant area of nuclear magnetic resonance used as a probe for confined spaces such as capillaries in the human body or pores in rock (NMR microscopy), the combination of the diffusive motion of the protons and their quantum reaction to applied so-called ‘gradient’ magnetic fields results in the appearance of the Torrey–Bloch equations [19, 20], which are formally identical to the evolution equations in the other areas mentioned. Finally, the emergence of closely connected considerations in the technologically important area of laser damage in materials [21, 22] is also worth comment.

2. Dynamic localization in electron transport in crystals

The simplicity of the underlying concepts has resulted in these resonance phenomena being rediscovered many times by investigators working independently of one another, often in widely separated time periods. We begin† the description with the finding made approximately 15 years ago by Dunlap and Kenkre, who observed that, when a time-periodic electric field was applied to a lattice, an initially localized electronic wavepacket would remain localized if the ratio between the strength of the electric field and its frequency took on certain discrete values [2–4]. For the specific case of a sinusoidally varying field, this ratio was found to be the root of the Bessel function of order zero. This phenomenon of dynamic localization has been studied independently, or extended to wider domains, by a number of authors including Krieger and Iafrate [8], Konotop *et al* [23], Cai *et al* [9] and Holthaus [5]. Important contributions have been made to the subject of Bloch oscillations, which occur for *constant* fields, by a number of other workers [6, 7] as well. We are interested here, however, only in the case of time-dependent fields.

The simplest way of explaining the resonance phenomenon is to consider a quantum particle (e.g. an electron) of charge q in a crystal under the action of an applied electric field of magnitude E and time dependence $f(t)$ and obeying the Hamiltonian

$$H = \hbar V \sum_{m=-\infty}^{\infty} (|m\rangle\langle m+1| + |m+1\rangle\langle m|) + \hbar q E f(t) a \sum_{m=-\infty}^{\infty} m |m\rangle\langle m|. \quad (1)$$

The evolution described is in the basis of Wannier states $|m\rangle$ localized on lattice sites m separated by the lattice constant a , the position operator having been assumed to be diagonal in the Wannier basis (single-band assumption), with V as the (nearest-neighbour) matrix element between the states. The analysis of Dunlap and Kenkre provides exact results for arbitrary time dependence of the field [2], specifies the explicit connection of the new phenomenon to Bloch oscillations by exploring the system in momentum space [3], treats the effects of scattering of the particle due to lattice imperfections by using the stochastic Liouville equation [4] and generalizes the treatment to arbitrary band shape (non-nearest-neighbour matrix elements V_{mn}) and arbitrary number of dimensions [2].

The basic evolution equation for the expansion coefficients $c_m(t)$ of the electron state in the Wannier basis which results from (1) is

$$i \frac{dc_m}{dt} = \mathcal{E} m f(t) c_m + V (c_{m+1} + c_{m-1}), \quad (2)$$

where (and henceforth) \hbar has been set to unity for simplicity, and $\mathcal{E} = qEa$. While the method of characteristics may be used in a straightforward fashion to solve this evolution explicitly, an interesting alternative is the following

† We have chosen this phenomenon to start our description only because of our own familiarity with the area rather than on the basis of any priority suggestions.

succession of transformations. We define $\eta(t) \equiv \int_0^t ds f(s)$ and remove the first term on the right-hand side of (2) by looking at the evolution of $C_m(t) \equiv c_m(t) e^{-im\mathcal{E}\eta(t)}$:

$$i \frac{dC_m}{dt} = V [e^{i\mathcal{E}\eta(t)} C_{m+1} + e^{-i\mathcal{E}\eta(t)} C_{m-1}], \quad (3)$$

and continue through a discrete Fourier transform $C^k \equiv \sum_m C_m e^{ikm}$ to obtain an equation for C^k , whose explicit solution is instantly obtained as

$$C^k(t) = C^k(0) e^{-i2V \int_0^t ds \cos[k + \mathcal{E}\eta(s)]}. \quad (4)$$

The lattice considered can be in arbitrary number of dimensions (m and k are then vectors) and its spatial extent may be either infinite, or finite in a ring topology, so that translational invariance is maintained. Inversion of (4) to obtain the time dependence in the direct lattice is effortless. Thus, for an infinite one-dimensional system, one obtains the amplitude propagator in terms of Bessel functions [2]. It is then straightforward to write, for initial occupation of site $m = 0$, the probability $P_m(t) \equiv |c_m(t)|^2$ and the mean square displacement $\langle m^2 \rangle \equiv \sum_m m^2 P_m(t)$:

$$P_m(t) = J_m^2 \left(2V \sqrt{u^2(t) + v^2(t)} \right), \quad (5)$$

$$\langle m^2 \rangle = 2V^2 [u^2(t) + v^2(t)], \quad (6)$$

where the key quantities are

$$u(t) = \int_0^t ds \cos[\mathcal{E}\eta(s)], \quad v(t) = \int_0^t ds \sin[\mathcal{E}\eta(s)]. \quad (7)$$

The phenomenon of (dynamic) localization of the electron is immediately evident from the nature of (7). For a sinusoidal applied field, i.e. for $f(t) = \cos \omega t$,

$$u^2(t) + v^2(t) = t^2 [J_0^2(\mathcal{E}/\omega) + f_1(t) J_0(\mathcal{E}/\omega) + f_2(t)], \quad (8)$$

where both the functions f_1 and f_2 decay at long times. The Bessel root condition for localization is now evident and means that the electron is dynamically localized whenever \mathcal{E}/ω is equal to a root of J_0 . It is important to note that the origin of the Bessel functions in (5) is the intersite connections on the one-dimensional lattice whereas the origin of the Bessel function in (8) is the sinusoidal dependence of the applied electric field.

