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Spatial memories and correlation functions in the theory of stress distribution in granular materials

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Abstract The problem of calculating stress distributions in granular materials is addressed via a formalism involving spatial memories. The stochastic origin of the memories is explained and the connection of the memories to correlation functions in the granular system are clarified in several ways: via stochastic considerations, through effective medium arguments, and by generalization of existing constitutive relations. It is indicated how to unify existing theories in the literature with the help of the memory formalism and how to apply the theory to compaction in dies to explain observed oscillations in the stress.

Keywords Granular materials, stress, memory kernel, correlation functions

1 Introduction

Stress distribution in static piles of granular material is, without doubt, a major area of current research in granular materials. The importance of the field stems from the desirability of understanding and control of stress distribution in various specific industrial contexts such as in pharamaceutical, agricultural and manufacturing operations. Peculiarities of granular matter [1–6] common to other phenomena such as avalanches, patterns in flow, and segregation, are notorious for the difficulties they present. Stress distribution investigations have their own additional severe difficulties that the theorist as well as the experimentalist must face. Thus, almost nothing is known definitively about the so-called constitutive relations among the stresses. And, while it is relatively easy to measure stress at the surfaces of a compact, values of stress in the interior must often be deduced from density distributions or other indirect observations.

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The present article provides a brief summary of a method we have developed recently [7–9] for the theoretical description of stress distribution, and indicates some important physical aspects of the method not published earlier. The method is based on spatially nonlocal equations for stress, said to involve 'memory functions'. It has two important ingredients. One is what is sometimes called the $t-z$ transformation: the singling out of one spatial direction in the granular material and treating it for the purpose of description as if it were time. Stress distribution in an n-dimensional system is then viewed as stress 'propagation' in an $(n-1)$-dimensional system. For instance, the study of the spatial variation of stress in a three-dimensional die becomes equivalent to the study of the 'time evolution' of disturbances in a two-dimensional membrane. Geometrical changes in the shape along the chosen direction are reinterpreted as temporal changes in the extent of the region under consideration. The $t-z$ transformation has appeared in investigations earlier than ours, notably in the work of Bouchaud, Cates and collaborators [10]. Its advantage is that it simplifies the mathematical treatment considerably and provides physical intuition based on knowledge of initial value problems in other fields. Its disadvantage is that approximation procedures, which are normally used along with the transformation, place restrictive limitations on the applicability of the analysis.

The other ingredient of our analysis, which is particular to our approach, is a spatially non-local formalism [7–9] based on integrodifferential equations of the Volterra type incorporating memory functions which characterize spatial correlations in the granular material. The memory formalism was suggested by extensive work in the rather distant area of exciton transport in molecular aggregates [11].

Once one has selected the z direction (to be thought of as time) as the direction of gravity and/or of the applied stress, one proceeds to seek a closed evolution equation for the scalar field σ_{zz} which represents the zz-component of the stress tensor. The characteristic ingredient of our approach is the use of an evolution equation which is non-local in z:

$$\frac{\partial \sigma_{zz}(x,y,z)}{\partial z} = D\int_0^z dz'\,\phi(z-z') \times \left[\frac{\partial^2\sigma_{zz}(x,y,z')}{\partial x^2} + \frac{\partial^2\sigma_{zz}(x,y,z')}{\partial y^2}\right] \quad (1)$$

The function ϕ which connects the derivatives of the stress at various depths z is the *memory function*. Along with the multiplying factor D, it is indicative of the spatial correlations of the granular material which arise from the granularity (variations in shape and size of the grains) and other properties such as friction. Generally, the memory functions can depend on x and y as well, but we suppress that dependence here to focus on the essential features of a memory description. The idea of utilizing a memory function approach arose originally from the need to address curious features such as oscillations in the values of observed stress down the center line in compacts [12–16]. The next section shows explicitly that memory functions are not mere phenomenological constructs but arise naturally from physical considerations.

2 Physical origin of the memory functions

There are several different ways one can understand the origin of memory functions in the description of stress distribution. We present three of them below.

2.1 Stochastic origin and connection to depth-dependent correlation functions

Consider, for simplicity, a two-dimensional granular compact(z along the vertical and x along the horizontal) consisting of weightless circular discs of a given radius arranged in perfect order. Let a vertical force be applied to the top of one of the discs lying on the top layer of the compact. Newtonian laws of statics dictate that the consequent force distribution, equivalently stress distribution, is down two lines in the compact, representative of the so-called light cones [7,10]. Viewed through the $t-z$ transformation, the representative point in the one-dimensional space of x travels ballistically with constant speed which we will call c. Here, 'travel' obviously represents changes in the x-coordinate of the representative point with changes in its depth z.

