

Hurst exponents for interacting random walkers obeying nonlinear Fokker–Planck equations

Niraj Kumar^{a,*}, G.M. Viswanathan^{a,b}, V.M. Kenkre^a

^a Consortium of the Americas for Interdisciplinary Science and Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131, USA

^b Instituto de Física, Universidade Federal de Alagoas, Maceió–AL, CEP 57072-970, Brazil

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ABSTRACT

Anomalous diffusion of random walks has been extensively studied for the case of non-interacting particles. Here we study the evolution of nonlinear partial differential equations by interpreting them as Fokker–Planck equations arising from interactions among random walkers. We extend the formalism of generalized Hurst exponents to the study of nonlinear evolution equations and apply it to several illustrative examples. They include an analytically solvable case of a nonlinear diffusion constant and three nonlinear equations which are not analytically solvable: the usual Fisher equation which contains a quadratic nonlinearity, a generalization of the Fisher equation with density-dependent diffusion constant, and the Nagumo equation which incorporates a cubic rather than a quadratic nonlinearity. We estimate the generalized Hurst exponents.

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1. Introduction

Random walks and anomalous diffusion [1–5] have traditionally been approached from the perspective of whether or not they include memory. Markoffian processes describe random walks with short (e.g., exponentially decaying) memory, whereas non-Markoffian walks describe the general case of random walks with memory of the (possibly complete) history [6]. Less work has gone towards the investigation of how interactions among random walkers can change the global behavior. The presence of even local interactions can dramatically alter global behavior. For example, when the diffusion equation for Brownian motion governed by a Wiener process is augmented with a convective term whose strength is linear in the local density of the walkers, the resulting Burgers equation leads to qualitatively different behavior: the principle of superposition breaks down and Gaussian solutions become unstable. A recent study [7] has generalized the formalism of Hurst exponents to address the problem in the context of Burgers equation arising from hydrodynamic models of vehicular traffic flow. Specifically, it was shown that initial conditions become important due to the breakdown of the principle of linear superposition. In the present paper we extend that study to several further cases of interacting random walkers.

The Hurst exponent quantifies how quickly particles diffuse. For the case of zero drift velocity, the Hurst exponent H describes how the mean squared displacement of a random walker $\langle x^2 \rangle \sim t^{2H}$ scales with time t . Normal diffusion gives $H = 1/2$ due to the central limit theorem, which guarantees convergence of the probability density function of the walkers' position to a Gaussian. For non-interacting particles, the Fokker–Planck equation for the probability density $P(x, t)$ of a particle is linear in P . Hence, the propagator of the Fokker–Planck equation contains all the relevant information concerning H .

* Corresponding author.

E-mail address: niraj@unm.edu (N. Kumar).

However, when particles interact, the resulting evolution equation might be nonlinear; hence Green's functions and propagators will obviously not exist.

In Section 2 we recall and discuss the findings of Ref. [7] that the usual method of estimating Hurst exponents can lead to spurious predictions as a result of non-negligible effects of initial conditions. In Sections 3–6 we apply the method to study an exactly solvable system as well as a number of nonlinear diffusion equations, including the Fisher and Nagumo equations.

2. Hurst exponents for interacting random walkers

2.1. Generalized Hurst exponents

A number of methods can be applied to quantify different aspects of anomalous diffusion. Anomalous diffusion has been studied using a number of formalisms and approaches. Continuous time random walks [2,8] and generalized master equations [9] (GMEs) are formally equivalent [10]. Fractional partial differential equations [11,12] are equivalent to GMEs. Here we use the formalism of Hurst exponents, which can be related to Hölder exponents [13–16], describing the degree of differentiability along the trajectories. This formalism has been used to study the anomalous dynamics of different systems. Recently, this has been applied in the field of finance as well [17].