Anisotropy in the electron mobility in a three-dimensional lattice is another of the dynamic localization effects. It is clear that the response to a dc field in a given direction can be modified profoundly by applying an ac field in that direction. Manipulation of the anisotropy properties of the mobility is thus possible through the application of *external* fields. For instance, in a cubic lattice with motion which is nearest neighbour and mutually independent in the three orthogonal directions, the effective matrix element in each direction r is found [2] to be

$$V_r^{\text{eff}} = V_r J_0(\mathcal{E}_r/\omega), \quad (9)$$

where \mathcal{E}_r is the product of the charge, the field component and the lattice constant along the r th direction.

A simple approach for addressing realistic effects by including scattering consists [4] of the use of the simplest

of stochastic Liouville equations which include dephasing of the density matrix [24]:

$$i \frac{d\rho_{m,n}}{dt} = \mathcal{E}f(t)(m-n)\rho_{m,n} + V(\rho_{m,n+1} + \rho_{m,n-1} - \rho_{m+1,n} - \rho_{m-1,n}) - i\alpha(1 - \delta_{m,n})\rho_{m,n}. \quad (10)$$

Here $\rho_{m,n} \equiv c_m^*(t)c_n(t)$ is the density matrix element of the electron and α describes the scattering (dephasing) rate. The effective diffusion constant, defined as $D \equiv \lim_{t \rightarrow \infty} [(m^2)(t) - (m^2)(0)]/(a^2/2t)$, is seen to be

$$D = D_0 \sum_m J_m^2(\mathcal{E}/\omega)\alpha^2/(\alpha^2 + (m\omega)^2), \quad (11)$$

where $D_0 = 2V^2a^2/\alpha$ is equal to the diffusion constant in the absence of the applied electric field. Of note is the *non-monotonic* behaviour of the diffusion constant, i.e. the oscillations obtained on varying \mathcal{E}/ω . For further analysis such as extensions for arbitrary band shapes [2], dynamics in reciprocal space showing connection to Bloch oscillations [3] and interband effects [25] we refer the reader to the respective references.

3. Excitation trapping in two-level atoms and memory formalism

In quantum optics, there has been considerable work on the two-level atom subjected to a time-dependent field [26]. A curious resonance phenomenon in this system was reported by Agarwal and Harshawardhan (AH) [10], who showed that there could be plateaus and sudden spurts of transfer in excitation transfer between atomic states. They also appeared to suggest that this phenomenon, which we will call the AH structure, was related to the root of the zeroth-order Bessel function. Consider a two-level atom subjected to optical excitation via an electromagnetic field of sinusoidal dependence. In notation consonant with that employed above for electron transport in crystals, we have the Hamiltonian as

$$H = \frac{\mathcal{E}}{2} \cos \Omega t (|1\rangle\langle 2| - |2\rangle\langle 1|) + V(|1\rangle\langle 2| + |2\rangle\langle 1|), \quad (12)$$

where \mathcal{E} is proportional to the strength of the electric field, ω is the frequency of the field and V is the dipole matrix element connecting the two atomic states.

Raghavan *et al* [11] investigated the connection between the AH structure [10] and dynamic localization [2–4] by extending numerically the latter results to finite chains (of which the two-site system representative of the atom is an example). Several interesting results emerge from that analysis. One is the expected reflection at the walls formed by the chain ends: it yields recurrences in the evolution. Another is that dynamic localization, while perfect for an infinite lattice, becomes imperfect for the finite lattice even when the Bessel root condition is obeyed exactly: on resonance, the mean square displacement oscillates at short times, signalling localization, but grows slowly at longer times, indicating escape from the localized region. A third is a criterion for determining the sensitivity of the localization phenomenon to the system parameters (particularly the finite size): the requirement for dynamic localization to be discernible turns out to be the smallness of the ratio $V/N\mathcal{E}$, where N is the

number of sites in the finite chain. The findings appeared to suggest almost total lack of sensitivity of the AH structure to the Bessel root condition as was shown graphically [12].

In order to investigate this sensitivity issue, and more generally to understand the origin of the *clean* separation of the AH structure into the plateau and sudden transfer, as well as the connection, if any, between the AH structure and dynamic localization, a memory formalism was developed recently [12]. The quantum optics system of [10] or the condensed matter system of [2–4], when the crystal considered is very small (consisting of two sites only!), obeys the Liouville–von Neumann equation for the density matrix ρ , which may be rewritten as an evolution equation for the elements p, q, r of the so-called ‘Bloch vector’:

$$\frac{d}{dt} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} 0 & -2V & 0 \\ 2V & 0 & -\mathcal{E}f(t) \\ 0 & \mathcal{E}f(t) & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0. \quad (13)$$

Here, $p(t) = \rho_{11} - \rho_{22}$, $q(t) = i(\rho_{12} - \rho_{21})$ and $r(t) = \rho_{12} + \rho_{21}$. Kenkre showed [12] that the use of Zwanzig projection operators [27] leads *without approximation* to the integro-differential equation

$$\frac{dp(t)}{dt} + 2 \int_0^t ds \mathcal{W}(t,s)p(s) = 0, \quad (14)$$

whenever the density matrix is initially diagonal in the representation of the states $|1\rangle$ and $|2\rangle$, as would happen, for instance, if the system were to occupy only one of the two states initially[†]. It is important that the memory function $\mathcal{W}(t,s)$ is known *explicitly* (and simply) in terms of the driving fields:

$$\begin{aligned} \mathcal{W}(t,s) &= 2V^2 \cos \left(\mathcal{E} \int_s^t dt' f(t') \right) \\ &= 2V^2 \cos[\mathcal{E}[\eta(t) - \eta(s)]]. \end{aligned} \quad (15)$$

