

Random-walk access times on partially disordered complex networks: An effective medium theory

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An analytic effective medium theory is constructed to study the mean access times for random walks on hybrid disordered structures formed by embedding complex networks into regular lattices, considering transition rates F that are different for steps across lattice bonds from the rates f across network shortcuts. The theory is developed for structures with arbitrary shortcut distributions and applied to a class of partially disordered traversal enhanced networks in which shortcuts of fixed length are distributed randomly with finite probability. Numerical simulations are found to be in excellent agreement with predictions of the effective medium theory on all aspects addressed by the latter. Access times for random walks on these partially disordered structures are compared to those on small-world networks, which on average appear to provide the most effective means of decreasing access times uniformly across the network.

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I. INTRODUCTION

There has been considerable recent interest in the statistical, topological, and dynamical properties of complex networks, important examples of which include the well-known scale-free and small-world networks (SFNs and SWNs). Justification for this interest stems from the increasing use of complex network theory to the understanding of a wide variety of phenomena in the social, biological, and physical sciences, a small sample of which include collaboration and acquaintance networks, protein interactions in the cell, power grids, entangled polymers, and information and communication networks [1–10]. In addition to the ongoing interest in static topological properties of such complex, generally disordered structures, there has been a parallel interest in various dynamical processes defined on them, and on how such processes differ from those that take place on ordered networks and regular lattices (RLs) [11–23].

In particular, a recent contribution to the study of random walks on Newman-Watts SWNs, which are structures formed by superimposing a classical random graph of randomly distributed links onto the sites of an ordered regular lattice [1–4], is a variation introduced by Parris and Kenkre [21] in which the random walk proceeds in continuous rather than discrete time, and the jump rate f for steps across the random shortcuts is different from the rate F for steps across regular lattice bonds. Such a modification has two primary advantages. First, it allows the strength of the regular and random graph components of the network to be independently adjusted (or even turned off) at fixed shortcut density by simply varying the ratio f/F . Second, it allows for the rational evaluation of schemes for upgrading or optimizing access times in existing information or communication networks, e.g., by randomly incorporating a relatively small number of newer, faster connections into an existing network already possessing a large number of older, perhaps slower, ones.

The analysis of Ref. [21] focused on the network traversal time $\tau = \tau_{N/2}$, defined as the mean first-passage time required

by a random walker initially at site n to reach the site at $m = n + N/2$, farthest from where it started on the regular one-dimensional (1D) lattice backbone (formed into a ring of N nodes), averaged over initial sites and network realizations. That study revealed an interesting collapse of traversal time data onto a universal curve for fast shortcuts $f/F \gg 1$ and low shortcut densities $n_{sw} = N_{sw}/N = 2\bar{k}$, defined as the ratio of the number N_{sw} of small-world shortcuts to the total number N of sites (or nodes) in the system. This ratio n_{sw} is equal to 2 times the average shortcut vertex degree \bar{k} .

The present paper explores a question that has naturally arisen from an extension [22] of the work of Ref. [21] on SWNs to the study of random walks on hybrid structures formed by superimposing scale-free shortcut networks onto regular lattices, carried out recently considering more general access times τ_m for a walker to reach a location m nodes away from where it started. One conclusion of Ref. [22] was that, at fixed shortcut degree \bar{k} and shortcut speed f , SFN-RL hybrids are always less efficient at reducing the mean traversal time τ , and result in more dispersion in the distribution of access times across the network, than their SWN counterparts. This conclusion, which simply reflects the more inhomogeneous nature of local environments in SFNs relative to SWNs, raises the question as to whether there exist other random structures whose embedding would generally enhance network traversal and access times more efficiently than SWNs.

To explore this question, we consider a new class of partially ordered random structures that are generally more ordered than the SWNs studied in Refs. [21,22], and that we generically refer to as traversal enhanced networks (TENs). Motivation for the development of such structures arises from the intuition that a walker's ability to traverse the system is not efficiently enhanced by shortcuts between nodes that are already close together. Indeed, in a given SWN system, if some of the shorter random links could be redirected into nodes farther away, the mean traversal and access times

for the entire structure would presumably decrease without changing the total number of shortcuts.

We consider here, therefore, random walks on a 1D ring of N nodes connected by nearest-neighbor hopping rate F , and associate with each possible link between two nodes separated by a backbone distance $|n| \leq N/2$ an *a priori* shortcut connection probability $q_n = q_{-n}$, with $1 \geq q_n \geq 0$, such that $q_0 = q_1 = 0$, and $\bar{k} = \sum_n 2q_n$. Two nodes separated by a particular backbone span $|n|$ are connected by a hopping rate f with probability q_n and are left unconnected with probability $1 - q_n$. The SWN hybrids previously studied in Ref. [21] are seen to be limiting cases of such a network with equal connection probabilities $q_n = q \sim \bar{k}/2N$. To optimize network traversal times, or times for access to an arbitrary site, on the other hand, TENs allow for the possibility of increasing connection probabilities q_n for large $|n|$ relative to those for small $|n|$, keeping fixed the total average shortcut degree \bar{k} . The general theory that we develop in this paper allows for the study of partially ordered networks with an arbitrary shortcut connection probability distribution q_n . For purposes of comparison with simulation, however, the main results of the present paper are applied to a specific subclass of such structures for which $q_n = q \delta_{|n|, n_0}$, i.e., for which shortcuts of a single span n_0 are randomly added. As might be expected, adding shortcuts of length n_0 dramatically decreases access times for pairs of sites separated by that distance. Perhaps less intuitively, however, we also find that the average access time to any point on the network fluctuates rapidly with n_0 , and depends, in particular, on whether shortcut lengths are commensurate with the 1D ring to which the shortcuts are added. Indeed, the obvious limiting case with $n_0 = N/2$ is found to maximally enhance the network traversal time, but is associated with the largest average access time, over all such networks with fixed average shortcut degree \bar{k} .

