



Analytic considerations in the theory of NMR microscopy

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Abstract

Nuclear magnetic resonance microscopy poses fundamental as well as practical challenges. The Torrey–Bloch equation obeyed by the magnetization density is basic to this field. We discuss exact solutions of the equation from a generalization that addresses the motion of the nuclear spins towards an attractive center with an arbitrary time dependence in the attractive (harmonic) potential. The solutions provide an analytic way to interpret observations made with gradient magnetic fields of arbitrary (finite) pulse duration and to study the simultaneous effect of spin diffusion and time-dependent magnetic fields.

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1. Introduction and background

This article consists of a brief introduction to some aspects of the theory of nuclear magnetic resonance (NMR) microscopy along with new theoretical predictions for a certain class of dynamic situations in the area. This field of research is at once practical and fundamental. Generally, the scope of NMR experiments in physics, chemistry and biology is vast. The use of NMR in magnetic resonance imaging (MRI) is well known in hospitals and clinical laboratories. The technique is also used in studies of molecular motion in heterogeneous media, where information is sought about the morphology of the media [1–4], and of biological tissues in various medical contexts [5]. Important also is the application, whereby information can be extracted through NMR measurements about the geometry of constricted spatial regions that restricts the motion of spins (such as those of protons that are part of water molecules). The procedures and their theoretical background may be learnt from standard texts [6,7].

In essence, the NMR technique in this context consists of the application of two parallel external magnetic fields to a large number of diffusing particles each possessing nuclear spin. The two magnetic fields are taken to point along the z -axis, one strong and homogeneous \mathbf{B}_0 , and the other weak and systematically inhomogeneous. The inhomogeneity is designed with a linear gradient $\mathbf{G} = Gx f(t) \hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the unit vector in the x direction, G the strength of the gradient and $f(t)$ its temporal shape function. The measured quantity is the transverse magnetization integrated over the space where the experiment take place called the NMR

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signal. The combined effect of the inhomogeneous external magnetic field and the diffusion of the spin-bearing particles leads to an attenuation factor of the NMR signal which is characterized according to the temporal shape of the field gradient. For instance, free diffusing spins under the constant field gradient manifest an attenuation of the NMR signal $M(t)$ which is cubic in time [8]. The famous expression is: $\ln[M(t)/M(0)] = -Dg^2t^3/3$, where g is defined as the product of the strength of the gradient field G and the gyromagnetic ratio of the particles γ , and D is the diffusion constant of the spins.

Original studies of diffusing spins with NMR were performed in unrestricted geometries under time-independent field gradients [8–10]. A significant advance was made through Stejskal and Tanner's pulsed-gradient spin echo (PGSE) method [11–13] which uses bipolar pulses as the temporal shape of the pulse gradient. The method led to a simple interpretation of the PGSE experiment in the limit as the pulse duration $\delta \rightarrow 0$ and $g \rightarrow \infty$ while δg remains finite, in analogy to a scattering process, and for many years served as the basis of \mathbf{q} -space MRI. In that narrow pulse approximation, analytical expressions for the attenuation factor can be obtained by just knowing the conditional probability, $P(\mathbf{r}, \Delta | \mathbf{r}_0)$, of finding a spin-particle at position \mathbf{r} at time Δ if initially the particle is at \mathbf{r}_0 . Thus, provided $\delta \rightarrow 0$ (and $g \rightarrow \infty$ in the manner described above), the normalized spin echo amplitude is given by

$$M(\Delta) = \int d\mathbf{r}_0 P(\mathbf{r}_0) \int d\mathbf{r} P(\mathbf{r}, \Delta | \mathbf{r}_0) \exp\{ig\delta \hat{\mathbf{x}} \cdot (\mathbf{r} - \mathbf{r}_0)\}. \quad (1)$$

This pretty but oversimplified analytic starting point fails in practice since δ is by no means infinitesimally small: a typical experimental value is of the order of milliseconds. The question arises as how to treat diffusion effects that may occur *during* the gradient pulses. Of the number of analytic attempts that have been made to obtain usable expressions for the echo signal that are valid beyond the assumption of vanishing δ , the most well known is based on the use of the method of cumulants [14,15]. It involves approximating the probability distribution of the quantity $\int_0^t dt' gf(t')x(t')$ as a Gaussian, followed by employing a truncation expansion. In the procedure, $x(t)$, the position of the spin as function of time, is analyzed as a random variable.