One can define the Hurst exponent $H(q)$ for a stationary stochastic process [14,16] in terms of the scaling of the absolute moments of the density:

$$\bar{x} \equiv \langle x \rangle \quad (1)$$

$$M_q(t) \equiv \langle |x - \bar{x}|^q \rangle \sim t^{qH(q)} \quad (2)$$

where the averages are taken over the propagator. Brownian motion and normal diffusion correspond to $H(q) = H = 1/2$, whereas anomalous diffusion corresponds to all other cases. As discussed in Ref. [7], one can generalize the concept to allow a scale dependence [18–20], such that $H = H(q, t)$:

$$M_q(t) \sim t^{qH(q,t)}. \quad (3)$$

For instance, the telegrapher's equation [21] has a mean squared displacement that grows quadratically for small times but linearly for larger times. The behavior is ballistic at small times, ($H(q, t) \approx 1$) but diffusive at large times ($H(q, t) \rightarrow 1/2$). This behavior can also be written in terms of the asymptotically defined Hurst exponent $H(q)$ and a scaling function f , such that $M_q(t) \sim t^{qH(q)}f(t/t^*)$, t^* being the typical crossover time and $H(q) = 1/2$ with $f \sim t$ for $t \ll t^*$ and $f \sim \text{constant}$ for $t \gg t^*$. However, we don't know, a priori, that such crossovers are generic, which limits the applicability of this scaling description. Also, for systems for which the Hurst exponent changes continuously in time, it is difficult to write proper scaling function and thus Eq. (3) gives a natural and more generic way to approach a problem.

2.2. Nonlinear Fokker–Planck equations

Standard methods of deriving Fokker–Planck equations from Langevin equations lead always to linear equations [7]. Consider, for example, the Boltzmann equation for gases, obeyed by the one-molecule distribution function. It is nonlinear, in contrast to the underlying linear Liouville equation for the N -molecule Liouville density. In the absence of inter-particle interactions, the Boltzmann equation would be linear. Similarly, if the gas molecules interact with a fixed system of random scatterers, the Boltzmann equation would be nontrivial but still linear. The standard manner of applying Hurst and Hölder exponents would work here. The case of interacting random walkers is different. If intermolecular interactions are turned on, nonlinearity enters the picture and immediately makes unavailable the superposition principle and propagator analysis. We recall the standard manner in which the Hurst exponent $H = H(2)$ is usually obtained from the behavior of the mean squared displacement for linear equations. In terms of the propagator or the Green function $\psi(x, x_0, t)$ and the initial distribution $P(x_0)$, the Hurst exponent is given by the scaling behavior of

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dx_0 (x - x_0)^2 \psi(x, x_0, t) P(x_0).$$

For a translationally invariant (homogeneous) system such as the one under consideration in this paper, this equation reduces to

$$\langle x^2 \rangle_{\delta} = \int dx x^2 \psi(x, t) \quad (4)$$

because the propagator is a function of the difference $x - x_0$. We use the suffix δ in the left hand side of Eq. (4) to emphasize that the $\langle x^2 \rangle$ used here can be considered to be the one calculated for an initially localized initial condition $P(x_0) = \delta(x_0)$. This can also be used for a collection of many random walkers provided they are non-interacting among themselves: the initial distribution P_0 is irrelevant in a linear system of non-interacting particles.

However, if the particles interact such that P satisfies a nonlinear equation, the nonlinearity prevents us from writing in Eq. (4) the linear superposition

$$P(x, t) = \int_{-\infty}^{\infty} dx_0 P_0(x_0) \psi(x - x_0, t). \tag{5}$$

Following Ref. [7], we compute $\langle x^2 \rangle_{P_0}$ by substituting x_0 with the center \bar{x}_0 of the distribution at time $t = 0$:

$$\langle x^2 \rangle_{P_0} = \int_{-\infty}^{\infty} dx (x - \bar{x}_0)^2 P(x, t, P_0), \tag{6}$$

where P now depends nonlinearly on the initial condition P_0 . We no longer expect $\langle x^2 \rangle_{P_0}$ to remain independent of P_0 . Other moments $\langle |x|^q \rangle$ also depend on initial conditions.

Non-normalized distributions and even systems that do not conserve probability can be studied, by defining the moments explicitly in terms of the distribution $P(x)$ via

$$M_q = \langle |x - \bar{x}|^q \rangle \equiv \frac{\int_{-\infty}^{\infty} dx P(x) |x - \bar{x}|^q}{\int_{-\infty}^{\infty} dx P(x)}, \tag{7}$$

thus allowing one to study, for instance, the Fisher equation.

3. Nonlinear diffusion coefficient

The classical diffusion equation has been widely used to study the dynamics of different population species. It assumes that there are no interactions between the random walkers. However, if the random walkers interact, the diffusivity might either increase or decrease in the presence of other random walkers. In other words, the diffusion coefficient D may increase as a result of population pressure or decrease because of mutual attraction.