Note that the memory is not of the *faltung* type because the Hamiltonian is time dependent, and is given equivalently by

$$\begin{aligned} \mathcal{W}(t,s) &= 2V^2[\phi_c(t)\phi_c(s) + \phi_s(t)\phi_s(s)] \\ \phi_c(t) &= \cos \left[\mathcal{E} \int_0^t ds f(s) \right] \\ \phi_s(t) &= \sin \left[\mathcal{E} \int_0^t ds f(s) \right]. \end{aligned} \quad (16)$$

For $f(t) = \cos \omega t$, the characteristic functions $\phi_c(t)$ and $\phi_s(t)$ are

$$\begin{aligned} \phi_c(t) &= \cos[(\mathcal{E}/\omega) \sin \omega t] \\ \phi_s(t) &= \sin[(\mathcal{E}/\omega) \sin \omega t]. \end{aligned} \quad (17)$$

If the parameter \mathcal{E}/ω is large w.r.t. 1, $\phi_c(t)$ and $\phi_s(t)$ oscillate rapidly. Because of these rapid oscillations, one may consider taking $p(s)$ out of the integral in (14) for short times (short w.r.t. the period of the field but long w.r.t. the time of oscillation of the memory) and solving fully the resulting differential equation for $p(t)$:

$$p(t) = e^{-2 \int_0^t dt' \int_0^{t'} ds \mathcal{W}(t',s)} = e^{-2V^2[h_c^2(t) + h_s^2(t)]}. \quad (18)$$

[†] For more general initial conditions, a term is appended to the right-hand side of (14).

The characteristic functions $h_c(t)$ and $h_s(t)$ appearing in the exponent, given by

$$\begin{aligned} h_c(t) &= \int_0^t ds \cos[(\mathcal{E}/\omega) \sin \omega s] \\ h_s(t) &= \int_0^t ds \sin[(\mathcal{E}/\omega) \sin \omega s], \end{aligned} \quad (19)$$

are identical to $u(t)$ and $v(t)$ of (7). A very rough approximation to the solution for times long w.r.t. the field period may be obtained by replacing the memory by its average \overline{W} over a period. One then obtains

$$p(t) = \cos\left(\sqrt{2\overline{W}}t\right) = \cos[2VJ_0(\mathcal{E}/\omega)t]. \quad (20)$$

In obtaining the second equality, use has been made of the fact that the averages of ϕ_c and ϕ_s are $J_0(\mathcal{E}/\omega)$ and 0 respectively, which follows from the well known expansion

$$e^{iz \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\theta}. \quad (21)$$

The short-time approximation (18) recovers the spurts and the plateaus of the exact solution simply from corresponding characteristics of the integrals h_c and h_s . The long-time approximation (20) follows the overall evolution, neglecting the short-time structure. Band collapse at the Bessel root condition is recovered effortlessly since the effective bandwidth $VJ_0(\mathcal{E}/\omega)$ vanishes when the condition is satisfied[†].

Immediate analytic insight into the connection of AH and dynamic localization as well as into the separation of the plateaus and the transfer spurts can thus be obtained by an examination of the behaviour of the characteristic functions ϕ and h *without* solving any evolution equation. For large values of \mathcal{E}/ω , $\phi_c(t)$ oscillates most rapidly around $\omega t = n\pi$ where $n = 0, 1, 2, \dots$ and least rapidly around $\omega t = (2n + 1)(\pi/2)$. Since $\sin \omega t$ behaves linearly w.r.t. ωt around the former locations and quadratically around the latter locations, $\phi_c(t)$ evolves with a characteristic time which varies as $1/\mathcal{E}$ in the rapid regions to $1/\sqrt{\mathcal{E}\omega/2}$ in the slow regions. Clearly discernible windows appear in the slow regions and that is where the sudden transfer occurs. The timescale separation is clean only when $\mathcal{E} \gg \omega$. Otherwise no AH structure appears.

The function $h_c(t)$, equivalently $u(t)$ in (7), oscillates rapidly elsewhere but itself undergoes transfers in the slow windows. The long-time approximation $h_c(t) \approx tJ_0(\mathcal{E}/\omega)$ corresponds to dynamic localization. We thus see at once that the two questions of interest are answered as follows. While the quiescent plateaus and sudden spurts (the AH structure) have little to do with the Bessel root condition, the AH structure and dynamic localization are interrelated in that the two represent *extreme* time limits. Indeed, both structures are

[†] It is important to remark here that the long-time replacement of the memory by its average invoked to pass from (14) to (20) neglects the accumulated effect of short-time transfer and leads to total band collapse, which is actually not correct for the two-state system (but correct for the infinite chain as in the analysis of [2]). However, the approximate result gives an idea of the tendency of the evolution and gives the correct qualitative description of bandwidth collapse in the infinite system.

controlled by the ratio \mathcal{E}/ω . The AH structure appears when the ratio is large, with consequent timescale separation in the ϕ and h functions, whereas dynamic localization arises when the ratio is equal to Bessel roots, with consequent collapse of the bandwidth. Incidentally, in the limit that \mathcal{E}/ω is *small*, the cosine of the cosine remains near the value 1 and rapid oscillations over the timescale of the field period do not occur. Thus, it is not merely sufficient for \mathcal{E} and ω to be widely different in value for the AH structure to be visible.