The rest of the paper is presented as follows: In the next section we introduce the master equation governing the evolution of the probabilities $P_m(t)$ for the walker to be at a given site on such a network at time t . For each network, the master equation can in principle be solved numerically in either the time or frequency (Laplace) domain, and the results averaged over an appropriate ensemble of networks to obtain observable quantities of interest. This becomes increasingly difficult for large N due to the size of the random matrices that must be manipulated in the process. Equivalent results can be obtained by performing Monte Carlo simulations that follow the trajectory of individual walkers, using the assigned transition probabilities to determine a specific random sequence of steps and transition times. Simulation results must then be doubly averaged, first over a sufficient number of walks, and then over an appropriate ensemble of networks. A useful fixed jump time simulation technique for computing mean traversal and access times for the underlying continuous time random walk is described in Sec. II for this purpose. The averaging process underscores the point that one is interested in statistical properties of the ensemble of networks associated with a particular set of connection probabilities q_n , which for large enough networks should be representative of typical properties associated with any given realization. Section III presents an analytic approach to the

problem with the construction of an approximate, but numerically accurate, effective medium theory (EMT) for describing the evolution of the ensemble averaged probabilities $p_m(t) = \langle P_m(t) \rangle$, from which the mean traversal times can be easily and quickly computed, even for very large networks. In Sec. IV we present and discuss the main numerical (Monte Carlo) results of the paper, comparing them in each case with the predictions of our analytic effective medium theory; we find them to be generally in excellent agreement with each other. The last section of the paper contains a summary.

II. NETWORK MODELS, MASTER EQUATIONS, AND SIMULATION TECHNIQUES

As is clear from the introduction, we are interested in random walks in continuous time as described by the master equation

$$\begin{aligned} \frac{dP_m}{dt} &= \sum_{n=\pm 1} F(P_{m+n} - P_m) + \sum_{n \neq \pm 1} f_{m,n}(P_{m+n} - P_m) \\ &= - \sum_n A_{m,n} P_n, \end{aligned} \quad (1)$$

where on the ring it is understood that $P_m = P_{m+N}$, and the second sum in the first line includes all values $n \in \{\pm 2, \pm 3, \dots, \pm(N/2-1), N/2\}$, but excludes nearest neighbors (N is assumed even). The transition rate $f_{m,n} = f_{m+n,m}$ between node m and node $m+n$ is a random variable equal to f with probability q_n and to zero with probability $1 - q_n$. In the second line we have expressed the linear relation on the right-hand side in terms of the transition matrix \mathbf{A} . Indeed, the solution to the master equation can be numerically obtained through an evaluation of the exponential of this matrix. Specifically, the probability to find the walker at node m at time t , given that it started at node n at $t=0$ is the Green's function or propagator [21]

$$G_{n,m}(t) = [e^{-\mathbf{A}t}]_{n,m}.$$

The functions $G_{n,m}(t)$, or their Laplace transforms

$$\tilde{G}_{m,n}(\varepsilon) = \int_0^\infty G_{m,n}(t) e^{-\varepsilon t} dt = [(\varepsilon + \mathbf{A})^{-1}]_{m,n} \quad (2)$$

are readily computed for moderately sized systems $N \leq 10^3$ using standard numerical techniques, but computation times and storage requirements can grow significantly with system size N .

Information about traversal and access times is straightforwardly obtainable from the Green's functions or their Laplace transforms, Eq. (2). For example, the mean first passage time $\tau_{m,n}$ for a walker to arrive at node m given that it started at n is computable through the expression [21]

$$\tau_{m,n} = - \lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} \left(\frac{\tilde{G}_{m,n}(\varepsilon)}{\tilde{G}_{m,m}(\varepsilon)} \right). \quad (3)$$

In this way, general traversal and access times are readily computed from the resolvent of the matrix \mathbf{A} for small values

of the Laplace variable ε ; a full solution of the problem requires only the operation of matrix inversion.

On the other hand, the full solution to the master equation contains more information than is really necessary for computing mean access times. It is also possible to simply perform a Monte Carlo simulation [22] that follows individual trajectories along an ensemble of random walks, with steps and dwell times governed by the transition probabilities assigned to a given network. Jump destinations of the random walker are chosen at each time step from the local transition probabilities at the site occupied by the walker, by means of pseudorandom variables (see, e.g., [24]). Specifically, a walker located at a given site on the network with shortcut degree k will make a transition to one of its two neighbors on the ordered ring with probability $p_{\text{lat}}=fT$, will make a transition to one of the k sites to which it is connected by a shortcut with probability $p_{\text{sh}}=fT$, and will stay at its present position with probability $p_{\text{stay}}=1-(2F+kf)T$. After performing n_{MC} Monte Carlo steps, the corresponding continuous time elapsed is $\tau=n_{\text{MC}}T$. As we show in the appendix, as long as we are interested only in computing mean first passage times (and not the probabilities themselves, or other observables), the choice of the fixed jump time T is arbitrary, provided all transition probabilities remain positive definite (i.e., provided $T \leq (2F+k_{\text{max}}f)^{-1}$). In the Appendix we prove, specifically, that this method (when averaged over a large enough ensemble of random walks) reproduces the mean access times of the master equation. The mean access time τ_m for a given network configuration, is then computed as the average of this quantity over random walks on the same network, and then over a sufficiently large ensemble of networks characterized by the same set of network parameters. Except where explicitly noted, simulation results shown in the present paper were obtained for networks of size $N=1000$, typically averaged over 100 different network configurations and 1000 different random walk trajectories per configuration. Specific simulation results appear in Figs. 1–8, together with predictions of the effective medium theory derived in the next section.