If the array is now considered to be realistic and therefore not perfectly periodic, we come upon irregularities stemming from changes in shape and size of the discs and/or presence of friction. The speed c now changes from location to location. The path of the representative point is jagged: c is a *stochastic* variable. Restricting attention to its z-variation only, we can write an equation of the kind one encounters in the description of Brownian motion:

$$\frac{dx}{dz} = c(z). \qquad (2)$$

An ensemble of representative points started at the top would evolve along the various paths, the distribution of the ensemble density being descriptive of the distribution of stress. Defining a Liouville density for the process, and averaging over all realizations of the stochastic process, it is possible to obtain evolution equations for the averaged probability density, equivalently for the average value of the stress $\sigma_{zz}(x,z)$, according to the stochastic characteristics of the process $c(z)$. The particular irregularities arising from the shape and size changes in the discs, or from their roughness, are reflected in $c(z)$ and thereby in the evolution of the stress.

The irregularities in the granular compact change the direction and magnitude of $c(z)$ as one goes down the depth coordinate, sometimes by small amounts and sometimes by large amounts. For simplicity, let us consider that $c(z)$ jumps between only two values, c and $-c$, with an exponential correlation characterized by depth l, and that the noise is a random telegraph. What this means is that the stochastic process is dichotomous, and that the number of jumps of $c(z)$ between the two (equal and opposite) values follows a Poissonian distribution. It is possible to show then [17] that the Liouville density, equivalently the stress, obeys

$$\frac{\partial \sigma_{zz}(x,z)}{\partial z} = c^2 \int_0^z dz'\, e^{-\alpha(z-z')} \left[\frac{\partial^2 \sigma_{zz}(x,z')}{\partial x^2}\right] \qquad (3)$$

The parameter $\alpha = 2/l$ describes the reciprocal of the depth scale over which the exponential correlation of the random process decays. If α vanishes, which means that $c(z)$ continues with its original value forever, the equation obeyed by $\sigma_{zz}(x,z)$ is a wave equation with wave speed c, and corresponds to constant memory. On the other hand, if the correlation of c's decays infinitely rapidly, specifically such that $\alpha \to \infty$, $c \to \infty$, $c^2/\alpha = D$, the memory $\phi(z)$ equals a delta-function $\delta(z)$, and the evolution for stress is a diffusion equation.

The above stochastic argument, while highly simplified, provides the essential understanding of the origin of the memory functions and of their connection to correlation functions in the granular system. The roughness of the granular particles, and the variation in their shape and size, produce a decay in the correlation of $c(z)$, and this correlation directly gives rise to the memory equation, the functional dependence of the correlation function and the memory function being identical in the simplified dichotomous case. Generally, as shown by Kuś [18], a stochastic process involving a sum of many random telegraphs may be approximately represented by a memory function in the stress equation which is the sum of a large number of exponentials.

We mention in passing that it is also possible to provide a stochastic foundation along the above lines to the diffusion equation *with z-dependent diffusion constant* used by Liu et al. [19] in mean field treatments of stress. If $c(z)$ is not a random telegraph but a stationary Gaussian process, with zero mean and a correlation function Δ:

$$\langle c(z)c(z')\rangle = \Delta(z-z'), \qquad (4)$$

the stress distribution can be shown to be governed [17] by

$$\frac{\partial}{\partial t}\sigma_{zz}(x,z) = D(z)\frac{\partial^2}{\partial x^2}\sigma_{zz}(x,z). \qquad (5)$$

The z-dependent diffusion constant $D(z)$ is given in terms of the integral of the correlation function as

$$D(z) = \int_0^z dz' \Delta(z-z'). \qquad (6)$$

If the correlation function decays extremely rapidly signifying that the stochastic process corresponds to a perfect random walk, the stress evolution equation is a simple diffusion equation (with constant diffusion coefficient). By providing an explicit derivation of the full diffusion equation of Liu et al. [19], this stochastic argument makes a contribution towards the clarification of the validity of that equation.

2.2 Effective medium considerations and memories from disorder

Some investigators find it natural to consider the stress distribution problem as entirely diffusive in the sense that the depth correlation of the c's decays extremely rapidly. This is often false in specific experiments as we have shown elsewhere [8]. Nevertheless, we will now show how memories can arise from granularity and disorder even when one begins with a simple diffusion equation.