Identification of disparate timescales within a physical process often enhances our physical understanding of the process. The memory formalism of [12] leads to the identification of four such timescales if $\mathcal{E} \gg \omega$. The first is the period of the applied field ($1/\omega$), the second is controlled by the magnitude of the applied field ($1/\mathcal{E}$), the third is essentially the geometrical mean of these two ($1/\sqrt{\mathcal{E}\omega}$) and the fourth is the bandwidth renormalized by the Bessel function ($VJ_0(\mathcal{E}/\omega)$). Internal oscillations in the probability transfer occur on the first timescale, transfer spurts occur on the second, repetition of the spurts occurs on the third and an overall transfer occurs on the fourth timescale.

The memory formalism clarifies the connection between the AH structure in two-level atoms and dynamical localization in electron transport in two ways. It shows, in particular, that the former does not, while the latter does, involve the Bessel root resonance. Moreover, it makes it explicit that \mathcal{E}/ω possesses crucial control over both phenomena, in one case simply by its magnitude relative to 1, and in the other through a precise coincidence with Bessel roots.

4. Dynamic localization in spin systems

The great deal of recent interest in the quantum tunnelling of magnetization in giant spin systems [13] and magnetic-field driven cold atoms [14] has arisen partly because quantum effects are easily observed in the magnetization of a macroscopic sample. Experimental [28–31] as well as theoretical [32, 33] work in this area has focused on the tunnelling between magnetization states in magnetic macromolecules [28–31] at very low temperatures. Resonance effects of the dynamic localization kind appear [15] in such systems because, for a collection of non-interacting spins coherently driven by crossed time-dependent magnetic fields, the evolution equation bears strong similarities to (2) with m , the quantum number for the z -projection of the spin, playing the role of the lattice site.

The Hamiltonian is

$$H = \Delta_0 f(t) \hat{I}_z + \Delta_x g(t) \hat{I}_x + \Delta_y h(t) \hat{I}_y, \quad (22)$$

where the operators \hat{I}_i are the standard angular momentum (spin) operators for a system of total spin j , and $\Delta_0 = g\mu_B B_z$, $\Delta_x = g\mu_B B_x$ and $\Delta_y = g\mu_B B_y$. Here g is Landé's factor, and μ_B is Bohr's magneton. It leads to

$$\begin{aligned} i \frac{dc_m}{dt} &= \Delta_0 f(t) m c_m(t) + \frac{1}{2} (\Delta_x h(t) \\ &\quad - i \Delta_y g(t) \sqrt{(j+m)(j-m+1)} c_{m-1}(t) + \frac{1}{2} (\Delta_x h(t) \\ &\quad + i \Delta_y g(t) \sqrt{(j-m)(j+m+1)} c_{m+1}(t) \end{aligned} \quad (23)$$

for the coefficients of expansion $c_m(t)$ for an arbitrary state expressed as $|\psi(t)\rangle = \sum_{m=-j}^{m=j} c_m(t)|j, m\rangle$. The time dependences of the x -, y -, z -directional fields are denoted by $g(t)$, $h(t)$ and $f(t)$ respectively.

The number of m -values in (23) is finite, and corresponds, in commonly encountered magnetic macromolecules such as Mn_{12}ac and Fe_8 , to a total spin j of the order of 10. As long as one is interested in the dynamics around the levels that are ‘distant from the ends’ of the spin-ladder, and in short enough times, it is possible to approximate the m -dependent square-root terms in (23) as constants: $\sqrt{j \pm m} \approx \sqrt{j}$. Numerical solutions of (23) under initial population of a state (or sublevel) m such that $|m| \ll j$ and $j \gg 1$ bear this approximation out rather well [15], provided strengths of the transverse fields are small compared with the longitudinal field, i.e. $\Omega_j, \Delta_1 j \ll \Delta_0$. We therefore analyse a much simpler form of (23):

$$i \frac{dc_m}{dt} = \Delta_0 f(t) m c_m + \frac{\Omega}{2} g(t) (c_{m+1} + c_{m-1}) + \frac{i\Delta_1}{2} h(t) (c_{m+1} - c_{m-1}), \quad (24)$$

where $\Delta_x j = \Omega$, $\Delta_y j = \Delta_1$.

We observe that (23) is nothing but the finite-chain equivalent of (2) with a time-dependent, complex nearest-neighbour matrix element. Resonance effects are immediately expected along the lines of dynamic localization and are indeed found numerically directly from the original equation (23). An analytic understanding may be given from the approximated equation (24) very simply.

For transverse field strengths weak compared with the longitudinal field strength, it is reasonable to take the extension of the spin-ladder to be infinite. Following the procedure of [2], the probability and mean square displacement expressions are found to be

$$|c_m(t)|^2 = J_m^2 \{ [(\Omega^2 + \Delta_1^2)(u(t)^2 + v(t)^2)]^{1/2} \} \quad (25)$$

$$\langle m^2 \rangle(t) = \frac{1}{2} [(\Omega^2 + \Delta_1^2)(u(t)^2 + v(t)^2)], \quad (26)$$

with

$$u(t) = \int_0^t dt' h(t') \cos \left[\Delta_0 \int_0^{t'} ds f(s) \right], \quad (27)$$

$$v(t) = \int_0^t dt' h(t') \sin \left[\Delta_0 \int_0^{t'} ds f(s) \right].$$

Equations (25)–(27) should be compared with their counterparts which arise in the dynamic localization of driven electrons (5)–(8) and to which they approach for the case of constant transverse fields.

