Effects of transport memory and nonlinear damping in a generalized Fisher's equation

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Memory effects in transport require, for their incorporation into reaction-diffusion investigations, a generalization of traditional equations. The well-known Fisher's equation, which combines diffusion with a logistic nonlinearity, is generalized to include memory effects, and traveling wave solutions of the equation are found. Comparison is made with alternative generalization procedures.

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I. INTRODUCTION

Fisher's equation [1] describes the dynamics of a field $u(x,t)$ subject to diffusive transport and logistic growth:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ku(1-u/K). \tag{1}$$

This kind of reaction-diffusion equation is relevant in chemical kinetics as well as in ecological contexts where u diffuses via a diffusion constant D and grows with a linear growth rate k while the environment imposes a carrying capacity K. Equation (1) belongs to a family of single component models of broad applicability. Fisher proposed it as a deterministic model of the spread of a favored gene in a population [2]. Other systems in which Eq. (1) plays a significant role include flame propagation, neutron flux in a nuclear reactor, and the dynamics of defects in nematic liquid crystals [1,3–6]. Work has been done on the propagation of wavelike structures in unbounded systems [6,7], as well as on the formation of spatial structures in finite geometries [8].

Diffusion is typically a limit of a more coherent motion interrupted by scattering events which is valid when the scattering events are extremely frequent. A useful manner of describing situations in which the frequency of scattering events is not infinitely large is through the incorporation of memories with finite decay times [9]. An analytical investigation of reaction-diffusion phenomena based on the introduction of exponential memory in the context of the logistic nonlinearity of the Fisher's equation, and a piecewise linearity approximation, was recently given by Manne, Hurd, and Kenkre (MHK) [10]. A number of interesting results appeared in that analysis, including a generalization of the relation between the minimum speed of shock fronts and the system parameters, and the appearance of a speed limit beyond which the fronts would exhibit spatial oscillations.

As stated in Ref. [10], the generalization of Fisher's equation is not unique, the one employed in Ref. [10] having been chosen for analytic tractability. A more natural generalization results in a more complex structure of the equation, involves a nonlinearity in the damping, and is the subject of the present paper. Our investigations show that, in the light of the present generalization, some of the interesting predictions of Ref. [10] are modified drastically, while others are only slightly changed.

Reference [10] first generalized the linear part of Eq. (1) by incorporating an exponential memory, converted the resulting integro-differential equation via differentiation into a differential (specifically the telegrapher's) equation, and finally added the nonlinear logistic term to obtain the starting point of the analysis. Here, however, we take as our point of departure

$$\frac{\partial u}{\partial t} = D\int_0^t \phi(t-\tau)\frac{\partial^2 u}{\partial x^2}d\tau + kf(u), \tag{2}$$

which reduces to Fisher's equation if the nonlinearity is logistic and if the memory function $\phi(t)$ is a δ function in time. Generally, the decay time of $\phi(t)$ is a measure of the time between scattering events: we will take $\phi(t)$ to have a simple exponential form $\phi(t)=\alpha e^{-\alpha t}$, where $1/\alpha$ represents the scattering time.

Transformation of Eq. (2) into a differential equation is trivially done by differentiating once with respect to time:

$$\frac{\partial^2 u}{\partial t^2} + [\alpha - kf'(u)]\frac{\partial u}{\partial t} = v^2\frac{\partial^2 u}{\partial x^2} + \alpha k f(u), \tag{3}$$

where, as in [10], we put $D\alpha = v^2$, the physical meaning of v being the speed dictated by the medium in the absence of scattering. This is the speed at which the underlying quasiparticle, whose number density or probability density is described by $u(x,t)$, moves ballistically (coherently) in between scattering events. If the coherent motion is interrupted too often by scattering, the motion looks diffusive and one returns to the Fisher limit.

Equation (3) becomes, in the absence of the nonlinearity, $k=0$, the well-known telegrapher's equation [11] suggested by Lord Kelvin for the description of propagation of transatlantic telegraph signals. Equation (3) differs from the starting point of the MHK analysis, viz.,

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} = v^2\frac{\partial^2 u}{\partial x^2} + \alpha k f(u), \tag{4}$$

in a significant way. The former shows that the damping coefficient multiplying $\partial u/\partial t$ is not constant, and can even be negative.