Granularity demands that one replace x by a discrete index m and randomness of shapes and sizes of the particles demands that the rates in the evolution equation be random functions. We start from a diffusive (but discrete) description represented, e.g., by

$$\frac{d\sigma_m(z)}{dz} = F_{m+1,m}\left[\sigma_{m+1}(z) - \sigma_m(z)\right] + F_{m,m-1}\left[\sigma_{m-1}(z) - \sigma_m(z)\right] \quad (7)$$

where σ_m is the value of σ_{zz}, and m is the discrete index representing the horizontal x (or y) coordinate. Generally m is a vector index, the evolution equation being appropriately modified, but we consider one horizontal dimension for simplicity. The randomness of the rates leads, via standard effective medium arguments, to a memory function that thus arises from *disorder* in the connections F:

$$\frac{d\sigma_m(z)}{dz} = \int_0^z dz' \mathbf{F}(z-z')\left[\sigma_{m+1}(z') + \sigma_{m-1}(z') - 2\sigma_m(z')\right] \quad (8)$$

Disorder in F's is replaced by memories in z, and $\mathbf{F}(z)$ is obtained from the random distribution $\rho(F)$ of the rates. Equation (8) is evidently equivalent to the diffusion equation in the continuum limit, $\mathbf{F}(z)$ being proportional to $D\phi(z)$. Effective medium arguments lead to the following prescription for converting the randomness in F's into memories.

$$\int_0^\infty \frac{df\,\rho(f)}{f + \left[\frac{\tilde{\mathbf{F}}\zeta(\tilde{\mathbf{F}})}{1-\tilde{\mathbf{F}}\zeta(\tilde{\mathbf{F}})}\right]} = \frac{1}{\tilde{\mathbf{F}}(\varepsilon)} - \zeta\left(\tilde{\mathbf{F}}(\varepsilon)\right). \quad (9)$$

The function ζ has a known dependence on ε and $\tilde{\mathbf{F}}$, and vanishes for small ε, yielding

$$\int_0^\infty \frac{df\,\rho(f)}{f} = \frac{1}{\tilde{\mathbf{F}}(0)} = \frac{1}{F}, \quad (10)$$

a simple and well known mean field result which states that the mean field rate is the reciprocal of the average of the reciprocal of the disordered rates. Memory functions which arise from such effective medium considerations are characterized by a sum of two parts with differing time constants and to stress distributions different from those predicted by a diffusion or telegrapher's equation [20].

2.3 Formal arguments via generalization of constitutive relations

Finally, we show how memory functions arise from mathematically natural generalizations of constitutive relations assumed in earlier theories of stress distributions. Typically, stress balance equations appearing from Cauchy relations (Newtonian statics) are three in number but involve six independent quantities [7,10] and therefore must be supplemented by additional relations known as constitutive or closure relations. Whether made explicit or implicit in the analysis, they are *ad hoc* in nature. The closure assumption of Janssen [21] and Thompson [22], particularly as expressed by Bouchaud et al. [10], postulates proportionality between the diagonal elements of the stress tensor (σ_{xx}, σ_{yy}, and σ_{zz}), as well as the vanishing of the shear components in the xy plane: $\sigma_{xy} = \sigma_{yx} = 0$. This relation is actually not used directly but only in the form of spatial derivatives [10]:

$$\frac{\partial\sigma_{xx}}{\partial x} = c^2\frac{\partial\sigma_{zz}}{\partial x}; \quad \frac{\partial\sigma_{yy}}{\partial y} = c^2\frac{\partial\sigma_{zz}}{\partial x}. \quad (11)$$

Let us extend this existing constitutive relation by generalizing it to incorporate the contributions of σ_{xz} and σ_{yz}. We represent these contributions through the addition of first-order terms in the sense of a Taylor's series expansion:

$$\frac{\partial\sigma_{xx}}{\partial x} = c^2\frac{\partial\sigma_{zz}}{\partial x} + \alpha\sigma_{xz}; \quad \frac{\partial\sigma_{yy}}{\partial y} = c^2\frac{\partial\sigma_{zz}}{\partial y} + \alpha\sigma_{yz}. \quad (12)$$

Combination of this extended form of the constitutive relation with other relations such as the Cauchy equations leads [7] to

$$\sigma_{xz}(z) = -c^2\int_0^z dz'\, e^{-\alpha(z-z')}\,\frac{\partial\sigma_{zz}(z')}{\partial x}$$
$$\sigma_{yz}(z) = -c^2\int_0^z dz'\, e^{-\alpha(z-z')}\,\frac{\partial\sigma_{zz}(z')}{\partial y}. \quad (13)$$

This at once yields the memory equation (1) with memory $\phi(z) = \alpha e^{-\alpha z}$. While they might have some formal (mathematical) appeal, assumed constitutive relations such as those in refs. ([10,21,22]), or their generalizations such as those presented here [7], have little or no physical content. Real justification of the memories is to be found in the stochastic or effective medium arguments given earlier in this section.