If we take the time dependence of the magnetic field in the z direction to be

$$f(t) = \cos \omega_0 t \quad (28)$$

and the time dependence of the fields in the transverse (x and y) directions is taken to be $g(t) = h(t) = \sin \omega_0 t$ or $g(t) = h(t) = \cos 2\omega_0 t$, we find that the probability of the initially occupied level executes oscillations such that the average value remains close to unity when the ratio of the longitudinal magnetic field energy to the longitudinal field frequency (Δ_0/ω_0) satisfies a resonance condition which

involves the root of Bessel functions. In the first case, in which the transverse time dependence is $\sin \omega_0 t$, the root is of the Bessel function of order unity, whereas in the second case, in which the transverse time dependence is $\cos 2\omega_0 t$, the root is of the Bessel function of order two.

It can be shown analytically from the approximated equation, and supported numerically from the exact equation, that, for a variety of transverse driving fields, the following resonance conditions for localization apply when the longitudinal field dependence is given by (28)

$$g(t) = h(t) = \cos(2n\omega_0 t) : \quad (29)$$

$$J_{2n} \left(\frac{\Delta_0}{\omega_0} \right) = 0 \quad n = 0, 1, 2, \dots$$

$$g(t) = h(t) = \sin((2n+1)\omega_0 t) : \quad (30)$$

$$J_{2n+1} \left(\frac{\Delta_0}{\omega_0} \right) = 0 \quad n = 0, 1, 2, \dots$$

Detailed control is thus possible by a manipulation of the time dependence of the three independent magnetic fields. The relationship of the frequencies and phases of the transverse fields to those of the longitudinal field dictates what order of Bessel functions is involved in the resonance condition.

It is interesting to observe that the translationally invariant equation (24), while an approximation for the magnetic systems we have considered, is an exact starting point for the analysis of the original dynamic localization effect [1, 2, 5] if the inter-site transfer integrals (bandwidths) are time-dependent and complex. Such a situation is difficult to arrange in the context of charges moving in a crystal under the action of an electric field. However, it appears possible, and, indeed, easily manageable, in optical lattices. The treatment we have presented above in the form of an analytic procedure, augmented to include interband effects, could be, thus, applicable directly for optical lattices. It is our hope that such experiments will be undertaken. Other experimental manifestations of the resonance phenomenon in spin systems would be apparent in observations of the spin correlation function and the scattering function of probe particles such as neutrons. While a detailed analysis of such suggested experiments will be given elsewhere, we mention in passing that the zero-frequency component of the van Hove scattering function for these systems turns out to be inversely proportional to the square root of $J_{2n}(\frac{\Delta_0}{\omega_0})$ and $J_{2n+1}(\frac{\Delta_0}{\omega_0})$ respectively for the two cases where the transverse fields are $g(t) = h(t) = \cos(2n\omega_0 t)$ and $g(t) = h(t) = \sin(2n+1)\omega_0 t$. It is clear that the singularity that the scattering function develops at zero frequency is representative of dynamic localization.

Zak [34] has derived expressions for the quasi-energies and found time-dependent solutions for a driven Bloch electron in a time-periodic electric field. These solutions generalize, to some extent, the solutions derived by Dunlap and Kenkre [2]. Recent work [35] has shown that it is possible to derive the most general condition for dynamic localization by combining the work of Raghavan *et al* [15] and Zak [34].

5. NMR microscopy: probing confined spaces via the Torrey–Bloch equation

The diffusion of particles possessing a nuclear spin, such as protons in water, moving in confined geometries such as capillaries or rock pores, presents a fascinating target for NMR microscopy. Investigations use pulsed-gradient spin-echo NMR as reviewed by Callaghan [19]. Two magnetic fields are applied to the sample: a homogeneous static magnetic field B_0 , and a time-dependent gradient magnetic field, the gradient being g and the time dependence $f(t)$. The equation of evolution for the density matrix ρ describing the spin and spatial variables of a single diffusing particle is

$$i \frac{d\rho(t)}{dt} = [H_r + I_z(\omega_0 - f(t)\gamma gx), \rho(t)], \quad (31)$$

where H_r is the Hamiltonian for spatial coordinates in the absence of spin, $\omega_0 = -\gamma B_0$ and γ is the gyromagnetic ratio. A conversion to the interaction picture, followed by an evaluation of the expectation value of the magnetization density, leads to the Torrey–Bloch equations [36] for the magnetization density:

$$\frac{\partial M(x, t)}{\partial t} = -igf(t)xM(x, t) + D \frac{\partial^2 M(x, t)}{\partial x^2}, \quad (32)$$

where D is the diffusion constant. We have displayed the evolution in one dimension only for simplicity; analysis in higher dimensions is straightforward.

Remarkably, (32) is almost identical to the dynamic localization equation (2). An obvious difference is that the former involves continuum space and diffusive motion whereas the latter deals with quantum motion on a discrete lattice. The fact that this difference is of little importance from a mathematical point of view was used by Kenkre *et al* [20] to obtain exact solutions for $M(x, t)$ for arbitrary initial conditions. In the NMR context (but not in the dynamic localization case) $M(t)$, the integral of $M(x, t)$ over all space, is important to calculate because it represents the total magnetization: it is, normally, the observed signal. This happy coincidence, which allows one to obtain observable quantities in NMR microscopy without performing a Fourier inversion, has no significant counterpart in dynamic localization of moving electrons: $\sum_m c_m(t)$, the sum of the amplitudes, has little value unless interest lies specifically in the occupation amplitude of the $k = 0$ Bloch state.