III. EFFECTIVE MEDIUM THEORY FOR COMPLEX NETWORKS

In this section we derive analytical expressions that define an effective medium theory (EMT) for the configuration averaged probabilities, or Green's functions

$$\langle \tilde{G}_{n,m} \rangle = \langle \tilde{G}_{n+...m} \rangle = \tilde{g}_{n-m}(\varepsilon) \quad (4)$$

that describe transport on the very general class of structures described in the introduction. We also specialize it to the analysis of partially ordered networks having a sharp distribution of link lengths, in which form it becomes very simple to implement. As suggested by Eq. (4) above, the key simplification that arises in EMT is that average transport properties of the ensemble are translationally invariant. In the present context, we assume that the functions $\tilde{g}_m(\varepsilon)$ obey the equations

$$\varepsilon \tilde{g}_m - \delta_{m,0} = \sum_{n=\pm 1} F(\tilde{g}_{m+n} - \tilde{g}_m) + \sum_{n=\pm 1} \tilde{w}_n(\varepsilon)(\tilde{g}_{m+n} - \tilde{g}_m). \quad (5)$$

Here, the $\tilde{w}_n(\varepsilon)$ are frequency-dependent rates, equivalently, memory functions in the Laplace domain, connecting pairs of sites on the network separated by a distance $|n|$. Notice that, without loss of generality, we consider that the random walker's initial position is at $m=0$. In the EMT equations, transport around the ring edge is characterized by the same rate F that obtains throughout the ensemble, but the $\tilde{w}_n(\varepsilon)$ must be determined from self-consistent considerations. The simplest (non-self-consistent) approximation, i.e., $\tilde{w}_n(\varepsilon) = \langle f_{m,n} \rangle = q_n f$, generally gives a very poor approximation to the dynamics, and is equivalent to an ‘‘annealed model’’ in which possible shortcut connections are determined anew at each step of the random walk [21], a situation very different from the actual ‘‘quenched disorder’’ model considered here, which has percolative possibilities lacking in an annealed model.

Note that once the effective medium parameters $\{\tilde{w}_n(\varepsilon)\}$ are determined, the rest of the problem is straightforward, since the translationally invariant set of equations (5) is easily solved by introducing discrete Fourier transforms $\tilde{g}^k(\varepsilon) = \sum_m \tilde{g}_m(\varepsilon) e^{-ikm}$, which diagonalize the EMT transition matrix, and lead to the solution

$$\tilde{g}_m(\varepsilon) = \frac{1}{N} \sum_k \frac{e^{ikm}}{\varepsilon + 2F [\cos(k)] + \tilde{w}_{N/2} [1 - \cos(kN/2)] + \sum_{n=2, N/2-1} 2\tilde{w}_n(\varepsilon) [1 - \cos(kn)]}, \quad (6)$$

where the outer sum is over all wave vectors $k=2\pi/N$, with $\in \{1, \dots, N\}$, where in the denominator we have separated out the single $n=N/2$ term, for which there is only one link, and where in the other terms (with $n < N/2$) contributions

from both links of a given length have been combined together. From these effective medium propagators it is also straightforward to evaluate access times. Indeed, analysis of the propagators (6) reveals that the mean access times (3) in

the effective medium can be simply calculated as

$$\tau_m = - \lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} \left(\frac{\tilde{g}_m(\varepsilon)}{\tilde{g}_0(\varepsilon)} \right) = N \tilde{\gamma}_m(0) \quad (7)$$

directly from the zero frequency value of the propagator difference

$$\tilde{\gamma}_m(\varepsilon) \equiv \tilde{g}_0(\varepsilon) - \tilde{g}_m(\varepsilon). \quad (8)$$

To determine the function $\tilde{w}_n(\varepsilon)$ self-consistently, we imagine that we have already performed the average over all links except for the one connecting the origin to the site n nodes away, so that the resulting equations of motion are not the completely averaged EMT equations (5), but are, instead, of the form

$$\varepsilon \tilde{p}_m - \delta_{m,0} = \sum_{n=\pm 1} F(\tilde{p}_{m+n} - \tilde{p}_m) + \sum_{n=\pm 1} \tilde{w}_n(\varepsilon)(\tilde{p}_{m+n} - \tilde{p}_m) + (\delta_{m,0} - \delta_{m,n})[f_{0,n} - \tilde{w}_n(\varepsilon)](\tilde{p}_n - \tilde{p}_0), \quad (9)$$

where $f_{0,n}$ is an actual rate (f or 0) drawn from the distribution governing it. For a given value of $f_{0,n}$ the solution to this defect problem can be formally written in terms of the propagators \tilde{g}_n for the effective medium. For example, the Laplace transformed probability to find the walker at the origin can be expressed in terms of the EMT quantities $\tilde{\gamma}_n$ in (8) through the relation

$$\tilde{p}_0 = \tilde{g}_0 - \frac{(f_{0,n} - \tilde{w}_n) \tilde{\gamma}_n^2}{1 + 2(f_{0,n} - \tilde{w}_n) \tilde{\gamma}_n}. \quad (10)$$

Averaging this over the binary distribution associated with this one remaining link must now generate the completely averaged effective medium, i.e., we self-consistently demand that $\langle \tilde{p}_0 \rangle = \tilde{g}_0$. Hence the average of the second term on the right-hand side of this last equation must vanish. Repeating for each node n gives the following set of self-consistent conditions:

$$\frac{q_n(f - \tilde{w}_n)}{1 + 2(f - \tilde{w}_n) \tilde{\gamma}_n} = \frac{(1 - q_n) \tilde{w}_n}{1 - 2 \tilde{w}_n \tilde{\gamma}_n}, \quad n = 2, \dots, N/2, \quad (11)$$

which are equivalent to the relations $2 \gamma_n \tilde{w}_n^2 - \tilde{w}_n(1 + 2f \gamma_n) + q_n f = 0$. By (e.g., numerically) solving this set of simultaneous equations at each value of ε , the set of functions $\{\tilde{w}_n(\varepsilon)\}$ can be determined, and the results used to evaluate the effective medium propagators, and thus the traversal and access times. Note that, in general, an analytic solution is not available since the functions $\tilde{\gamma}_n(\varepsilon)$ themselves depend in a complicated way on the complete set of effective medium parameters $\{\tilde{w}_n(\varepsilon)\}$, through (6) and (8).