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Whereas Eq. (4) is obviously simpler in form than Eq. (3), the latter could be argued to stem from a more natural generalization of the Fisher equation, involving a simple replacement of a δ-function memory by a finite decay time memory in the transport term $D\partial^2 u/\partial x^2$ in Eq. (1). We are interested in comparing the physical consequences of the two generalizations in the context of traveling wave solutions. We do this in two steps. In Sec. II, we give a general analysis of traveling wave solutions without approximating the equation in any way. We investigate the minimum speed of wave fronts, the existence of spatially oscillatory fronts given by MHK, and give a qualitative description of all the kinds of front shapes that the system can support as a function of the parameters. In a certain regime in parameter space, a dynamical stabilization of the unstable state of the logistic equation is observed, with fronts of state $u=0$ invading the state $u=K$. In Sec. III, we replace the nonlinearity by a piecewise linear approximate form and obtain analytic solutions following the method of MHK. We end with concluding remarks in Sec. IV.

II. NONLINEAR ANALYSIS

We look for waves moving in the direction of increasing x: $u(x,t)=KU(x-ct)=KU(z)$, where c is the speed of the nonlinear wave, generally different from the speed of linear waves v, dictated by the medium. We are also renormalizing with respect to the carrying capacity K to simplify the reaction term in the logistic case. With this ansatz, we obtain from Eq. (3) the following ordinary differential equation:

$$(v^2-c^2)U''+c[\alpha-kf'(U)]U'+\alpha kf(U)=0. \tag{5}$$

For a logistic reaction term $f(U)=U(1-U)$ and $f'(U)=1-2U$, so that Eq. (5) becomes

$$mU''+c(\alpha-k+2kU)U'+\alpha kU(1-U)=0, \tag{6}$$

where we have followed the notation of MHK, $m=v^2-c^2$, to emphasize the formal similarity between Eq. (6) and the equation of motion of a damped oscillator of mass m, subject to the nonlinear force $-\alpha kU(1-U)$.

A convenient way to analyze the solutions of Eq. (6) is to write it as a first order system:

$$U'=V\equiv f(U,V),$$

$$V'=\frac{1}{m}[(k-\alpha-2kU)cV-\alpha kU(1-U)]\equiv g(U,V). \tag{7}$$

The system (7) has two equilibria: $(U^*,V^*)=(0,0)$ and $(U^*,V^*)=(1,0)$. We can analyze the character of these by looking at the linear behavior in the neighborhood of the equilibria.

At the equilibrium (0,0) we have the eigenvalues

$$\lambda_{\mp}=\frac{c(k-\alpha)\mp\sqrt{(k-\alpha)^2c^2-4k\alpha m}}{2m}. \tag{8}$$

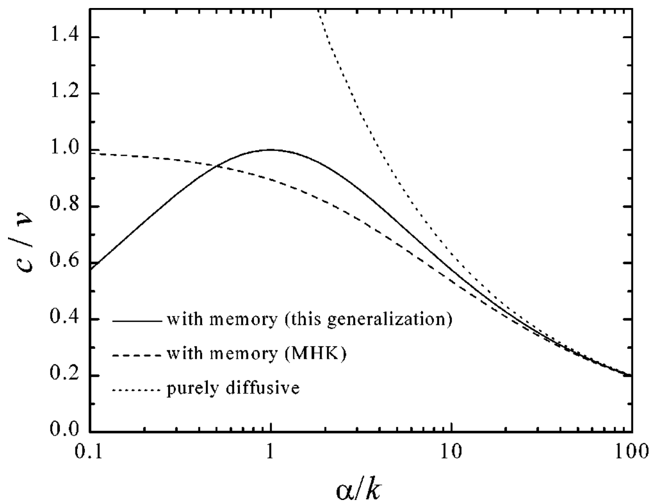

FIG. 1. The ratio of the minimum speed of nonlinear waves c_{min} to the medium-dictated speed v, as a function of the system parameter α/k, which is the ratio of the scattering rate at which motion coherence is interrupted to the growth rate. Lower values of α/k represent more coherent motion. Three curves are shown: the present generalization (full line), the MHK generalization of Ref. [10] (dashed line), and the purely diffusive Fisher limit (dotted line) with $D=v^2/\alpha$.