The explicit solution of (32) for arbitrary initial conditions may be written down as follows [20]. The initial magnetization $M(x, 0)$ is Fourier transformed to obtain $M^k(0) = \int M(x, 0)e^{ikx} dx$. We denote by $M^{g\eta(t)}(0)$ the value of $M^k(0)$ obtained by replacing k by $g\eta(t)$, where $\eta(t) = \int_t^0 ds f(s)$. The magnetization at arbitrary times is then

$$M(t) = [M^{g\eta(t)}(0)]e^{-Dg^2 \int_0^t ds [\eta(t) - \eta(s)]^2}. \quad (33)$$

Given that NMR microscopy and dynamic localization of electrons in crystals are rather unrelated to each other in the underlying physics, it is interesting to compare and contrast the two. What are the similarities? In both cases, every point in space has associated with it a rotating vector. In NMR

microscopy, the vector is the spin which precesses around the static homogeneous field B_0 at a frequency proportional to the time-dependent (and space-dependent) gradient field. For the electron in the crystal, it is the quantum mechanical amplitude vector rotating at a frequency proportional to the site energy whose gradient is proportional to the time-dependent electric field. With the exception of the presence or absence of a factor of i , evolution equations are essentially identical, although the evolving quantity is the expectation value of the magnetization (in a certain sense a largely classical entity) in the NMR case, and the quantum mechanical amplitude in the electron case. What are the differences? The continuum versus discrete lattice difference has no significance. Indeed, a theory of NMR microscopy on a discrete lattice of possible interest to optically detected magnetic resonance experiments of Harris, Zewail and collaborators [37, 38] has already been given on the basis of the above considerations [20]. There are two important differences. (i) The measurable in NMR microscopy is controlled by the *phase difference* of the relevant rotating vectors at different locations while in dynamical localization of electrons it is determined by the *magnitude difference* of the rotating vectors. (ii) The motion is coherent in the dynamic localization case unless scattering predominates but incoherent in the NMR case unless ballistic motion is included along with the diffusive component. Under normal conditions, sharp resonances are not observable easily in the NMR case but are in the case of the electron moving quantum mechanically in the crystal. Comparison of the NMR treatment with the analysis of giant-spin magnetic materials systems (see, e.g., (23)) given above reveals the important difference that motion is in real space in the former but only in the space of the magnetic quantum number m in the latter.

Work is under way on including ballistic motion in NMR microscopy and porting resonance insights obtained in the dynamic localization context to the NMR domain to facilitate design of the time dependence of pulses to probe confinement.

6. Related phenomena in other fields

We make brief mention in this section of some other fields in which identical or related resonance effects appear. Dunlap and collaborators have obtained very similar equations of evolution in Josephson junction arrays, analysed the possibility of designing frequency to voltage converters [17], and have investigated [25] the dynamics of Bose condensates falling under the action of the gravitational field and simultaneously subjected to a time-dependent field obtained by optical lattice modulation. That work, as well as related studies by Niu and collaborators, promise an exciting marriage of dynamic localization concepts and the new field of Bose condensation.

Intriguing overlaps exist between dynamic localization systems and interacting electron–phonon systems described by the discrete nonlinear Schrödinger equation (DNLSE). The so-called nonlinear dimer described by the DNLSE [39, 40] may be looked upon as the two-state system undergoing dynamic localization but with an intrinsic ‘externally applied field’. This field is the displacement of a harmonic oscillator

interacting so strongly with the two-state system that it is slaved by the evolution of the two-state system. The quantity $\mathcal{E}f(t)$ is then proportional to the probability difference $p(t)$ rather than externally determined. The evolution matrix Λ in (13) is now nonlinear:

$$\Lambda = \begin{pmatrix} 0 & -2V & 0 \\ 2V & 0 & -\chi p \\ 0 & \chi p & 0 \end{pmatrix}. \quad (34)$$

The nonlinearity leads to the physical pendulum equation for the *time integral* of $p(t)$, and to Jacobian elliptic solutions for $p(t)$, explicitly showing self-trapping [12, 39, 40].

This nonlinear dimer involving polaronic transfer on the one hand, and the dynamic localization dimer under sinusoidal driving fields treated earlier on the other, can be viewed as extreme limits of a third system: the simplest interacting electron–phonon system in the approximation wherein phonons are treated classically but the electron is treated quantum mechanically. Known as the semiclassical dimer, this system evolves under the action of a matrix Λ dependent on $\mathcal{E}(t)$ as in (13), rather than its nonlinear counterpart (34), in which $p(t)$ appears explicitly, but with $\mathcal{E}(t)$ given by

$$\frac{d^2\mathcal{E}(t)}{dt^2} + \omega^2\mathcal{E}(t) = \omega^2\chi p(t). \quad (35)$$

The limit in which $\mathcal{E}(t)$ is independent of $p(t)$ (which would occur if χ were to vanish) describes the dynamic localization problem. The limit in which $\mathcal{E}(t)$ is slaved by $p(t)$, i.e. the situation in which the second time derivative in (35) can be neglected as the result of a time disparity argument [39, 40], leads to the discrete nonlinear Schrödinger equation and the phenomenon of self-trapping.

Resonance effects are less obviously present in the above two cases. However, there exists yet another physical system which unifies the dynamic localization system and the nonlinear dimer through the equation

$$\frac{d}{dt} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} 0 & -2V & 0 \\ 2V & 0 & -\mathcal{E}f(t) - \chi p \\ 0 & \mathcal{E}f(t) + \chi p & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0. \quad (36)$$

It represents a system in which the strong electron–phonon interaction and the slaving assumption have already produced a nonlinear dimer to which an independent sinusoidal field is applied externally. This field can indeed produce resonance effects controlled by the Bessel root condition. A study of this system constitutes the generalization of dynamic localization to the nonlinear domain of the DK analysis and has been carried out by Bishop and collaborators [9] numerically for arbitrarily large systems. Questions sometimes raised as to why the numerical investigation of this system shows the Bessel root condition to continue to apply unmodified in spite of the nonlinearity have been answered [12] recently for the dimer system.