We study the case where D becomes a function of the density u , leading to the following nonlinear diffusion equation [22]:

$$\frac{\partial u}{\partial t} = D_m \frac{\partial}{\partial x} \left(\left(\frac{u}{u_0} \right)^m \frac{\partial u}{\partial x} \right). \tag{8}$$

This nonlinear diffusion equation shows that for $m \neq 0$, diffusion is density dependent. Here the sign of D_m will dictate whether the population are moving away from or moving towards each other. If $D_m > 0$, individuals in a population move away from each other resulting from the population pressure and $D_m < 0$ corresponds to a situation where they move towards each other. The strength of such attraction or repulsion is $D_m(u/u_0)^m$, which is a nonlinear function of density. Here u_0 is the initial density of population. We also note that for $m = 0$, Eq. (8) reduces to the standard diffusion equation. Taking $\alpha = 1/(2 + m)$, the above equation allows the following exact solution:

$$u(x, t) = f(t)g\left(\frac{x}{t^\alpha}\right), \quad |x| \leq r_0(t/t_0)^\alpha, \\ = 0, \quad |x| > r_0(t/t_0)^\alpha \tag{9}$$

where

$$f(t) = u_0 \left(\frac{t_0}{t} \right)^\alpha, \\ g\left(\frac{x}{t^\alpha}\right) = \left(1 - \left(\frac{t_0^\alpha x}{r_0 t^\alpha} \right)^2 \right)^{1/m}, \tag{10}$$

with

$$r_0 = \frac{Q \Gamma\left(\frac{1}{m} + \frac{3}{2}\right)}{\pi^{1/2} u_0 \Gamma\left(\frac{1}{m} + 1\right)} \quad t_0 = \frac{r_0^2 m}{2D_m(m + 2)},$$

and Q is the initial number of populations released at the origin. Now, substituting $x/t^\alpha = z$, we can write

$$\langle x^q \rangle = \frac{t^{\alpha q} \int dz z^q g(z)}{\int dz g(z)}. \tag{11}$$

From the above equation the Hurst exponent is $H(q) = 1/(2 + m)$. For $m = 0$ we recover the classical diffusion problem with constant diffusion coefficient.

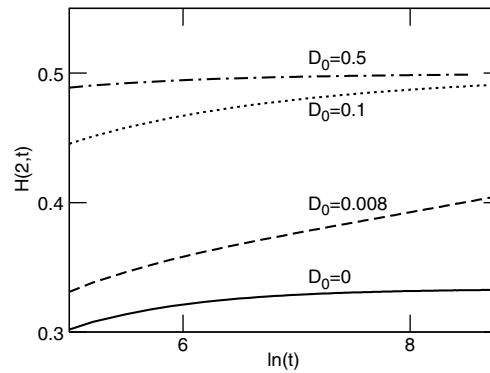


Fig. 1. Evolution of the Hurst exponent $H(2, t)$ for $m = 1$, $u_0 = 1$, $D_m = 1$ and for different values of D_0 . For $D_0 = 0$, the asymptotic dynamics is subdiffusive with $H(2) = 1/3$ as one expects from the exact analysis. For $D_0 \neq 0$, the dynamics is always diffusive in the long time limit. However, in the short time limit one finds subdiffusive behavior. The larger the value of D_0 , the faster the convergence to the asymptotic diffusive behavior. Parameters are in arbitrary units.

For the combined case of a diffusion coefficient that contains a constant part and a part dependent on the density, the dynamics evolves as

$$\frac{\partial u}{\partial t} = D_m \frac{\partial}{\partial x} \left(\left(\frac{u}{u_0} \right)^m \frac{\partial u}{\partial x} \right) + D_0 \frac{\partial^2 u}{\partial x^2}. \quad (12)$$

We solve this equation numerically using a semi-implicit method with spatial steps 0.05. In Fig. 1, we plot the Hurst exponent for $q = 2$, for $m = 1$. We observe that for $D_0 \neq 0$, the dynamics is always diffusive in the long time limit. However, in the short time limit we observe subdiffusive behavior. The temporal stretch of this subdiffusive behavior decreases with increasing value of D_0 , i.e., convergence to its asymptotic diffusive behavior becomes faster with increasing D_0 values.