If we assume that U is a density or a concentration, solutions where U oscillates below $U=0$ are not allowed. This imposes the condition that the eigenvalues $\lambda_{\mp}$ be real, from which we obtain

$$c\geqslant c_{min}=v\frac{1}{\sqrt{1+\frac{1}{4}(y-1/y)^2}}, \tag{9}$$

where $y=\sqrt{\alpha/k}$. This relation states that there is a minimum value of the speed of the nonlinear waves. It is to be compared to the known result in the context of Fisher's equation: $c_{min}=2\sqrt{kD}$ (see, for example, [1]). If we make the formal identification $D\alpha=v^2$, as in Eq. (3), we have

$$c\geqslant c_{min}=2v\sqrt{1/y^2}. \tag{10}$$

Equation (9) is also to be compared to the previously found result of Ref. [10]:

$$c\geqslant c_{min}^{MHK}=v\frac{1}{\sqrt{1+\frac{1}{4}y^2}}, \tag{11}$$

Figure 1 displays a comparison of the minimum speeds c_{min} as a function of the system parameter α/k in the three cases: purely diffusive (Fisher limit), the MHK generalization, and the present generalization. It can be seen that the two generalizations approach asymptotically the behavior of the purely diffusive situation for large values of α/k. This is to be expected since $\alpha\rightarrow\infty, v\rightarrow\infty, v^2/\alpha=D$ is the diffusive limit of Eq. (2). At low values of the ratio α/k, however, there is a

TABLE I. Character of the equilibria for the possible combinations of parameters and wave velocity, according to the generalization studied in this paper. The "case number" refers to the discussion in the text.

Case no.	α,k	c	(0,0)	(1,0)
(1)	$\alpha>k$	$c<c_{min}$	stable spiral	saddle
(2)		$c_{min}<c<v$	stable node	saddle
(3)		$v<c$	saddle	unstable node
(4)	$\alpha<k$	$c<c_{min}$	unstable spiral	saddle
(5)		$c_{min}<c<v$	unstable node	saddle
(6)		$v<c$	saddle	unstable node

TABLE II. Character of the equilibria for the possible combinations of parameters and wave velocity, according to the generalization studied in [10].

Case no.	α,k	c	(0,0)	(1,0)
(1)	$\alpha>k$	$c<c_{min}$	stable spiral	saddle
(2)		$c_{min}<c<v$	stable node	saddle
(3)		$v<c$	saddle	unstable node
(4)	$\alpha<k$	$c<c_{min}$	stable spiral	saddle
(5)		$c_{min}<c<v$	stable node	saddle
(6)		$v<c<c_{osc}$	saddle	unstable node
(7)		$c_{osc}<c$	saddle	unstable spiral

notable difference. The finite correlation time generalizations allow waves with lower speed than those present in Fisher's equation and, in addition, there is no divergence when $\alpha/k \to 0$. Moreover, the present generalization predicts a sharply different behavior in the region $\alpha<k$, due to the antidamping present in Eq. (6). This growing branch of the c_{min} function turns out to have drastic consequences in the nature of the traveling fronts when the system is far from the diffusive regime. On the one hand, it can be seen that waves with speed $c=v$ are obtained at a finite value of the memory constant α, namely, $\alpha=k$. When approaching the ballistic limit beyond this point, traveling waves are allowed with speeds lower than v. The nature of these will be analyzed in the following discussion.

Around the equilibrium (1,0) the eigenvalues are

$$\mu_{\mp}=\frac{-c(k+\alpha)\mp\sqrt{(k+\alpha)^2c^2+4k\alpha m}}{2m}. \qquad (12)$$

Oscillations around this equilibrium are in principle possible, as observed in [10], since there is no impediment to the solution growing beyond $U=1$. The condition for oscillations is found from the radicand of the eigenvalues $\mu_{\mp}$:

$$(k+\alpha)^2c^2+4k\alpha(v^2-c^2)<0, \qquad (13)$$

which, after some manipulation, becomes

$$(\alpha-k)^2c^2+4k\alpha v^2<0. \qquad (14)$$

Both terms in this relation are positive, implying that the relation cannot be satisfied by the wave speed for any set of parameters.

A similar analysis can be carried out with Eq. (4), corresponding to the MHK model. The character of the equilibria for both generalizations is summarized in Tables I and II. Note that there is a c_{osc} in Table II Case (7), not present in the results in Table I because Eq. (14) cannot be satisfied. Different combinations of parameters allow a variety of traveling fronts connecting one equilibrium to the other. Cases (1) to (7) in Table II are the fully nonlinear equivalents of the piecewise linear situations analyzed in [10]. The situations resumed in Table I are discussed below.