A number of attempts have been made to go beyond the truncated cumulant approximation [16,17]. One way is to return to the so-called Torrey–Bloch equation (TBE) [8] for the transverse magnetization density $M(\mathbf{r}, t)$. In its one-dimensional version, the equation has the form:

$$\frac{\partial M(x, t)}{\partial t} = -igf(t)xM(x, t) + D \frac{\partial^2 M(x, t)}{\partial x^2}. \quad (2)$$

The first term in the right-hand side of (2) refers to the arbitrary time-dependent linear gradient field, where $g = G\gamma$ and $f(t)$ is the temporal shape of the gradient field taken to be in the x direction. The second term in the right-hand side of (2) arises from the diffusion of spins with diffusion constant D . Once $M(x, t)$ is known, the total, spatially unresolved, NMR signal $M(t) = \int dx M(x, t)$ may be computed. Exact analytical expressions in free space have been obtained [18] in this manner for arbitrary time dependence of the field gradient and for arbitrary initial spin distribution in space.

In the present paper we discuss a generalization of Eq. (2) and its solution that takes into account a possible tendency that the spins might have in certain systems to be constrained to attractive centers. In the next two sections we formulate the problem and show the general solution, and analyze the situation for time-dependent potentials *and* time-dependent gradient fields.

2. Generalization of the Torrey–Bloch equation and solution

The Fokker–Planck generalization of Eq. (2) which takes into account the effects of an attractive external potential of the form $\mathcal{U}(x, t) = A(t)x^2/2$ is given by

$$\frac{\partial M(x, t)}{\partial t} = -igf(t)xM(x, t) + \frac{\partial}{\partial x} \left[D \frac{\partial M(x, t)}{\partial x} + A(t)xM(x, t) \right] \quad (3)$$

and involves adding to the diffusion process a drift term given by $A(t)(\partial/\partial x)[xM(x, t)]$. The term takes into account the effects of an external harmonic potential with a time-dependent spring constant.

Eq. (3) is not straightforward to solve analytically. One of our new results is the exact solution we provide in Fourier-space [19] for any initial condition and any time dependence of the gradient field and the spring constant, respectively, of $f(t)$ and $A(t)$:

$$\hat{M}(k, t) = \hat{M} \left(k e^{-\int_0^t ds A(s)} + g \int_0^t dt' f(t') e^{-\int_0^{t'} ds A(s)}, 0 \right) \times \exp \left\{ -D \int_0^t dt' \left[k e^{-\int_0^{t'} ds A(s)} + g \int_{t'}^t ds f(s) e^{-\int_{t'}^s d\rho A(\rho)} \right]^2 \right\}. \quad (4)$$

Here, $\hat{M}(k, t)$ is the Fourier transform of the magnetization density $M(x, t)$.

Inversion of Eq. (4) can be difficult *but is unnecessary* if our interest is in the total signal $M(t)$. By taking the limit $k \rightarrow 0$ of Eq. (4), we get

$$M(t) = \hat{M} \left(g \int_0^t dt' f(t') e^{-\int_0^{t'} ds A(s)}, 0 \right) \times \exp \left\{ -D \int_0^t dt' \left[g \int_{t'}^t ds f(s) e^{-\int_{t'}^s d\rho A(\rho)} \right]^2 \right\}. \quad (5)$$

This *exact* solution allows us to address a diversity of experimental situations and to obtain various particular cases known earlier. For the PGSE experiment, we have here an exact analytical treatment of pulses of arbitrary duration with the added advantage that the initial spin distribution may also be arbitrary. It is also found, that the initial spin distribution plays an important role in the attenuation of the NMR signal. In particular, given an attractive potential, oscillations in the signal are observed when the initial distribution is localized at specific positions. The oscillations disappear for wide distributions but add up to an attenuation of the signal. A variety of possible time dependences of the signal occurs depending on the values of the dimensionless ratio $\alpha = A/(Dg^2)^{1/3}$. The physical significance of this quantity can be understood from the well-known cubic dependence of the logarithm of the NMR signal on time if spins diffuse freely. The characteristic time in that free diffusion case is $(Dg^2)^{-1/3}$. The quantity α is simply the ratio of that free diffusion signal time to $1/A$, the characteristic time for the spin to be pulled in to its attractive center.

Consider an initially localized magnetization density. The signal has the characteristic free diffusion form, with an exponent cubic in time, for small At : for $At \rightarrow 0$, the signal goes as $e^{-1/3 Dg^2 t^3}$. In the opposite limit of large At , one has a simple exponential decay: $e^{-(Dg^2/A^2)t}$. This exponential decay also occurs in the limit $\alpha \rightarrow \infty$. By contrast, for $At \ll 1$ in the initial Gaussian case, one obtains a quite different time dependence: $e^{-(Dg^2/2A)t^2}$. For a detailed analysis leading to the general expression (4) derived by Sevilla and Kenkre, and the reduction to specialized cases, we refer the reader to Ref. [19].

