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Effects of finite probing windows on the interpretation of the multifractal properties of random walks

L. Giuggioli[1], G. M. Viswanathan[1,2], V. M. Kenkre[1], R. R. Parmenter[3,4] and T. L. Yates[4]

[1] *Consortium of the Americas for Interdisciplinary Science and Department of Physics and Astronomy University of New Mexico - Albuquerque NM 87131, USA*
[2] *Instituto de Física, Universidade Federal de Alagoas - 57072-970, Maceió-AL, Brazil*
[3] *Valles Caldera Trust, Jemez Springs - NM, 87025, USA*
[4] *Department of Biology, University of New Mexico - Albuquerque, NM 87131, USA*

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Abstract – We investigate the general problem of how the finiteness of a probing window for measurements of the movements of a random walker can lead to spurious detection of multifractality as well as to incorrect values of Hurst exponents, and propose a method for correcting for these effects. We also study the case in which the roaming region of the walker is itself of limited extent, when a nonlinear interplay occurs between the roaming area and the window size. In the context of animal movements, we describe briefly an application of these ideas to mark-recapture observations in a mouse population, of interest to the important topic of the spread of the Hantavirus epidemic.

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A classic problem in physics concerns the analysis of the scaling properties of random walks [1–3]. Indeed, random walks find wide application in diverse fields. They may be simple, as in the case of Brownian diffusion [1,2], or more complex, as in the case of anomalous diffusion [3]. Fractal analysis of these random walks aims at describing in a quantitative manner scaling behavior observed in physical [3], biological [3], ecological [4], physiological [5], sociological [6] and economic [7] phenomena. Accurately quantifying such scaling behavior helps since parallels may be drawn among classes of processes where similar scaling is observed. Our interest is in showing how interpreted fractal dimensions of a random walk are modified when observations are limited in space.

Spatially selective observations often arise in ecology where a random walking animal strays beyond the spatial region in which the experimental set-up allows detection, *e.g.*, the line of sight of a camera, a radio transmitter, a microscope or a telescope. In ecological studies the linear dimensions of the window may range from a few kilometers, as in the study of tree diversity in a forest [8], to a few hundred meters when observing animals via trapping methods [9], to a few centimeters in the case of the motion of microorganisms in sea water [10]. The measurements are obviously limited by the sampled area in that data gathered contain cutoffs beyond which no information can be collected. Consequently, incorrect conclusions might be drawn on the walker motion characteristics if the effects of the window are not accounted for.

To study the effects of the finiteness of the probing window, we consider a random walk that may be normal or anomalous. We address the general case of fractional Brownian motion [11], *i.e.*, assume that the probability distribution function of the walker, governed by a diffusion equation with a time-dependent diffusion coefficient [12] $D(t)$, is $P(x,t) = (\pi\rho(t))^{-1/2}\exp(-x^2/\rho(t))$ wherein $\rho(t) = 4\int_0^t D(t')\mathrm{d}t'$. For simplicity, we consider throughout this paper a walker in 1D space but our analysis can be trivially extended to higher dimensions. We limit our considerations to an algebraic dependence of $D(t)$ on t, $D(t) = \alpha D_\alpha t^{\alpha-1}$, where D_α is a generalized diffusion coefficient of dimensions $[Length]^2[Time]^{-\alpha}$, with $0<\alpha<1$ for subdiffusion, $1<\alpha<2$ for (sub-ballistic) superdiffusion and $\alpha = 1$ for normal diffusion.

To characterize the fractal dimension d of a random walker we follow standard usage in the recent literature [13,14] and introduce a quantity known as the Hurst exponent H [15] which is related [13] to the fractal