Note also that in the SWN hybrids studied in Ref. [21], where $q_n = q$ for all shortcuts, and where a single effective medium parameter $\tilde{w}_n = \tilde{w}$ was assumed, the value of \tilde{w} was identified as the root of the equation

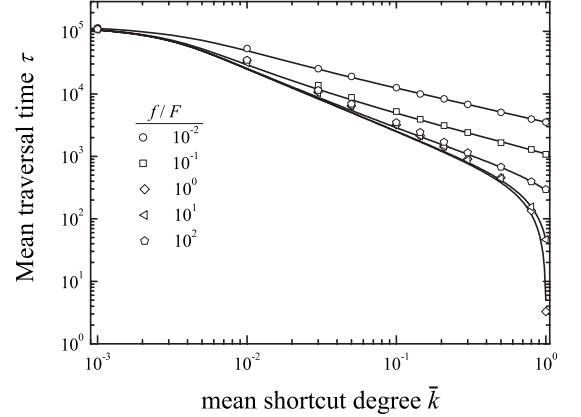


FIG. 1. Mean traversal time as a function of the mean shortcut degree \bar{k} on a TEN of size $N=1000$, with links of maximal length $n_0=N/2$, for different values of f/F , as indicated.

$$\sum_{n=2}^{N/2} \frac{q(f - \tilde{w})}{1 + 2(f - \tilde{w}) \tilde{\gamma}_n} = \sum_{n=2}^{N/2} \frac{(1 - q) \tilde{w}}{1 - 2 \tilde{w} \tilde{\gamma}_n}, \quad n = 2, \dots, N/2 \quad (12)$$

obtained by summing (or averaging) each of the equations (11). In the present context, we anticipate generally having to simultaneously solve the set of $(N-2)/2$, Eqs. (11), for the corresponding number of effective medium parameters \tilde{w}_n .

Of course, in situations in which $q_n=0$, the vanishing of the left-hand side of (11) leads to a vanishing of the associated effective medium rate $\tilde{w}_n(\varepsilon)$ as well, and hence a reduction in the number of coupled equations that must be solved. Thus, in the simplest such network, for which $q_m = q \delta_{m,n_0}$, and in which only shortcuts of a single fixed length n_0 are added to the ring, the set reduces to one equation for determining the single nonzero effective medium parameter $\tilde{w}_{n_0}(\varepsilon)$, which is easily determined numerically, and from which the corresponding access times can then be calculated using (7). In the next section we present the results of Monte Carlo simulations on such networks along with the corresponding predictions of the effective medium theory derived above.

IV. NUMERICAL RESULTS AND DISCUSSION

In Fig. 1 we show the mean traversal time $\tau = \tau_{N/2}$ (normalized to the mean hopping time $\tau_F = F^{-1}$ for hops around the edge of the ring) as a function of the mean shortcut degree \bar{k} on a TEN of $N=1000$ sites with links of maximal length $n_0=N/2$, for different values of f/F . Open symbols are the results of simulation, the solid curves running near or through them are the predictions of EMT. As expected, the mean traversal time decreases monotonically as the mean degree of the network increases, since a larger density of shortcuts naturally shortens the time needed to cross the network. The decrease in traversal time is modest for small values of the ratio f/F , but becomes quite large for $f \geq F$ and for shortcut densities close to the maximum value, which for this network occurs when the mean shortcut vertex degree

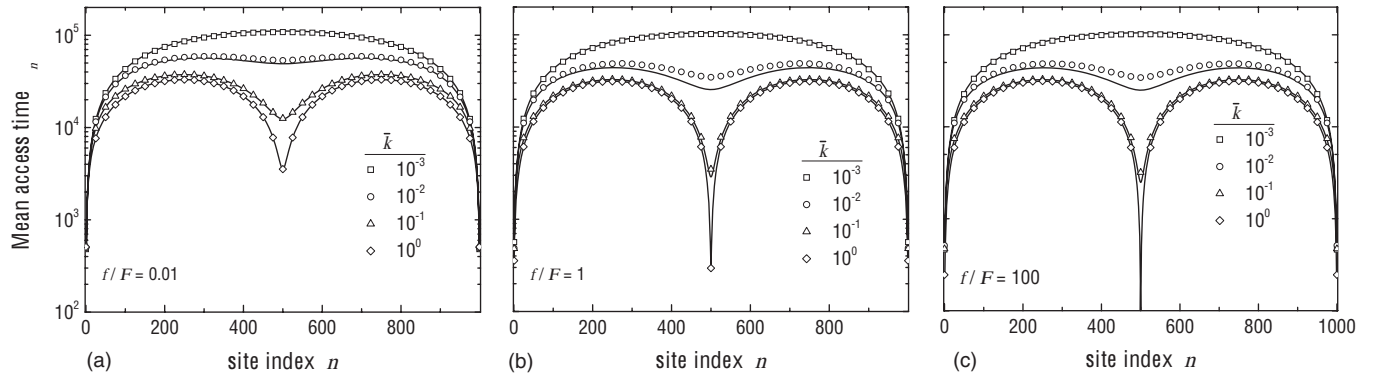


FIG. 2. Access times for TENS of size $N=1000$, with $n_0=N/2$, as a function of the site position n (where the random walk's initial position is $n=0$). The plots correspond to different shortcut densities in the range $0.01 \leq \bar{k} \leq 1$, as indicated, and different jump rate ratios: (a) $f/F=10^{-2}$; (b) $f/F=1$; (c) $f/F=10^2$. Vertical scales are the same in (a)–(c).