4. The Fisher equation with fixed diffusion constant

The well-known Fisher equation [23] describes the dynamics of a field $u(x, t)$ subject to diffusive movement and a logistic reaction term:

$$\frac{\partial u}{\partial t} = D_0 \frac{\partial^2 u}{\partial x^2} + au - bu^2. \quad (13)$$

Here D_0 is the diffusion constant of the interacting species, a is the growth rate and b controls the competition for the resources. This equation was originally proposed for the study of an advantageous gene in a population. Eq. (13) allows a family of travelling wave solutions with speed $V \geq 2\sqrt{aD_0}$ to invade the unstable phase $u = 0$ from the stable phase $u = a/b$. However, for a steep enough initial condition the selected asymptotic speed is the minimum speed $V_0 = 2\sqrt{aD_0}$ [24].

In order to study invasion dynamics in terms of the Hurst exponent formalism, we numerically solve Eq. (13) using the method used in Section 3 and taking $D_0 = 1$, $a = 0.1$, $b = 1$ in appropriate (arbitrary) units. The initial density has the form of a step. It is zero everywhere except in the localized patch in the middle where it is uniform and nonzero. In Fig. 2, we have shown the result of this simulation. The familiar Fisher equation behavior wherein the density step rises to the saturation value a/b with a subsequent evolution of the front at the Fisher velocity is evident in Fig. 2(a). We see clearly from Fig. 2(b) and (c), particularly from the inset in the latter, that the Hurst exponent increases to the *super*-ballistic value (i.e. larger than 1) and then drops to the ballistic (i.e., 1) value at large times. The motion, which is nearly that characteristic of standard diffusion (i.e., $H = 0.5$) at short times, increases as the reaction terms kick in, becomes faster than ballistic and settles to the latter at long times when the system evolution is completely described by the two fronts traveling at constant (Fisher) velocity. This behavior of the Fisher equation, made apparent by focusing on the time-dependent Hurst exponent, is worthy of note.

5. Density-dependent diffusion constant

In many ecological systems the movement of animals depends on the local density of individuals [22,25]. Thus the diffusion constant in the Fisher equation needs to be replaced by a density-dependent function. This kind of dispersal arises from the possibility that individuals in a population may prefer to migrate from crowded regions to sparsely populated regions as mentioned in Section 3. The reverse effect may also occur as a result of attraction between members of a species.

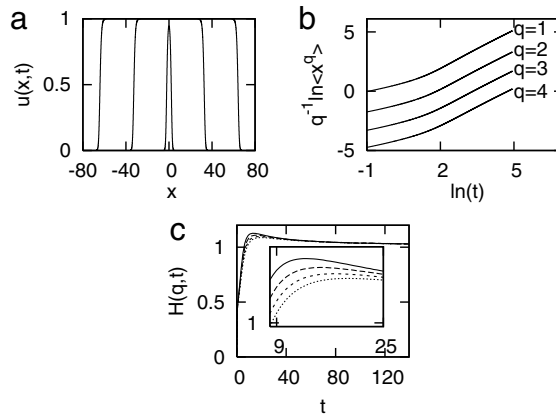


Fig. 2. (a) Solutions of the Fisher equation at different times starting with a step function initial density. Density has been expressed in the units of a/b and distance in terms of b/a . It is obvious that the stable phase invades the unstable phase. (b) Moments $M(q, t)$ with time expressed in terms of $1/a$. The plots have been scaled appropriately for the sake of visualization. (c) Local Hurst exponents $H(q, t)$ of solutions of the Fisher equation. Inset: The zoomed plot in a given time interval for $q = 1, 2, 3$ and 4 from top to bottom respectively.

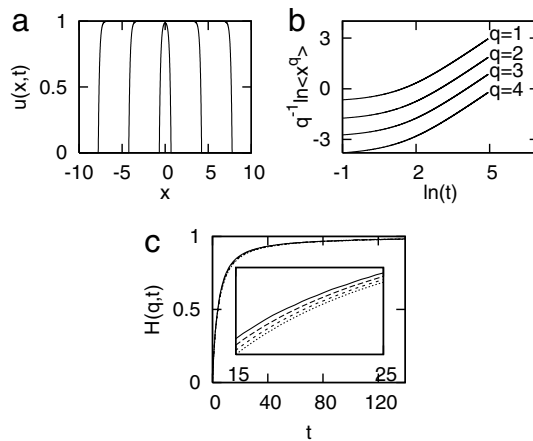


Fig. 3. Plots similar to those of Fig. 2 but with a diffusion constant that depends linearly on density. The density is expressed in units of a/b while distance is in terms of b/a and t in units of $1/a$ in this figure as well as in Fig. 2.