dimension via $d=2-H$. The Hurst exponent describes the time evolution of the mean-square displacement of the walker: $\int_{-\infty}^{\infty} x^2 P(x,t)\mathrm{d}x = \langle x^2 \rangle \sim t^{2H}$. It is well known that, if an arbitrary moment of the probability distribution $P(x,t)$ can be expressed in terms of the mean square displacement via $\langle |x|^q \rangle \sim \langle x^2 \rangle^{q/2} \sim t^{qH}$, a single (constant) scaling exponent H describes the entire process. In such a case, the motion is said to be *monofractal*. Simple Brownian diffusion has a Hurst exponent H equal to $1/2$ while anomalous diffusion has H smaller for subdiffusion, $0<H<1/2$, and larger for superdiffusion, $H>1/2$. In contrast to monofractals, the quantitative description of *multifractal* [16] random walks [17] requires multiple scaling exponents. Specifically, the higher and lower moments may not possess identical scaling, so that $\langle |x|^q \rangle \sim t^{qH(q)}$, with the (generalized) Hurst exponent being in such a case q-dependent.

By comparing the q-th moment of a random walker at two different times ($t_1>0$ and $t_2>t_1$), it is possible to calculate H:

$$H(q) = \ln\left[\frac{\langle |\Delta x|^q (t_2)\rangle}{\langle |\Delta x|^q (t_1)\rangle}\right] \frac{1}{q \ln\left(\frac{t_2}{t_1}\right)}, \tag{1}$$

which reduces, for monofractal walks, to a (constant) exponent. For fractional Brownian walkers we thus have $H=\alpha/2$.

If a walker is observed over all space, the evaluation of eq. (1) is straightforward since the q-exponent is factored out and H becomes a constant. On the other hand, the evaluation is nontrivial when $\langle |\Delta x|^q (t)\rangle$ is observed through a probing window. The presence of a limited spatial window of observation affects the evaluation of the q-th moments by cutting off the tails of $P(x,t)$:

$$\langle\langle |\Delta x|^q (t)\rangle\rangle = \frac{\int_{-G/2}^{G/2} \mathrm{d}x_0 \int_{-G/2}^{G/2} \mathrm{d}x\, |x-x_0|^q P_{x_0}(x,t) I(x_0)}{\int_{-G/2}^{G/2} \mathrm{d}x_0 \int_{-G/2}^{G/2} \mathrm{d}x P_{x_0}(x,t) I(x_0)}. \tag{2}$$

The presence of the integral over x_0 in eq. (2) stems from the fact that the observations are made over a collection of random walkers, *i.e.*, many members of the ensemble. The motion is characterized by averaging over individual behaviors which translates thus to an average over the initial location of each individual. $I(x_0)$ is the probability per unit distance of a walker to be located initially in x_0 somewhere inside the window of width G centered around the origin. The cutoff length G in eq. (2) represents the extent of the finiteness of the probing window. Our study focuses on the effects of $G<\infty$.

In most situations, $I(x_0)$ is a constant either because of natural homogeneity or because of our own ignorance of the initial condition of the walker. Equation (2) can then be simplified and eq. (1) reduces to

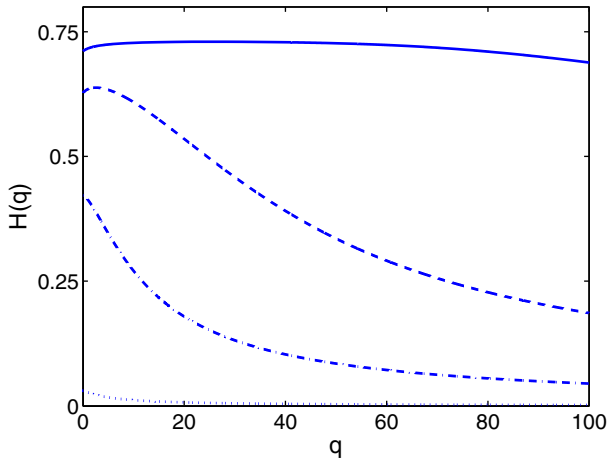

Fig. 1: Hurst exponent $H(q)$ as function of q for a collection of fractional Brownian walkers ($\alpha=3/2$ and $b=2^{3/2}$) observed inside a spatial window of size G. From top to bottom the curves are shown with decreasing value of the extent of the probing window: the values of y (which is proportional to G, see text) are 6.25, 2.5, 1.25, 0.25, respectively.