$\bar{k}=\bar{k}_{\max}=1$. The decrease seen in Fig. 1 is quite similar to that previously encountered in the SWN case [21], where, for low shortcut densities and $f \geq F$, the data were found to collapse onto a universal curve, while, for higher shortcut densities near $\bar{k} \approx 1$, traversal times scaled as $\tau \sim (f/F)^{-1}$. In this regime of the SWN, as in the present case, by increasing the ratio f/F the mean traversal time can be made arbitrarily small [21]. Not surprisingly, the behavior seen in the SWN and in the TENS of Fig. 1 arise for a similar reason. In the limit $\bar{k} \ll 1$, the shortcut density is very low and the traversal time tends to the diffusive limit, where $2F\tau_{\text{diff}}=(N/2)^2$. Even when $f/F \gg 1$, if $\bar{k} \ll 1$, the shortcut connections are very sparse, and there will not be a substantial reduction in the traversal time, as the walker still spends most of its time on the slower steps it must take around the ring. In this regime, therefore, the traversal time is largely insensitive to the value of f/F , as long as it is large enough for the shortcuts not to be rate limiting. However, when the number of shortcuts approaches a threshold near $\bar{k} \approx 1$, percolation of the underlying shortcut network leads to a change in the transport mechanism; in this regime particles need few if any slow steps to get across the system, and can move mainly along the fast shortcut connections.

In order to further compare characteristic features of transport on different network structures, we also consider the distribution and spatial variation of access time τ_n for going from an arbitrary initial site to one located n nodes away along the 1D backbone. Figure 2 shows access times τ_n as a function of node position n for the $n_0=N/2$ TEN at different shortcut densities in the range $1 \geq \bar{k} \geq 0.01$, as indicated, and for different jump rate ratios: (a) $f/F=10^{-2}$, (b) $f/F=1$, and (c) $f/F=10^2$. Again, open symbols indicate simulation results, and solid lines the predictions of the EMT, which is generally quite good, except in a few cases where it tends to underestimate the access time. For small mean shortcut degrees, $\bar{k} \ll 1$, access times grow monotonically with distance, as transport is dominated by diffusion along the regular lattice bonds. For denser networks on the other hand, one observes a local minimum taking place at $n=n_0=N/2$, reflecting the traversal time enhancement that might be expected of the TEN structure.

Figures 2(a)–2(c) show the effect of increasing the speed of shortcut connections. As expected, in the sparse network limit ($\bar{k} \ll 1$) access times are essentially universal and are insensitive to shortcut speeds, while for denser networks access is enhanced dramatically around $n=n_0=N/2$ for large values of f/F . Indeed, at $f/F=10^2$ [see Fig. 2(c)], an improvement of almost 3 orders of magnitude in $\tau_{N/2}$ is achieved by increasing the shortcut density from $\bar{k}=0.1$ to $\bar{k}=1$. The improvement is dramatic, but not very democratic, and is largely confined to a region in the immediate vicinity of n_0 .

To highlight this point, we show in Fig. 3 a comparison of access times for TEN and SWN structures, for $\bar{k}=1$ and for different jump rate ratios f/F , as indicated [25]. In this figure we see a sharp contrast between access times in TEN structures (which clearly reflect their specific design goal of improving access to sites directly across the network) and SWN access time distributions, which are remarkably flat, and which, for the same number of links, do a better job of providing faster access times to most network nodes.

Further insights on the different transport properties of TEN and SWN structures can be gained by considering

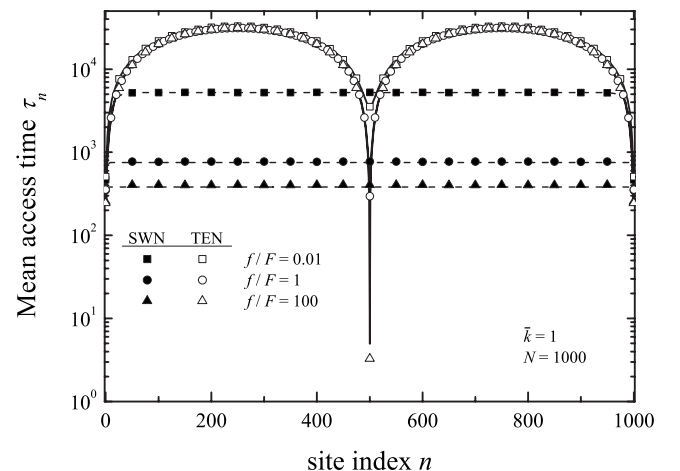


FIG. 3. Comparison between access times for the SWN and the TEN (with $n_0=N/2$) for $\bar{k}=1$, $N=1000$, and different jump rate ratios, as indicated.