We try to capture the first of these density dependences by taking a simple linear relation $D(u) = D_0u$. This dependence modifies the diffusion term in Eq. (13) and the corresponding evolution can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(D_0u \frac{\partial u}{\partial x} \right) + au - bu^2. \tag{14}$$

We study the invasion dynamics in this case starting with an initial profile similar to that used for the Fisher equation with constant diffusion and keeping similar values of the parameters D_0, a and b . The result is shown in Fig. 3. The characteristic front evolution with inputs from the Fisher nature as well as from the power dependence of D is seen in Fig. 3(a). The resulting $H(q, t)$ as shown in Fig. 3(b) and (c) displays monotonic increase in contrast to the case of Fig. 2. The motion appears to remain always sub-ballistic.

6. The Nagumo equation

The Fisher equation assumes that the birth term is linear and the competition term for the resources is quadratic in u . A number of biological and ecological systems require that instead of the Fisher equation we use the Nagumo equation:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - au + bu^2 - cu^3. \tag{15}$$

Here the birth term is quadratic in the density and the competition term is cubic in the density. The Nagumo equation provides an additional zero in the nonlinearity relative to the Fisher case. The physical content behind such a term is the

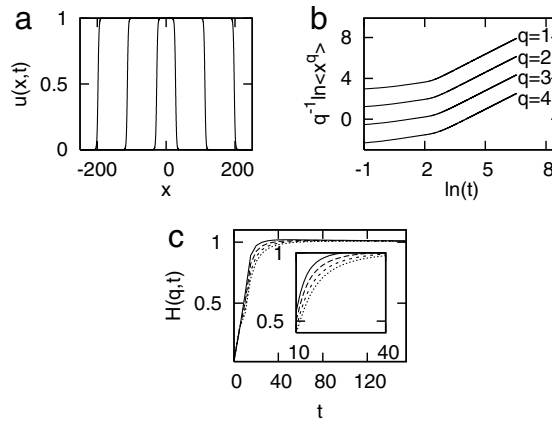


Fig. 4. (a) Solutions of the Nagumo equation at different times starting with a step function initial density. (b) Moments $M(q, t)$: The plots have been scaled appropriately for the sake of visualization. (c) Local Hurst exponents $H(q, t)$ of solutions of the Nagumo equation. Inset: The zoomed plot in a given time interval for $q = 1, 2, 3$ and 4 from top to bottom respectively.

Allee effect, in the presence of which, unlike in the logistic case, the zero- u solution is stable. If the density of population u is small initially, it is attracted towards the vanishing value and if large, it is attracted to the nonzero value. The physical origin of the Allee effect is the possible increase of survival fitness as a function of population size for low values of the latter. Existence of other members of the species may induce individuals to live longer whereas low densities may, through loneliness, lead to extinction. There is a great deal of evidence for such an effect in nature [26,27] and there have been recent reports [28,29] of theoretical work addressing the effect.

There are three possible homogeneous steady state solutions for u : $u_0 = 0$, $u_{\max} = (b + \sqrt{b^2 - 4ac})/2c$ and $u_{\min} = (b - \sqrt{b^2 - 4ac})/2c$. By performing linear stability analysis of these solutions, we find that u_0 and u_{\max} are stable while u_{\min} is unstable. It is convenient to reduce Eq. (15) to dimensionless form by performing the following substitutions:

$$\begin{aligned} x &\rightarrow x/\sqrt{u_{\max}^2 c/D_0}, \\ t &\rightarrow t/u_{\max}^2 c, \\ u &\rightarrow u/u_{\max}. \end{aligned} \quad (16)$$

Introducing a quantity $\alpha = u_{\min}/u_{\max}$ and using $b/u_{\max} = (\alpha + 1)c$ and $a/u_{\max}^2 = \alpha c$, we get the following equation in the dimensionless form:

$$\frac{\partial u}{\partial t} = \frac{D}{D_0} \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\alpha). \quad (17)$$

It can be shown that for the steady state homogeneous solution of Eq. (17), $u = 0$ and $u = 1$ are stable while $u = \alpha$ is unstable. However, the selection of one of the stable solutions out of two stable solutions depends on the initial condition. We study the invasion dynamics starting with an initial profile consisting of nonzero uniform density in the middle ($u = 0.4$) and zero on both sides of it and take $\alpha = 0.1$, $D = D_0 = 1$. The results are shown in Fig. 4. Similarities with Fig. 2 are clear except for the non-monotonic evolution of the Hurst exponent in the Fisher case.