Equation (6) can be exploited directly to understand the nature of the solutions of the traveling wave problem, without obtaining analytical solutions via simplifications. We will base the following discussion on the mechanical interpretation of Eq. (6), which can be reinterpreted, following MHK, as describing the motion of a particle of mass m in a nonlinear potential, subject to a state-dependent damping that can be positive or negative. Since the mass of the particle can also be positive or negative, depending on whether the velocity c is lower than v or not, we analyze the two cases separately.

A. $c<v$

In this case a particle of mass $m>0$ is moving in a potential $\phi(U)=\alpha k(U^2/2-U^3/3)$ [see Fig. 2(a)]. This potential has a minimum at $U=0$, but we have ruled out solutions that oscillate around it based on the positivity of U [these are cases (1) and (4) in Table I]. Another possible solution is an overdamped trajectory that connects the equilibrium at $U=1$ with that at $U=0$. This corresponds to a traveling front of the state 1 invading the system at state 0. The damping coefficient is $\gamma=c(\alpha-k+2kU)$. We can see that if $\alpha>k$ then $\gamma>0$ for all values of U along this trajectory. This is a solution that connects the saddle with the stable node shown in case (2) of Table I.

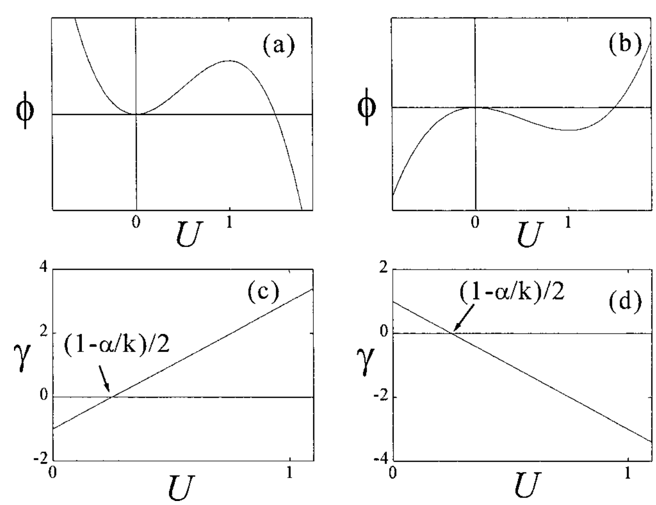

FIG. 2. The potentials and the damping coefficients of the mechanical analog of Eq. (6). Cases (a) and (c) correspond to $m>0$, while (b) and (d) correspond to $m<0$.

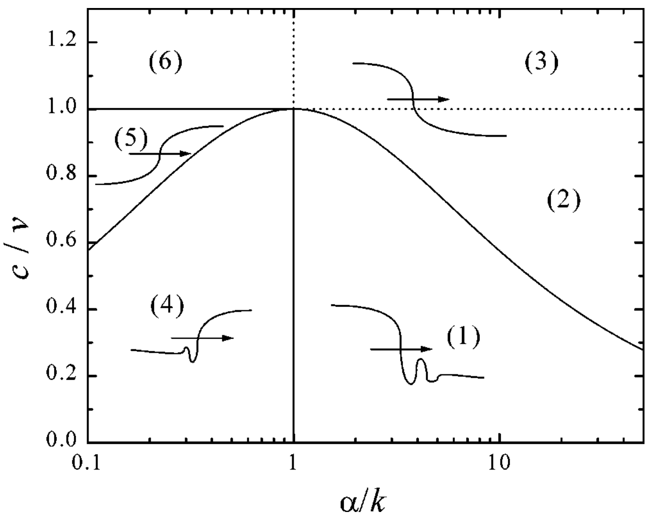

FIG. 3. The regions in parameter space where different traveling fronts are found. The front shapes are schematic. The numbers refer to the cases shown in Table I.

Now, if $\alpha<k$, the motion is damped on part of the trajectory [when $U>(1-\alpha/k)/2$, see Fig. 2(c)] and antidamped on the rest, specifically near $U=0$. This means that the equilibrium at $U=0$ is unstable, and there is a solution connecting it to the equilibrium at $U=1$. It is a trajectory connecting the unstable node at (0,0) to the stable manifold of the saddle at (1,0) [see Table I, case (5)]. This trajectory corresponds to a front of state 0 invading the state 1.