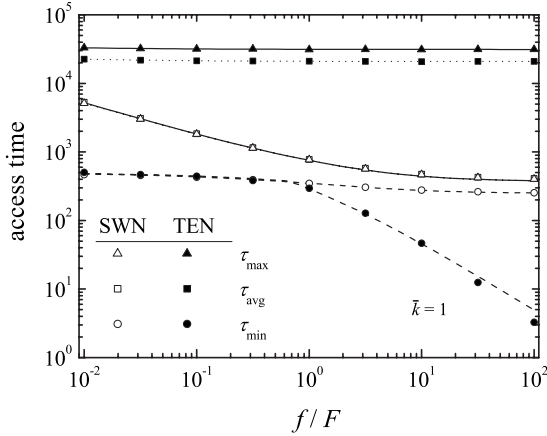


FIG. 4. Comparison between the minimum, average, and maximum mean access times for the SWN and the TEN, respectively. As in Fig. 3 we consider $\bar{k}=1$, $N=1000$, and $n_0=N/2$.

simple statistical properties of the access time distribution, such as the minimum, average, and maximum access times, defined by $\tau_{\min}=\min\{\tau_n\}$, $\tau_{\text{avg}}=(N-1)^{-1}\sum_n\tau_n$, and $\tau_{\max}=\max\{\tau_n\}$, respectively.

From our previous discussion regarding the results in Fig. 2, it should be noted that, for TENs in the parameter region $\bar{k}\approx 1$ and $f/F\gg 1$, the mean traversal time is actually the minimum access time for the network, i.e., $\tau_{\min}\sim\tau_{N/2}$. For SWNs, on the other hand, the mean and maximum access times are essentially the same, and well approximated by the traversal time, $\bar{\tau}\sim\tau_{\max}\sim\tau_{N/2}$. A comparison of τ_{\min} , τ_{avg} , and τ_{\max} for TENs and SWNs as a function of the rate ratio f/F is shown in Fig. 4. As in Fig. 2, we consider dense networks with $\bar{k}=1$. For both types of network, when $f/F\leq 1$ the minimum access time τ_{\min} corresponds to the time to arrive at one of its nearest neighbors. However, in the TEN case a cross-over takes place around $f/F\approx 1$ to a regime in which τ_{\min} corresponds to the traversal time for going all the way across the network. Indeed, for $f/F\gg 1$ it is observed that $\tau_{\min}\ll\tau_{\text{avg}},\tau_{\max}$, reflecting the large traversal enhancement effect characteristic of TENs in this parameter region. The differences among τ_{\min} , τ_{avg} , and τ_{\max} are much less pronounced in the case of SWNs, which is again a manifestation of the nearly flat distribution of this type of network.

In order to investigate further the properties of TEN structures, in particular, we now consider the case in which shortcut connections have lengths different from the ring's diameter d . If we consider shortcuts connecting sites separated by a distance n_0 (measured along the backbone ring in units of lattice bond length) as straight rods or wires of length L , then

$$L=d\sin(\pi n_0/N). \quad (13)$$

Figure 5 shows access times for TEN structures of $N=1000$ sites with mean degree $\bar{k}=1$, and shortcuts of different length: (a) $n_0=125$, (b) $n_0=250$, and (c) $n_0=375$. Here the mean vertex degree is now one-half the maximum value it can attain, since there are now two possible links of length n_0 connected to any given site. For these structures, an interesting oscillatory pattern arises for $f/F\gg 1$, with a characteristic repeat length related to the shortcut length n_0 and to the commensurability of n_0 with the total number of sites in the ring. Notice the similarity between the patterns for $n_0=125$ and $n_0=375$, which presumably arises due to the fact that when one moves around the ring with steps of length $n_0=125$, the path closes after one circuit, and revisits the same set of eight sites, separated by a distance n_0 , while with steps of length $n_0=375$, the path closes after two circuits, over which it visits the same set of sites (modulo 1000) as visited in one circuit with steps of length $n_0=125$. As with the system in which $n_0=N/2$, however, there are clearly significant regions of the network for which the speed-up in access time is not very significant. Are such gaps inevitable? Perhaps not. In these examples the shortcut length is commensurate with the circumference of the ring. One might imagine, based upon the description above for $n_0=375$, that the periodic structure might be “filled in,” by choosing a step length that is highly incommensurate with the circumference of the ring. Indeed, an investigation of this effect confirms that a slight alteration in the shortcut length can lead to dramatic changes in the spatial distribution of mean access times. We display this effect in Fig. 6, where we show the mean access time distribution for four systems, consisting of two pairs with shortcut lengths that differ by a relatively small number of sites. The dramatic change in the shape, and the overall mean access time, suggests that the latter quantity is a strongly fluctuating function of the shortcut length n_0 for this class of networks. In Fig. 7 we present a plot showing this strong

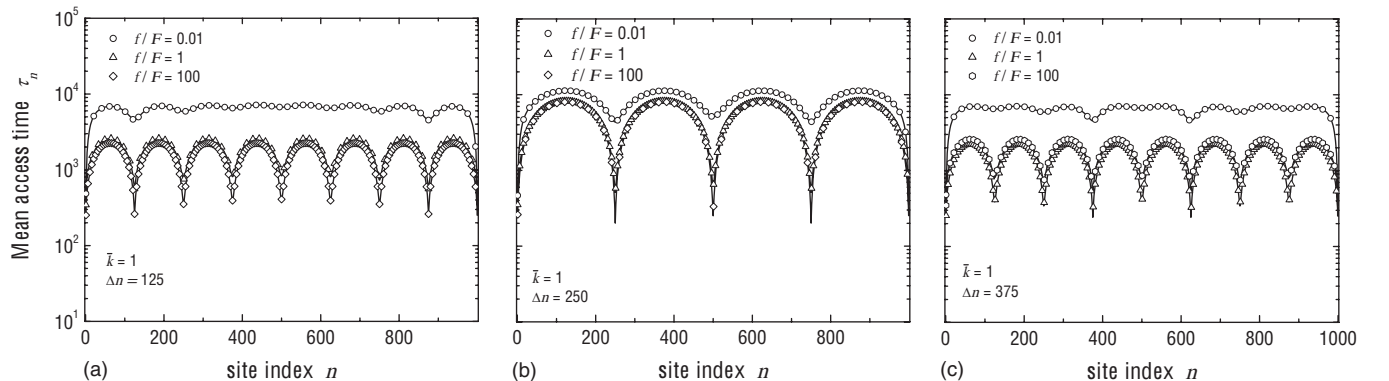


FIG. 5. Access times for TEN structures of size $N=1000$, mean degree $\bar{k}=1$, and shortcuts of different length: (a) $n_0=125$, (b) $n_0=250$, and (c) $n_0=375$. Vertical scales are the same in (a)–(c).