$$\begin{aligned} H(q) &= \frac{\alpha}{2} + \frac{\ln\left[\frac{y\gamma\left(\frac{1+q}{2},y^2\right)-\gamma\left(\frac{2+q}{2},y^2\right)}{by\gamma\left(\frac{1+q}{2},b^2y^2\right)-\gamma\left(\frac{2+q}{2},b^2y^2\right)}\frac{g(by)}{g(y)}\right]}{q\ln(t_2/t_1)}, \\ y &= G/\sqrt{4D_\alpha t_2^\alpha}, \quad b=(t_2/t_1)^{\alpha/2}, \end{aligned} \tag{3}$$

wherein $g(z)=\sqrt{\pi}z\,\mathrm{erf}(z)+e^{-z^2}-1$, and $\gamma(\nu,x)=\int_0^x \mathrm{d}t e^{-t}t^{\nu-1}$ is the incomplete Gamma function.

Equation (3) along with eq. (5) below represent the primary expressions we have derived and form the basis of the main analysis in this paper.

It is clear from eq. (3) that the appearance in it of y, therefore of the extent G of the probing window, makes $H(q)$ depend on the size of the window. A monofractal walk which in extended space would actually have a constant (q-independent) H would be interpreted as multifractal through an analysis on the basis of eq. (3) precisely for this reason. To make this apparent, we have plotted in fig. 1 $H(q)$ for different values of G, which decreases as we go down the curves from top to bottom. It is evident from the plot how monofractal behavior can be erroneously interpreted as multifractal. None of the curves is constant as q is varied. For large q values the average $\langle |\Delta x|^q (t)\rangle$ gives information about the tails of $P(x,t)$. Finite G cuts off those tails. Higher moments of $P(x,t)$ quickly saturate to the size of the window ($\propto G^q$). For sufficiently large q, the moments at time t_2 change only slightly from those at time t_1, making $H(q)$ decay to zero proportionally to $1/q$. As G increases, $H(q=0)$ approaches $\alpha/2$ and the region where the curve is essentially flat gets larger, pushing the inflection point further out. Eventually $H(q)$ reduces to the constant $\alpha/2$ for any q if $G\to\infty$. On the other hand,

for G sufficiently small, the maximum disappears and $H(q)$ decays monotonically to zero for large q.

A common characteristic of all the curves in fig. 1 is $H(0) < \alpha/2$. Even if the spatial window strongly affects the higher moments, the renormalization of the window cut-off also modifies the peak of $P(x,t)$. Its small-q moments are modified as well. This effect can be verified analytically by calculating the limit $q \to 0$ in eq. (3). We get $H(0) = \alpha/2 + 2\left[k(y)/g(y) - k(by)/g(by)\right]/\ln(t_2/t_1)$, where $k(z) = \int_0^z \mathrm{d}s e^{-s^2} \ln(s)(z-s)$. Since $k(z)/g(z)$ is a monotonic function of z and $b > 1$, $k(y)/g(y) - k(by)/g(by) < 0$ and thus $H(0) < \alpha/2$. Bounding information on the possible values of α and D_α may also be obtained by studying the Hurst exponent derivative $H'(q) = \mathrm{d}H(q)/\mathrm{d}q$ at $q = 0$.

A thorough theoretical analysis of the effects of limited spatial observations in the context of animal movements was carried out recently by some of the present authors to determine the home range of mice [18,19]. That analysis led to the evaluation of the home range area, as well as the diffusion constant, in a mouse population observed in mark-recapture experiments in Panama [20] and New Mexico [21]. We generalize here that analysis to include arbitrary moments. The presence of the window makes the q-th moment saturate at large times. However, saturation may also occur if the walkers themselves roam inside a confined area. The classic example of such confinement is when each individual of an animal population lives inside its own home range [22]. In evaluating the q-th moment one has to take into account the fact that each walker is confined to a different region of space relative to the observation window. For this one takes an additional average over all possible center positions x_c of the confining area [18], which we will consider here to be homogeneously placed, for simplicity. We generalize here the findings of ref. [18] to fractional Brownian motion by studying the full time dependence of the moments and using it to estimate the Hurst exponent.