Finally, we remark on connections of the considerations described in this paper to laser damage in materials. Strong nonlinearities and feedback effects arise when materials are bombarded by intense laser radiation. Linear response theory is inapplicable, the absorption coefficient is itself dependent

on light intensity and electron avalanches can occur, leading to explosive acquisition of energy. An understanding of the underlying physical processes requires an explanation of at least two processes: the complex nonlinear absorption process whereby electrons are transferred from the valence to the conduction band, and their subsequent evolution as they move within the conduction band, participating in multiple phenomena such as impact ionization and recombination. Both require a nonperturbative analysis and both have interesting overlaps with issues that arise in dynamic localization. To illustrate, we mention Volkov states as an example of the first of these overlaps and the nature of long-range master equations for electron transitions as an example of the second.

It is well known [21] in the laser damage literature that the exact solutions for the Schrödinger equation for a free electron in the presence of a classical electric field of arbitrary strength or time dependence are given by incorporating the so-called Volkov phase in the plane wave solutions $e^{i(xp - tp^2/2m)/\hbar}$. It is easy to see that this incorporation of the Volkov phase is nothing else but the replacement of k by $k + \mathcal{E}\eta(s)$ that we observe in the dynamical localization solution in (4). Indeed, (4) results in precisely the Volkov form for the wavefunction if we allow for the simple fact that the free electron band is parabolic rather than of cosine form as is characteristic of the tight-binding chain considered in (4) for dynamic localization!

The second example of overlap arises from the rates used in the master equations obtained nonperturbatively for the scattering of electrons in the conduction band by Epifanov *et al* [22]. The process considered is the transition of an electron as a result of the electron–phonon interaction, but in the presence of a strong time-varying (classical) electromagnetic field. Even if the electron–phonon interaction is weak enough to involve a single phonon in the elemental event, the process is generally of a multi-*photon* nature. For sinusoidal time variation of the electric field, the rates involve Bessel functions of an argument proportional to \mathcal{E}/ω . The order of the multiphoton process (the number of photons involved in the process) appears as the index of the Bessel function. As the light intensity is increased, or the frequency decreased, this argument increases. The first order process can thus be weaker than the second order process ($J_1 > J_0$) for appropriate conditions. This can have curious effects in the evolution.

It is to be hoped that the recognition of such overlaps will help one to solve the difficult problems that exist in laser damage by borrowing from known concepts and techniques in the other areas mentioned.

7. Concluding remarks

We have described a variety of physical systems† which exhibit resonance phenomena in most cases interrelated through the existence of Bessel root conditions. Bessel

† Additional studies relevant to the discussion but not mentioned above include the work of Haroche *et al* [41] in atomic physics, the theoretical semiclassical analysis of Pegg and Series [42] and experimental work by Chapman [43], and investigations of two-level atoms driven by a rf field by Yabuzaki *et al* [44].

functions arise in many contexts. Cylindrical symmetry in Laplace's equation, recursion relations for the wavefunction coefficients in a site-localized basis and the appearance of trigonometric functions in the exponent are three common sources. Of these, the last two are relevant to the resonance phenomena discussed here. The Bessel root condition arises, however, from only the third of these sources, and requires that the externally applied field be trigonometric (sinusoidal). Other time variations may be expected to give rise to conditions involving the roots of other special functions.

Quantum control [45] is a subject of great recent interest. The availability of simple analytic expressions provided in this overview suggests that such control could be achieved by designing the time dependence of the appropriate driving fields. Manipulation of the electromagnetic fields forming the optical lattice or condensate traps in order to achieve desired time dependence of the driving field [1], and independent variation of the time dependence of mutually orthogonal applied fields in magnetic macromolecules appear to be particularly good experimental candidates for such quantum control.