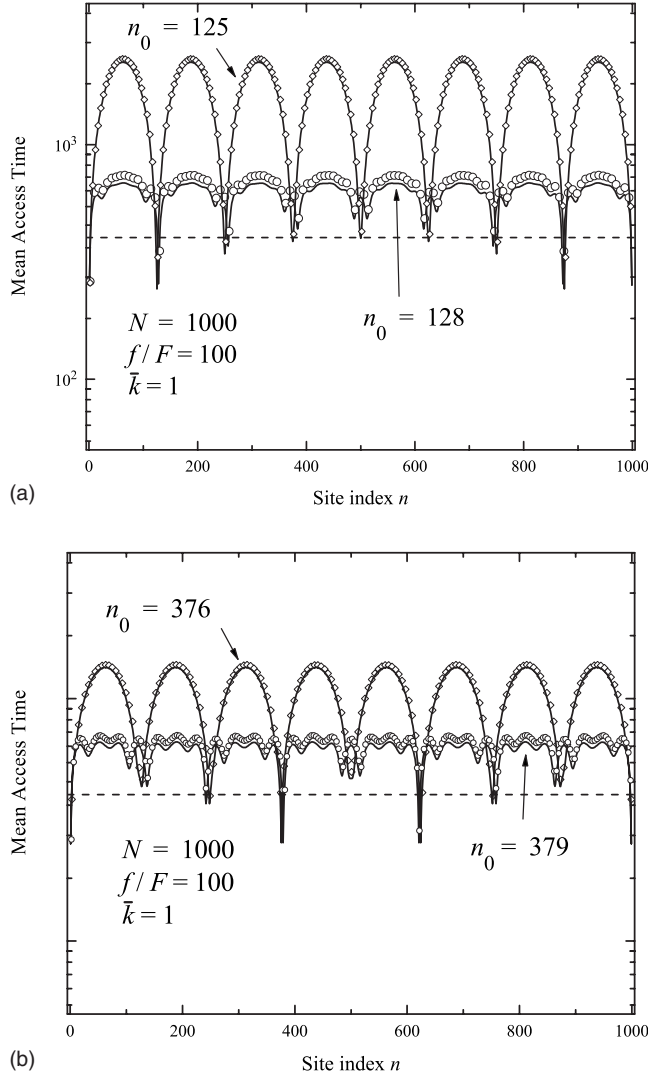


FIG. 6. Access times for TEN structures of size $N=1000$ and rate ratio $f/F=100$. (a) Results for connections of length $n_0=125$, and 128, and (b) of length $n_0=376$ and 379. Symbols are the results of numerical simulations, solid lines the predictions of effective medium theory. Horizontal dashed line is the mean access time for SWNs at the same shortcut density.

variation of the mean access time of the entire network as a function of the shortcut length n_0 , for a fixed average number of shortcuts. In Figs. 6 and 7 the solid lines are the result of the effective medium theory, the symbols are the result of numerical simulation. Effective medium theory does an excellent job capturing the highly fluctuating access time spectrum. The results show that in addition to speeding up access times to the specific sites connected by shortcuts of a given chosen length, a significant reduction in the overall access time is also possible, provided that the shortcut length is not commensurate with existing structures in the network to which the shortcuts are added.

Finally, as discussed in the introduction, one of the main interests of studying transport properties on different network topologies (with the addition of shortcuts to an underlying ordered structure) is aimed at optimizing access times and signal transmission across the network, where added

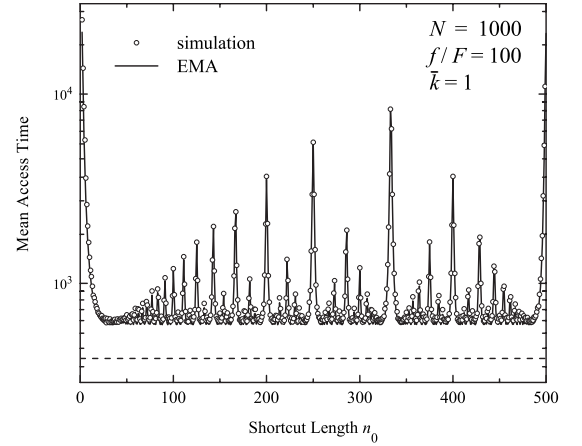


FIG. 7. Average network access times for TEN structures of size $N=1000$ and rate ratio $f/F=100$, as a function of the length n_0 of the shortcuts, at fixed shortcut density. Horizontal dashed line is the mean access time for SWNs at the same shortcut density.

shortcuts might represent fast, high quality connections ($f/F \gg 1$). In this context, let us consider a comparison at fixed “cost” [as represented by the total length of shortcut wires, i.e., $L_{\text{total}} = L \times (N\bar{k}/2)$] and compare access times obtained on TEN configurations for different values of L . As an example, Fig. 8 shows access times for two TEN structures of size $N=1000$, where fast connections ($f/F=100$) of length $n_0=250$ and $n_0=500$ are compared for corresponding values of the shortcut densities that keep L_{total} fixed. From Eq. (13) and the condition of equal total length, it turns out that $\bar{k}(n_0=N/4) = \sqrt{2} \times \bar{k}(n_0=N/2)$. One can see that the use of a large number of shorter connections is generally more effective in reducing access times to most of the network nodes, except for sites located in the immediate neighborhood of the region spanned by the longer connections.