The reduced randomness associated with the confining home range can be modelled [18] via a Fokker-Planck equation [23] of the form

$$\frac{\partial P(x,t)}{\partial t} = D(t)\frac{\partial}{\partial x}\left\{\frac{\partial P(x,t)}{\partial x} + \frac{\mathrm{d}U(x-x_c)}{\mathrm{d}x}P(x,t)\right\}, \tag{4}$$

where the dimensionless U is the confining potential and the time dependence of the diffusion constant $D(t)$ represents the anomalous diffusion of the walker.

At short times, and independently of the form of U, the mean-square displacement grows as $\sim t^\alpha$, while at long times, it saturates to a value associated with the confining potential and proportional to $\int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 \exp\left[-U(x-x_c)\right]$. In general, the choice of U should be assessed in each case on the basis of *a priori* knowledge of the specific animal behavior. In light of our application to the *Peromyscus Maniculatus* mark-recapture data [21] of New Mexico, where the distribution of mice displacements over time could be approximated with a Gaussian, we consider the parabolic potential $U(x) = x^2/L^2$, where L is the home range dimension. In that case,

$$\begin{aligned} H(q) &= \frac{\ln\left[\frac{f(t_2)}{f(t_1)}\right]}{2\ln(t_2/t_1)} \\ &\quad + \frac{\ln\left[\frac{z\gamma\left(\frac{1+q}{2},z^2\right)-\gamma\left(\frac{2+q}{2},z^2\right)}{b'z\gamma\left(\frac{1+q}{2},b'^2z^2\right)-\gamma\left(\frac{2+q}{2},b'^2z^2\right)}\frac{g(b'z)}{g(z)}\right]}{q\ln(t_2/t_1)}, \\ z &= G/\left(L\sqrt{2f(t_2)}\right), \quad b' = \sqrt{f(t_2)/f(t_1)}, \end{aligned} \tag{5}$$

where $f(t) = 1 - \exp\left[-2D_\alpha t^\alpha/L^2\right]$. Notice that eq. (5) is a more general form of eq. (3) in that the quantities b, y and α in the latter are replaced by b', z and $\ln\left[f(t_2)/f(t_1)\right]/\left[\ln(t_2/t_1)\right]$, respectively. As the home range extent L becomes much larger than the length $\sqrt{D_\alpha t^\alpha}$ which characterizes the (fractional Brownian) motion, specifically, in the limit $L^2/D_\alpha t^\alpha \to \infty$, eq. (5) reduces to the simpler eq. (3).

As an illustrative application of our theory we consider the study of animal motion via mark-recapture observations, which involves animals being trapped (and released) periodically [9]. Traps are placed within an area which is obviously finite. When an animal is captured for the first time, it is tagged and its position is recorded. If that animal is subsequently recaptured at t_1, we obtain the first displacement $\Delta x(t_1)$. One more recapture of the same animal at t_2 gives the second displacement $\Delta x(t_2)$. Our theory can thus be applied. Details on how to renormalize the distribution of displacements given the discrete trap locations may be found in ref. [21].

Data we have chosen for our illustrative application are from a long-term (1994–2003) field work study at four sites in New Mexico [24] of the principal host of the Hantavirus Pulmonary Syndrome, the deer mouse, *Peromyscus maniculatus*. Animals were captured on a monthly basis for three consecutive nights on each occasion. Only those (adult) mice that were captured during two consecutive nights (1070) or during the first and third night (530) constitute the data we will use. We focus here on the analysis of the mice motion along the East-West direction [21]. This is because the data statistics along the North-South direction turn out to be rather poor and do not allow us to draw meaningful conclusions on the anomalous aspects of their motion. Our analysis of the motion should thus be considered only as an illustration. Further observations would be necessary in order to draw finalized conclusions about the anomalous aspects of *Peromyscus maniculatus* in the New Mexico landscape.