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References

- [1] Madison K W, Fischer M C, Diener R B, Niu Q and Raizen M G 1998 *Phys. Rev. Lett.* **81** 5093
- [2] Dunlap D H and Kenkre V M 1986 *Phys. Rev. B* **34** 3625
- [3] Dunlap D H and Kenkre V M 1988 *Phys. Lett. A* **127** 438
- [4] Dunlap D H and Kenkre V M 1988 *Phys. Rev. B* **37** 6622
- [5] Holthaus M 1992 *Phys. Rev. Lett.* **69** 351
- [6] Rabinovitch A and Zak J 1971 *Phys. Rev. B* **4** 2358
Luban M 1985 *J. Math. Phys.* **26** 2386
Luban M and Luscombe H H 1986 *Phys. Rev. B* **34** 3674
Emin D and Hart C F 1987 *Phys. Rev.* **36** 2530
Emin D and Hart C F 1987 *Phys. Rev.* **36** 7353
- [7] Zeuner S, Keay B J, Allen S J, Maranowski K D, Gossard A C, Bhattacharya U and Rodwell M J W 1995 *Phys. Rev. Lett.* **75** 4102
- [8] Krieger J B and Iafraite G J 1986 *Phys. Rev. B* **33** 5494
- [9] Cai D, Sánchez A, Bishop A R, Falo F and Floría L M 1994 *Phys. Rev. B* **50** 9652
Cai D, Bishop A R and Grønbech-Jensen N 1994 *Phys. Rev. Lett.* **72** 591
Cai D, Bishop A R, Grønbech-Jensen N and Salerno M 1995 *Phys. Rev. Lett.* **74** 1186
- [10] Agarwal G S and Harshawardhan W 1994 *Phys. Rev. A* **50** R4465
- [11] Raghavan S, Kenkre V M, Dunlap D H, Bishop A R and Salkola M I 1996 *Phys. Rev. A* **54** R1781
- [12] Kenkre V M 2000 *J. Phys. Chem. B* **104** 3960
- [13] Gunther L and Barbara B (ed) 1995 *Quantum Tunneling of the Magnetization (NATO ASI Series E 301)* (Dordrecht: Kluwer)
- [14] Xu G and Heinzen D J 1999 *Phys. Rev. A* **59** R922
Xu G and Heinzen D J 1999 SB13.01 *Proc. APS March Meeting (Atlanta, 1999)*
- [15] Raghavan S, Kenkre V M and Bishop A R 2000 *Phys. Rev. B* **61** 5864
- [16] Vitanov N V and Suominen K A 1997 *Phys. Rev. A* **56** R4377
Karczmarek J, Stott M and Ivanov M 1999 *Phys. Rev. A* **60** R4225
- [17] Dunlap D H, Kovanis V, Duncan R V and Simmons J 1993 *Phys. Rev. B* **48** 7975
- [18] Anderson B P and Kasevich M A 1998 *Science* **282** 1686
Choi D and Niu Q 1999 *Phys. Rev. Lett.* **82** 2022
- [19] Callaghan P T 1991 *Principles of Nuclear Magnetic Resonance Microscopy* (Oxford: Oxford University Press)
Stejskal E and Tanner J 1965 *J. Chem. Phys.* **49** 1768
- [20] Kenkre V M, Fukushima E and Sheltraw D 1997 *J. Magn. Reson.* **128** 62
- [21] Keldysh L V 1965 *Sov. Phys.-JETP* **21** 1135
Milonni P 1992 *Phys. Rev. A* **45** 2138
Reiss H R 1992 *Phys. Rev. A* **45** 2140
Becker W, Davidovich L and McIver J K 1994 *Phys. Rev. A* **49** 1131 and references therein
- [22] Epifanov A S, Manenkov A A and Prokhorov A M 1976 *Sov. Phys.-JETP* **43** 377
Epshtein E M 1970 *Sov. Phys.-Solid State* **11** 2213
See also Kenkre V M, McIver J K and Kovanis V I 1987 *Bull. Am. Phys. Soc.* **32** FQ11
- [23] Konotop V V 1993 *Phys. Rev. E* **47** 1423
Konotop V V, Chubykalo O A and Vazquez L 1993 *Phys. Rev. E* **48** 563
- [24] Kenkre V M and Reineker P 1982 *Exciton Dynamics in Molecular Crystals and Aggregates* (Berlin: Springer)
- [25] Catanzariti M and Dunlap D H unpublished
- [26] Eberly J H and Popov V D 1988 *Phys. Rev. A* **37** 2012
Sanjay Kumar M, Li Pons M and Eberly J H 1991 *Phys. Rev. A* **44** 1995
Shirley J H 1965 *Phys. Rev. B* **13** 979
- [27] Zwanzig R W 1960 *J. Chem. Phys.* **33** 1338
Nakajima S 1958 *Prog. Theor. Phys.* **20** 948
- [28] Friedman J R, Sarachik M P, Tejada J and Ziolo R 1996 *Phys. Rev. Lett.* **76** 3830
- [29] Thomas L, Lioni F, Ballou R, Gatteschi D, Sessoli R and Barbara B 1996 *Nature* **383** 145
- [30] Sangregorio C, Ohm T, Paulsen C, Sessoli R and Gatteschi D 1997 *Phys. Rev. Lett.* **78** 4645
- [31] Wernsdorfer W and Sessoli R 1999 *Science* **284** 133
- [32] Dobrovitski V V and Zvezdin A K 1997 *Europhys. Lett.* **38** 377
- [33] Gunther L 1997 *Europhys. Lett.* **39** 1
- [34] Zak J 1993 *Phys. Rev. Lett.* **71** 2623
- [35] Raghavan S and Kenkre V M unpublished
- [36] Torrey H C 1959 *Phys. Rev.* **115** 575
- [37] Harris C B and Zwemer D A 1978 *Ann. Rev. Phys. Chem.* **29** 473
- [38] Burland D and Zewail A 1979 *Adv. Chem. Phys.* **40** 369
- [39] Kenkre V M 1989 *Singular Behavior and Nonlinear Dynamics* ed S Pnevmatikos, T Bountis and S Pnevmatikos (Singapore: World Scientific)
- [40] Christiansen P L and Scott A C (ed) 1990 *Davydov's Soliton Revisited: Self-Trapping of Vibrational Energy in Protein* (New York: Plenum)
- [41] Haroche S, Cohen-Tannoudji C, Audoin C and Schermann J P 1970 *Phys. Rev. Lett.* **24** 861
For a review of this and related work see
Cohen-Tannoudji C, Dupon-Roc J and Grynberg G 1992 *Atom-Photon Interactions* (New York: Wiley)
- [42] Pegg D T and Series G W 1970 *J. Phys. B: At. Mol. Phys.* **3** L34
- [43] Chapman G J 1970 *J. Phys. B: At. Mol. Phys.* **3** L39
- [44] Yabuzaki T, Nakayama S, Murakami Y and Ogawa T 1974 *Phys. Rev. A* **10** 1955
- [45] Shi S and Rabitz H 1990 *J. Chem. Phys.* **92** 2927
Shi S and Rabitz H 1992 *J. Chem. Phys.* **97** 276
Grifoni M and Hänggi P 1998 *Phys. Rep.* **304** 229
Deutsch I H and Jessen P S 1998 *Phys. Rev. A* **57** 1972