B. $v<c$

In this situation m is negative, so we multiply Eq. (6) by -1, to obtain

$$|m|U''-c(\alpha-k+2kU)U'=\alpha kU(1-U). \qquad (15)$$

Now the particle of mass $|m|$ moves in the potential $\phi(U)=-\alpha k(U^2/2-U^3/3)$ [refer to Fig. 2(b)], where there is a stable equilibrium at $U=1$. The sign of the damping coefficient is also reversed, and the motion is always antidamped when $\alpha>k$. When $\alpha<k$, it is always antidamped in the vicinity of $U=1$. It can be seen in Table I [cases (3) and (6)] that the equilibrium at (0,0) is always a saddle, and the equilibrium at (1,0) is always an unstable node. There is a trajectory that connects the unstable node to the stable manifold of the saddle, corresponding to a front of state 1 invading the state 0.

We note that, contrary to the generalized telegrapher's equation studied in [10], we do not have here oscillating wave shapes [corresponding to the unstable spiral at (1,0) shown in Table II]. Also to be remarked are the solutions where the state $U=0$—which is an unstable state of the logistic equation—invades the state $U=1$. This stabilization of the null state has been made possible by interplay of memory and reaction, through the state-dependent damping coefficient of Eq. (3).

A phase diagram summarizing these results is shown in Fig. 3. A qualitative indication of the nature of the front wave is given, together with a reference to the cases enumerated in Table I.

III. PIECEWISE LINEARIZATION

Explicit solutions corresponding to the front waves described above cannot be found in the fully nonlinear situation. However, a piecewise linearization of Eq. (3) can be made, as in related models of reaction-diffusion processes (see, for example, [12–14] for excitable systems, [15] for an electrothermal instability, and the generalization of Fisher's equation found in [10]). Following [10], we take the following as a reaction function:

$$f(U)=\begin{cases} U/a, & U\leqslant a,\\ (b-U)/(b-a), & U\geqslant a,\end{cases} \qquad (16)$$

where $a<b$. With this reaction term, which, incidentally, generalizes somewhat the logistic form (which corresponds to $a=b$), the oscillator equations in the traveling wave ansatz (5) become

$$mU''+2\gamma_1 U'+k_1^2U=0, \qquad U\leqslant a, \qquad (17)$$

$$mU''+2\gamma_2 U'+k_2^2(b-U)=0, \qquad U\geqslant a, \qquad (18)$$

where

$$\gamma_1=\frac{c}{2}\left(\alpha-\frac{k}{a}\right), \quad \gamma_2=\frac{c}{2}\left(\alpha-\frac{k}{b-a}\right), \qquad (19)$$

$$k_1^2=\frac{\alpha k}{a}, \quad k_2^2=\frac{\alpha k}{b-a}. \qquad (20)$$

Suppose that we are looking for a front solution that interpolates from $U=b$ as $z\to-\infty$ to $U=0$ as $z\to\infty$. Without loss of generality, we take $U=a$ at $z=0$. We look for solutions of two oscillators: one to the right of $z=0$, and one to the left of $z=0$. We define the following variables:

$$U_R(z)=U(z), \qquad z\geqslant 0, \qquad (21)$$

$$U_L(-z)=b-U(z), \quad z\geqslant 0.$$

Equations (17) and (18) become

$$mU_R''+2\gamma_1 U_R'+k_1^2U_R=0, \qquad (22)$$

$$-mU_L''+2\gamma_2 U_L'+k_1^2U_L=0. \qquad (23)$$

The wave front shape, which is a solution to Eqs. (17) and (18), can be easily found in terms of exponentials, by matching solutions of Eqs. (22) and (23) and their derivatives on either side of $z=0$. In some cases, one of the solutions of Eqs. (22) or (23) contains a growing exponential, and the corresponding factor has to be set to zero to avoid unbounded growth. Examples of all the cases described in the previous section are shown in Fig. 4, within this piecewise linear scheme. The solutions have been arbitrarily shifted in