3 Relation of the memory equation to other evolution equations

In this section we point out how the present formalism unifies different approaches existing in the literature, specifically the wave approach and the diffusive approach, and show how the memory equation is related to equations with z-dependent diffusion constants, which form the point of departure of the analysis of ref. [19].

3.1 Unification of wave and diffusive approaches to stress distribution

A powerful feature of the memory approach is its ability to unify diverse existing approaches by showing them to emerge as particular cases for special forms of the memory, as well as its added potential to describe the entire intermediate regime. This is at once evident from the fact that the simple diffusive approach (with constant D), which we may call the simplified Liu approach [19], represented by

$$\frac{\partial \sigma_{zz}(x,y,z)}{\partial z} = D\nabla^2 \sigma_{zz}(x,y,z), \tag{14}$$

and the wave approach of Bouchaud et al. [10], represented by

$$\frac{\partial^2 \sigma_{zz}(x,y,z)}{\partial z^2} = c^2\nabla^2 \sigma_{zz}(x,y,z), \tag{15}$$

are extreme limits of (1) for the respective cases when the memory is a delta-function, $\phi(z) = \delta(z)$, and when the memory is a constant, $\phi(z) = c^2/D$. The former limit is $\alpha \to \infty$, $c \to \infty$, $c^2/\alpha = D$, whereas the latter limit is $\alpha = 0$. Furthermore, when neither parameter limit is operative, wavelike behavior is apparent for depths smaller than $1/\alpha$, while diffusive behavior predominates for depths larger than $1/\alpha$. The parameter c is, of course, directly related to the slope of the so-called light cones, and α measures the decay of the correlation function. As remarked earlier, the perfect memory situation represents the fact that the stress applied on one particle is transmitted along the lines of contact between particles without loss of information about the original strength and direction of the applied force, while the delta-function memory situation describes Markoffian behavior, i.e., complete loss of information at every step.

The diffusive limit of the evolution has been used in the past for developing mean field treatments [19] and addressing the magnitude distribution of the stresses rather than their spatial variation. The wave limit has been discussed primarily via ray tracing arguments [10] in what may be termed the geometrical limit of the wave equation. Our own approach has been quite different. We have obtained actual solutions of these equations for the propagators (Greens functions) through explicit initial value and boundary value treatments and, with their help, attempted to address the *spatial* distribution of stress in granular systems. Our especial emphasis has been to present an intermediate starting point which combines the physics inherent in the extreme limits of wave-like and diffusive behavior and is capable of describing the *entire* range in between. Therefore, we focus attention on memory functions which are neither constant nor have infinitely fast decay. We reserve for a future discussion rich features which arise from algebraic memories, and consider here the simple intermediate situation which corresponds, as shown above, to random telegraph noise, i.e., to a form for the memory which is exponential $\phi(z) = \alpha\exp(-\alpha z)$. With respective limits for small and large α as the wave and the diffusive cases, this intermediate memory gives the telegrapher's equation, which, suppressing y, has the form

$$\frac{\partial^2 \sigma_{zz}(x,z)}{\partial z^2} + \alpha\frac{\partial \sigma_{zz}(x,z)}{\partial z} = c^2\frac{\partial^2 \sigma_{zz}(x,z)}{\partial x^2}. \tag{16}$$

The easy unification of the extreme limits provided by our memory approach may be appreciated either directly as explained above, or through explicit solutions such as those of (16). Consider stress propagation in an unbounded medium and take the applied stress $\sigma_{zz}(x,0)$ at the 'surface' $z = 0$ to be a delta function $\delta(x)$. With the notation that T vanishes identically for $cz \leq x$, and equals, for $cz \geq x$,

$$T = \left(\frac{\alpha}{4c}\right)\left\{I_0\left(\frac{\alpha}{2c}\sqrt{c^2z^2-x^2}\right) + \frac{cz}{\sqrt{c^2z^2-x^2}}I_1\left(\frac{\alpha}{2c}\sqrt{c^2z^2-x^2}\right)\right\}, \tag{17}$$

the I's being modified Bessel functions, the solution of (16) is given by

$$\sigma_{zz}(x,z) = e^{-\alpha z/2}\left[\frac{\delta(x+cz)+\delta(x-cz)}{2} + T\right]. \tag{18}$$