Since the time during the night at which the various mice were captured has an uncertainty at most of ± 6 hours, we obtain three different fits to the experimental curve. One corresponds to the case in which all mice are assumed to have been recaptured halfway through the night both on the second ($t_1 = 1$ day) and third day

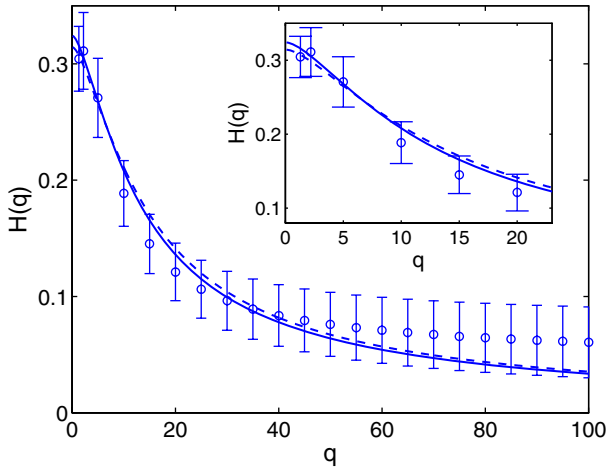

Fig. 2: Hurst exponent $H(q)$ for the deer mouse, *Peromyscus maniculatus* in New Mexico, obtained from mark-recapture experiments. Corresponding to 3 different recapture scenarios (see text), 3 different curves —two of them identical to each other as the solid line— are obtained from a least-squares fit of eq. (5). The fitting parameters are α and D_α with the restriction that $\alpha < 2$. Values of α that emerge (see text) show that the motion is superdiffusive ($\alpha > 1$). The inset provides a zoomed-in view of the $0 \leqslant q \leqslant 23$ region. The error bars are obtained by using three different binning procedures.

($t_2 = 2$ days). The second case corresponds to all mice assumed being recaptured at the beginning of the second night ($t_1 = 3/4$ day) and at the end of the third night ($t_2 = 9/4$ days). The third situation may occur if all the mice were recaptured at the end of the second night ($t_1 = 5/4$ day) and at the beginning of the third night ($t_2 = 7/4$ days). In fig. 2 we have plotted the Hurst exponent with 3 different fits. The solid, dashed and dotted curves correspond, respectively, to the three possible recapture scenarios discussed above. The circles in fig. 2 represent the mean values of the experimental data with their corresponding error bars. Three different binning procedures have been used to estimate the distribution of experimental displacements. From the least-squares fit the α parameters for the three recapture scenarios are 1.95, 2.00 and 1.71 and their corresponding D_α's are, respectively, 2568, 3285 and 2185 expressed in units of $\mathrm{m}^2/\mathrm{day}^\alpha$.

The fitting apparently shows superdiffusive behavior. However, even though all fitted curves are within reported error bars, the χ^2 is relatively large (≈ 7–$8 \cdot 10^{-3}$), and none of the fitting exercises captures an important qualitative feature of the data —the presence of a maximum particularly evident in the inset for small q-values. Surely, it is also possible that the mice motion is governed by some kind of anomalous diffusion different from fractional Brownian motion [25–27].

In summary, we have constructed the theory for interpreting data for fractional Brownian random walkers when observations are confined within a region of space, and considered effects of the walkers themselves being physically constrained within a limited region of space. We have shown how the Hurst exponent depends analytically on α and D_α, and on the home range dimension and the size of the probing window. In our application to data, the statistics happen not to be clear enough to allow unambiguously useful conclusions be drawn. Nevertheless we have shown explicitly how to carry out the procedure and have extracted the parameters α and D_α. Apparently, the results tend to point to the mouse walks being superdiffusive in the case studied and over the short time scale of 1–2 days analyzed. We hope that the theory we have reported will be useful in characterizing the motion of animals as well as other random walkers in a wide variety of field contexts. Limited spatial observations, whether of the mark-recapture type or of more sophisticated nature, are often an inescapable fact of field observations. There should be no doubt that this characteristic of realistic observations must be incorporated in estimation and parameter extraction.

The new techniques for studying multifractality that we have developed here have other important physical applications. We note, for example, the recent interest in distinguishing genuine anomalous diffusion from Markovian correlated random walks [28]. In this context, due consideration of the scales of the relevant probing windows might lead to greater insight into such questions.

* * *

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