7. Concluding remarks

In summary, we have extended the formalism of Hurst exponents to cases where the random walkers are in interaction with one another, the interactions being described via nonlinear Fokker–Planck equations. We have also applied the formalism to several cases. One of them involves the physical situation in which the diffusion coefficient of the walkers increases with their density as a result of population pressure. With an assumed power law dependence of the diffusion coefficient on the density, we have examined the exact analytic expression that can be written down for the Hurst exponent. We have also studied numerically three different reaction–diffusion equations which are not analytically solvable: the Fisher equation with diffusion that does and does not depend on the density of the walkers, and the Nagumo equation which has higher order reaction nonlinearities (Fig. 5). The generalized Hurst exponents obtained from these equations are, needless to say, consistent with the known behavior of their solutions.

The extension to nonlinear evolution equations of the concept of generalized Hurst exponents allows one to interpret the behavior of the solutions in terms of the concepts of anomalous diffusion. It would be helpful to have a single method of defining such generalized Hurst exponents, whether the focus of study is a linear or a nonlinear equation. Attempts that

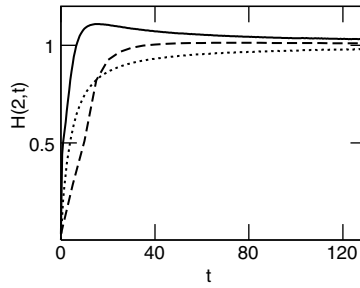


Fig. 5. Comparison of the behaviors of the three equations studied. The solid line corresponds to Fisher equation with fixed diffusion constant, the dotted line is for the density-dependent diffusion while the dashed line corresponds to the Nagumo equation.

we have made to develop such a unified approach have not yet succeeded. It might be worthwhile nevertheless to report our efforts in passing. For the estimation of Hurst exponents, we note that

$$H(q, t) = \frac{\ln M_q(t)}{q \ln t} \tag{18}$$

so the Hurst exponents represent slopes on double-log plots of the moments $M_q(t)$ and our results in this paper are based on this definition. However, even for linear problems, this definition can lead to errors, due to the effects of the initial conditions. As an illustrative example, consider the diffusion equation

$$\frac{\partial}{\partial t} P = D \frac{\partial^2}{\partial t^2} P, \tag{19}$$

whose integration leads immediately to

$$\langle x^2 \rangle = 2Dt + \langle x^2 \rangle_0 \tag{20}$$

where $\langle x^2 \rangle_0$ denotes the initial value. Except for the special case $\langle x^2 \rangle_0 = 0$ corresponding to a δ -function initial condition, Eq. (18) gives an incorrect value $H \neq 1/2$.

This error is an artifact of the method of estimation. Although Hurst exponents do in fact represent the slope in log–log plots of the moments, the initial value of the moment inside the logarithm leads to spurious estimation of anomalous diffusion. Only in the long time limit does one obtain the correct value.

Motivated by this methodological problem, one might attempt to estimate the Hurst exponents in a manner which is effectively immune or insensitive to the initial conditions (at least for the case of linear problems). Consider, for example, the following “new” definition for estimating Hurst exponents:

$$H(2, t) = \frac{1}{q} \left[1 + \frac{\ln \partial M_2(t) / \partial t}{\ln t} \right]. \tag{21}$$

By taking a time derivative inside the logarithm, we eliminate the initial values of the moments, since they are constants. But this reduces the power law exponent qH_q by unity; hence we compensate by explicitly adding 1 to get the correct value for qH_q . The above method for estimating H may not work if $qH > 1$. In this case, in addition to a constant, M_q might have a nonconstant dependence on initial conditions. However, in terms of scaling function f , as discussed in Section 2.1, one can define, in the long time limit, $H(q) = [\ln M_q(t) - \ln f(t/t^*)] / (q \ln t)$ for $t \gg t^*$. In this limit the scaling function f saturates, so one can estimate $H(q)$ in the long time (infinite time) limit which works even for nonlinear systems. More generally, we could try the following definition:

$$H(q, t) = \frac{1}{q} \left[\gamma + \frac{\ln \partial^\gamma M_q(t) / \partial t^\gamma}{\ln t} \right], \tag{22}$$

with $qH - 1 \leq \gamma \leq qH$. A good choice would be the integer part: $\gamma = [qH]$. Our reasoning is that for sufficiently large γ , all effects due to initial conditions can be removed. However, even this method does not always work. The question remains open.

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