In the limit $\alpha = 0$,

$$\sigma_{zz}(x,z) = (1/2)\left[\delta(x+cz)+\delta(x-cz)\right] \tag{19}$$

as in ref. [10] and we recover the phenomenon of 'light cones'. Our solution shows that, in addition, there is a nonvanishing stress distribution *within* the light cones. This stress is given by our term T. In the limit which reduces our theory to the opposite extreme of Liu et al. [19], the light cones spread out to coincide with the surface $z = 0$, and the entire region experiences stress. A stress distribution $f(x)$ applied at the top surface causes, throughout the medium,

$$\sigma_{zz}(x,z) = \int_{-\infty}^{\infty}\frac{e^{-(x-x')^2/4Dz}}{(4\pi Dz)^{1/2}}f(x')dx'. \tag{20}$$

3.2 Relation of the memory equation to equations with z-dependent diffusion coefficients

Interesting comments can be made on the relationship between the memory equation (1) and an equation with z-dependent diffusion coefficient such as (5) which forms the point of departure of ref. [19]. We have seen that such an equation arises naturally for Gaussian processes when the stochastic process which characterizes $c(z)$ is Gaussian with a correlation function $\Delta(z)$, and that the z-dependence of $D(z)$ is obtained by integrating twice the z-dependence of $\Delta(z)$.

If $\sigma^k(z)$ is the Fourier transform of the stress, and $\widetilde{\sigma^k}(\varepsilon)$ is the Laplace transform of $\sigma^k(z)$, i.e.,

$$\widetilde{\sigma^k}(\varepsilon) = \int_0^\infty \sigma^k(z)\exp(-\varepsilon z)dz = \int_0^\infty\left[\int_{-\infty}^{\infty}\sigma_{zz}(x,z)\exp(ikx)dx\right]\exp(-\varepsilon z)dz \tag{21}$$

the memory equation results in

$$\frac{\widetilde{\sigma^k}(\varepsilon)}{\sigma^k(0)} = \frac{1}{\varepsilon + Dk^2\widetilde{\phi}(\varepsilon)} \tag{22}$$

whereas the z-dependent diffusion equation results in

$$\frac{\sigma^k(z)}{\sigma^k(0)} = \exp\left[-k^2 \int_0^z dz' \int_0^{z'} dz'' \Delta(z'')\right] \tag{23}$$

The two results can be made identical to each other if the Gaussian correlation function and the memory are both proportional to Dirac delta functions $\delta(z)$. Furthermore, if the memory and the normalized Gaussian correlation are equal to each other, such that

$$D(z) = D \int_0^z dz' \int_0^{z'} dz'' \phi(z''), \tag{24}$$

the 'mean square displacement of the stress', defined as

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 \sigma_{zz}(x,z) = -\left[\frac{\partial^2 \sigma^k(z)}{\partial k^2}\right]_{k=0} \tag{25}$$

is *identical* in the two cases, no matter what the z-dependence of the correlation function! An explicit example may be given for the exponential memory: $\phi(z) = \alpha \exp(-z)$. One gets, in both cases,

$$\langle x^2 \rangle = \frac{2c^2}{\alpha}\left[z - \frac{1-\exp(-\alpha z)}{\alpha}\right]. \tag{26}$$

The equality of the mean square displacement might tempt one to conclude that the two cases are identical. Such a conclusion would be quite incorrect. Higher moments are not identical, and the full solution of the stress can exhibit oscillatory behavior from the memory equation but never from the z-dependent diffusion equation for a case such as the one with exponential memory. Indeed, if an exponential correlation is assumed for the Gaussian process, and a memory equation which results in the same solution for the stress is demanded by equivalence, the memory is found to develop a k-dependence in Fourier space. Time and space components of the memory are then inseparable in the memory equation even if they are separable for the $D(z)$ equation. It is straightforward to write an equivalence between two generalized memory and $D(z)$ equations in which space and time components are combined. Despite this formal similarity, one is led to conclude that oscillatory behavior such as one observed in compacts [12–16] is more natural to memory equations.