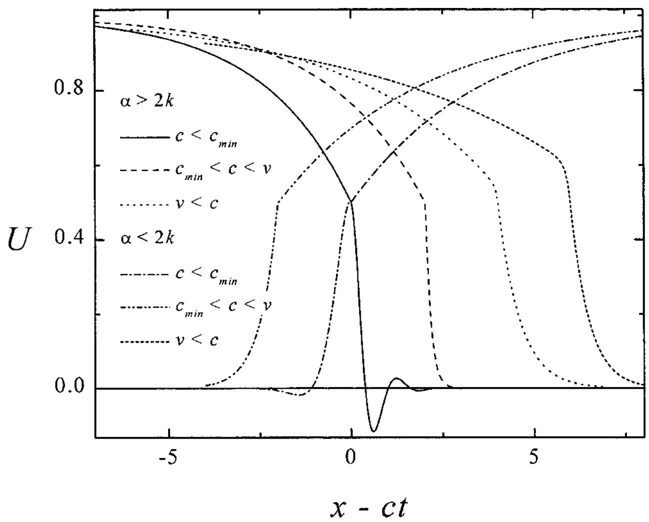

FIG. 4. Front waves in the piecewise linear version of the model. $\alpha < 2k$ and $\alpha > 2k$ correspond to the two distinct regimes of damping or antidamping in this linearized model. We have taken $a = 1/2$ and $b = 1$.

the variable $z = x - ct$ to avoid superposition of the curves. In the model linearized according to Eq. (16), the regimes of damping and antidamping of Eq. (5) are separated by the condition $\alpha = 2k$, which is used in the figure caption to classify the curves. (The condition separating the damping from the antidamping regimes is $\alpha = k/a$ in the piecewise linear model, instead of the $\alpha = k$ of the fully nonlinear model.)

IV. CONCLUSIONS

Reaction-diffusion systems in which the transport process is wavelike at short times and diffusive at long times are the focus of the present investigation. This passage of the character of the motion from coherent to incoherent is a general feature of all physical systems and may be represented by a memory function whose decay time (or correlation time) represents the demarkation. We have analyzed here a generalization of Fisher's reaction-diffusion equation which has a logistic nonlinearity describing the reaction process. Our study has centered on traveling wave solutions. From arguments without approximation we have found a generalization of the known Fisher's equation result regarding the minimum speed of the traveling waves. While the generalization converges to the Fisher's equation result in the limit of diffusive transport (memory that decays infinitely fast), sharp differences occur in the wavelimit (see Fig. 1): solutions are found to be possible involving "inverse" fronts, in which the state $U = 0$ invades the state $U = 1$ (see Fig. 3). We have also found explicit analytic solutions (typical cases plotted in Fig. 4) via the piecewise linearization introduced for this problem elsewhere [10] and pointed out a number of differences in the predictions arising in our present analysis. Some of these differences are expected but at least one of them is surprising: solutions that oscillate spatially when they have a speed above a certain value (called c_{osc} in Ref. [10]; see also Table II) predicted in Ref. [10] are found to disappear in the present analysis, which can be argued to be a more natural generalization of Fisher's equation to include wavelike transport. There are a number of issues, such as the stability of the solutions against perturbations introduced into the system, which will be studied in future investigations.

The introduction of memory functions [9] into the analysis of reaction-diffusion systems was made by Manne, Hurd, and Kenkre [10] to allow the description of coherent aspects of motion which are always present at short enough times, and equivalently to avoid problems associated with the infinite propagation speed characteristic of diffusion equations. A more recent analysis [16] with the same motivation has proposed two generalizations of Eq. (1)—one in discrete time and one in continuous time—that involve a long-range interaction or spread in the field u. In that analysis the problem of the asymptotic speed of fronts was studied: it was shown that the speed of propagating fronts (in the discrete system) is also smaller than the result of Fisher's equation, and indeed it is bounded instead of diverging when the reaction rate diverges.

We mention in passing the problem of what kind of initial conditions eventually evolve into a traveling front. The general problem can be very difficult, but a simplified treatment is possible as follows. Consider that the initial state has the behavior $u(x,0) \sim Ae^{-ax}$ for $x \to \infty$. Correspondingly, we suppose that the leading edge of the traveling wave has the form

$$u(x,t) = Ae^{-a(x-ct)}, \tag{24}$$

with a and A arbitrary.

We substitute this into the differential equation, and suppose that, at the leading edge, $u \approx 0$:

$$(ac)^2 u + \alpha acu - kacu + o(u^2) = v^2 a^2 u + \alpha ku + o(u^2). \tag{25}$$

Disregarding the terms in u^2 we arrive at a dispersion relation between c and a:

$$c = \frac{k-\alpha}{2a} \pm \sqrt{\frac{(\alpha+k)^2}{4a^2} + v^2}. \tag{26}$$

The negative sign before the square root in this expression leads to negative c, and can be disregarded. The expression with the positive sign gives the speed of the front wave that will eventually develop from an initial condition that has an exponential decay of coefficient a.

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