V. SUMMARY

In this paper we have investigated the transport of walkers in a class of networks similar to the Newman-Watts small-

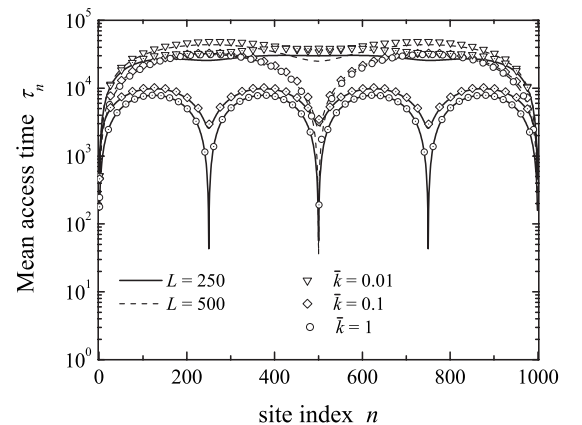


FIG. 8. Average network access times for TEN structures of size $N=1000$ and rate ratio $f/F=100$, and different shortcut densities are compared for systems with the same total length of all shortcuts.

world networks on the basis of an effective medium theory. The primary motivation has been to study the effect of varying the shortcut distribution in a well-defined way. The intuition that replacing shorter links in a small-world network with an equal number of longer ones might improve access times across the network is only partially verified. In keeping with straightforward intuition, it has been found that, if the goal is to simply improve access between nodes lying at specific distances, then randomly adding links of that length will indeed improve the specific subset of access times between nodes located at those distances, sometimes dramatically so. In cases where the shortcut lengths are commensurable with the underlying size of the system, this reduction in access times is not generally reflected in the access time of the network as a whole, with large regions of the network retaining surprisingly large access times. In situations where the shortcut length is highly incommensurate with the size of the system, however, the reduction in access times expected at distances associated with the shortcut length are also at least partially shared by most of the remaining sites on the network. Presumably this would reflect the more typical behavior that one would expect in adding shortcuts of a fixed length to an already disordered structure. It also suggests a strategy for probing hidden structures in apparently disordered networks.

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APPENDIX

In this appendix we show how the numerical procedure described in the main part of this paper is able to correctly compute mean access and traversal times, using a time step T that is chosen arbitrarily, provided the associated transition probabilities remain positive. We hasten to point out that if we were interested in numerically simulating (or integrating) the master equation (1), such an approach would not work: For numerical accuracy the time step T would have to be chosen much smaller than the smallest hopping time in the system. On the other hand, when enough walks are employed the process described above does correctly compute mean access times.

To see this, we consider a walker that has just arrived at some site m on the network. This site will have two neighbors on the ring connected to it by transition rates F , and k_m shortcut sites to which it is connected with rates f . From the master equation it is well known, and straightforward to

show, that a walker will leave such a site at a random time τ drawn from the exponential jump time probability distribution

$$\psi_m(\tau) = \Gamma_m e^{-\Gamma_m \tau}, \quad \Gamma_m = 2F + k_m f.$$

When it does make a transition, the probability p_s for it to go to one of the shortcut sites to which it is connected, and the probability p_R for it to go to one of its two neighboring sites on the ring, are given by

$$p_s = \frac{f}{2F + k_m f}, \quad p_R = \frac{F}{2F + k_m f}.$$

In the numerical procedure described in Sec. II, these branching ratios are preserved each time a jump occurs, as is easily verified. Thus, each random walk path generated by our numerical procedure has the same statistical weight as it would if governed by the exact jump time distribution given above. It suffices therefore to consider the ensemble of random walks that take place on any given random walk path, and to show that our numerical procedure correctly predicts the correct average duration of walks occurring on such a path. If the initial site and final site of such a path are taken to be the ones whose access time is of interest, then such a demonstration is equivalent to showing that the numerical procedure correctly calculates these quantities of interest.

We note, however, that the average duration of walks occurring on a particular path is just the sum of the average times required for each step along the way. The average time that the particle waits at a given site m , as described above, before hopping is

$$\tau_m = \int_0^\infty d\tau \tau \psi_m(\tau) = \Gamma_m^{-1}.$$

In the numerical procedure described in Sec. II, on the other hand, the hopping time distribution function is not a continuous exponential distribution, but a discrete one, i.e., the particle can only leave site m at integral multiples $T = T$ of the basic time step T . The probability p that it does so at the n th such step is

$$P = p^{-1} q,$$

where $p = p_{\text{stay}} = 1 - (2F + k_m f)T$ and $q = 1 - p = (2F + k_m f)T$. The average time that it spends at this site, with the numerical procedure we have implemented, is then easy to compute. We find

$$\tau_m = \sum_{n=1}^{\infty} n T P^n = (2F + k_m f)^{-1} = \Gamma_m^{-1},$$

which is the same as for the (correct) continuous exponential jump time distribution function, and that it is, as claimed, independent of the size of the time step, provided that all transition probabilities remain positive. It is clear that for the computation of access times, one could implement any hopping time distribution that has the correct mean pausing time at each site.

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- [25] Notice that the effects of adopting different network structures are more noticeable for large shortcut densities (close to $\bar{k} \simeq 1$), since otherwise the behavior tends to the universal diffusion limit previously discussed.