4 Boundary value problem for compaction in dies

The original motivation for our investigations [7,8] was provided by reported observations of curious features such as spatial oscillations in stress down the center line in compacts. These features are apparent in recent experimental results [12] as well as in data that have been available in the literature for many years [13–16]. Experimental information about the distribution of stress in a powder compact has been difficult to obtain unambiguously. Observations have employed, in some cases, direct measurement of the stress with the use of sensors or strain gauges [15,25] within, or at the edge of, a compact to measure the forces that evolve during pressing. Other cases have involved indirect deduction of the stress distribution from the density distribution within the compact. The first approach suffers from a lack of accuracy and the second from the need for specific assumptions of a local stress-density relation at every point in the compact [22–24]. Nevertheless, it is quite clear that characteristic unexplained features such as the non-monotonic variation of the stress with depth along the centerline of the compact emerge regularly (but not universally), and that a theoretical description of these features is not trivial. Indeed, Aydin et al. [12] have referred to the *failure* of existing theories to account for the oscillatory behavior. The reader is referred to ref. [8] for a detailed description of the experimental background.

Spatial oscillations of stress in compacts was the primary problem which motivated the memory approach. We address this issue by developing a boundary value analysis of (1) in the compact. Details of the theory may be found in [7] and applications to experiment in [8]. Under the assumption that the extent in the z-direction is large, (16) can be solved through the application of the method of separation of variables. Oscillatory variation of stress arising from the wave element in cases in which c/α is not negligible becomes obvious from the solutions. For space reasons, we refer the reader elsewhere [8] for the details of the application of our theory to compaction in dies. With given distributions of stress along the top surface and the side walls, explicit solutions are found for the stress in the interior, and are compared successfully to experiment. Oscillations down the center line emerge naturally but not always, the factor governing their appearance being the ratio c/α. Closed contours signifying true wavelike behavior appear in some cases but not in others, also depending on the value of c/α. The analysis gives a satisfactory explanation of existing observations. Furthermore, it provides prescriptions based on a study of practical matters such as the effect of lubrication of the walls and of changing the profile of the applied stress at the top of the compact [8]. The wave ingredient of memory equations is found to be essential to explain some of the data (as in uranium dioxide) where oscillations are clearly visible. Furthermore, even for cases which exhibit no such oscillations (as in magnesium carbonate and alumina), a careful analysis based on our predictions lead to the conclusion that the diffusive limit is inadequate for most observations [12,15,16].

5 Conclusions

The formalism of memory functions for the description of stress distribution described in the present paper represents a small advance in a difficult area of research. It has some virtues and a number of shortcomings in its present stage. The latter include the assumption that the present does not influence the past (in the sense of the $t-z$ transformation) which means that stresses at smaller depths are considered as not influenced by stresses at larger ones. This assumption is not always valid as the stochastic paths representing the variable $c(z)$ can in some cases turn upwards in a granular system. Indeed, stress distribution cannot be looked upon universally as an initial value problem. Related to this problem is the evident restriction that the stress analysis presented above for dies be used only in long pipes or media without a bottom. Termination in the z direction as in a compact introduces 'boundary conditions in time' which appear difficult to treat from evolution equations. In the true time evolution situation, we predict behavior at a later time, given spatial boundary conditions for all time and an initial condition. The incorporation of a 'final' condition, i.e., a boundary condition at large values of time seems difficult to implement. Another notable absence from the formalism is the incorporation of features peculiar to the granular system such as isostaticity [26], mentioned elsewhere in this volume. Indeed, our approach does not even touch upon complexities specific to granular matter such as the dependence of stresses on history [27,28]. The spirit in which the investigations reported in this paper have been undertaken is that the subject is forbiddingly difficult, but interesting and fundamental, and that even small advances are important to attempt in this fascinating area of research.

Achievements of the formalism include unification of diverse approaches such as those applicable in the extreme diffusive and wave limits, treatment of the entire range in between, explanation of observed features such as oscillations in stress distribution and the potential to address observations such as the burial problem [9].

It is perhaps useful to emphasize that memory functions are not mere phenomenological constructs but may be computed from given stochastic properties of the granular system arising from the varying shapes and sizes of the grains and from the grain-grain interaction. Information about these stochastic properties themselves may be obtained in principle from a combination of scattering experiments, and of computer simulations such as those reported by Endicott, and by Vidales et al. [29]. Oher directions of related work include extensions [30] to include nonlinearities of the kind relevant to reaction diffusion systems.

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