3. Analysis for time-dependent constraining potentials

We report new results in the present paper, not contained in Ref. [19] or earlier work, by focusing on the case when $A(t)$ is time dependent. This is a special situation in which the constraining walls or potential are not static but *have their own dynamics*. A possible application of considerable experimental relevance is the study, via the NMR technique, of time-dependent elasticity in colloidal gels [20]. For simplicity, let us consider the situation when $A(t)$ and the dimensionless field gradient temporal shape $f(t)$ are proportional to each other, i.e., $A(t) = Af(t)$. While we do this for analytic tractability, if the time dependence of the potential is known, or theorized, it could correspond to *experimentally* choosing in the laboratory a corresponding time dependence in the turn-on of the gradient field. Then, from Eq. (5), the NMR signal for initially localized spins is given by

$$\frac{-A^2 \ln M(t)}{Dg^2} = e^{-2\mathcal{F}(t)} \int_0^t dt' e^{2\mathcal{F}(t')} - 2e^{-\mathcal{F}(t)} \int_0^t dt' e^{\mathcal{F}(t')} + t, \quad (6)$$

where $\mathcal{F}(t)$ is defined by

$$\frac{d\mathcal{F}(t)}{dt} = Af(t). \quad (7)$$

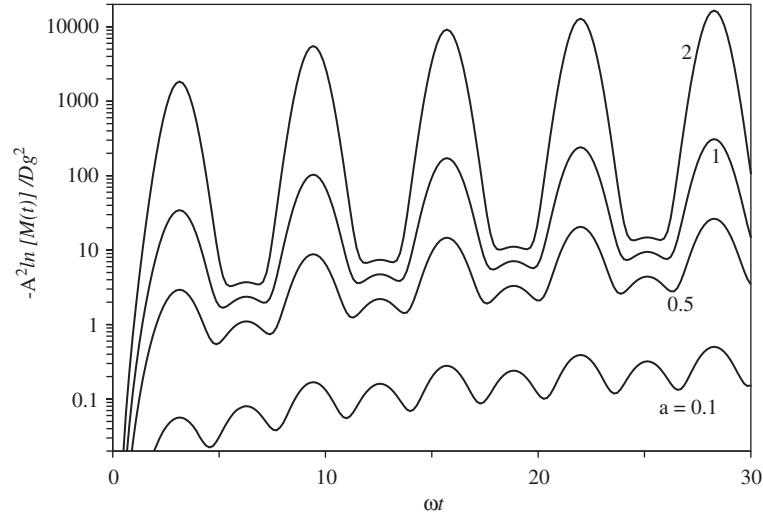


Fig. 1. Time dependence of the NMR signal for the case of time-dependent gradient field and also a time dependent attractive potential. Both time dependences are sinusoidal with identical frequency ω . Plotted is $-A^2 \ln[M(t)]/Dg^2$ versus ωt for various values of $a = A/\omega$ as shown: 0.1, 0.5, 1 and 2.

This exact analytical expression for $\ln M(t)$ can be studied further for a sinusoidal variation of $A(t)$, with frequency ω and amplitude A . For this specific case we get

$$\begin{aligned} \frac{-A^2 \ln M(t)}{Dg^2} = & t[I_0(2A/\omega)e^{-(2A/\omega) \cos \omega t} - 2I_0(A/\omega)e^{-(A/\omega) \cos \omega t} + 1] \\ & + \frac{2}{\omega} e^{-(A/\omega) \cos \omega t} \sum_{k=1}^{\infty} \frac{\sin \omega k t}{k} [e^{-(A/\omega) \cos \omega t} I_k(2A/\omega) - 2I_k(A/\omega)], \end{aligned} \quad (8)$$

where $I_k(z)$ is the modified Bessel function I of order k and argument z . The dimensionless parameter A/ω is simply the ratio of the time period of the potential to the time it takes the spins to get to the center of the trap. For $A/\omega \ll 1$, expression (8) reduces to the free case reported in Ref. [18]. In Fig. 1, expression (8) is plotted against ωt for various values of the ratio A/ω . We note that as A/ω increases, the behavior of the maxima of (8) changes: odd-numbered maxima increase their height while even-numbered maxima diminish in size. We hope that this and many other related predictions we have made will find potential use in the laboratory.

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